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Research Article

Endpoint Estimates for Generalized Commutators of Hardy Operators on H^1 Space

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We study the H^1-boundedness of the generalized commutators of Hardy operator with a homogeneous kernel as follows: $\mathscr{H}^m_{\Omega,A,\beta}f(x) = (1/|x|^{n-\beta})\int_{|y|<|x|}(\Omega(x-y)/|x-y|^{m-1})R_m(A;x,y)f(y)dy$, where $R_m(A;x,y) = A(x) - \sum_{|\alpha|<m}(1/\alpha!)D^\alpha A(y)(x-y)^\alpha$ with $m \in Z^+$, $0 \le \beta < n$ and $\Omega \in \mathrm{Lip}_1(S^{n-1})$. We prove that, when $m \ge 1$, $\mathscr{H}^m_{\Omega,A,\beta}$ is not bounded from H^1 to $L^{n/(n-\beta)}$ unless $\mathscr{H}^m_{\Omega,A,\beta} \equiv 0$. Finally, we prove that $\mathscr{H}^m_{\Omega,A,\beta}$ is bounded from H^1 to $L^{n/(n-\beta),\infty}$ with $m \ge 1$.

1. Introduction

Let $f \in L^p(\mathbb{R}^+)$ with $1 < p < \infty$; the classical Hardy operator is defined by

$$Hf(x) = \frac{1}{x}\int_0^x f(t)\,dt, \quad x \neq 0. \tag{1}$$

A famous result proved by Hardy [1] can be stated as follows;

$$\|Hf\|_{L^p(\mathbb{R}^+)} \le \frac{p}{p-1}\|f\|_{L^p(\mathbb{R}^+)}. \tag{2}$$

Hardy [1] also pointed out the fact that the constant $p/(p-1)$ in (2) is the best possible. Later, Hardy operator was studied by many mathematicians; please see [2, 3] for more details.

In 1995, Christ and Grafakos [4] studied the following n-dimensional Hardy operator;

$$\mathscr{H}f(x) = \frac{1}{\nu_n|x|^n}\int_{|y|<|x|} f(y)\,dy, \quad x \in \mathbb{R}^n \setminus \{0\}, \tag{3}$$

where ν_n is the volume of the unit ball in $\mathbb{R}^n$, and they proved the following inequality:

$$\|\mathscr{H}f\|_{L^p} \le \frac{p}{p-1}\|f\|_{L^p}, \quad 1 < p < \infty. \tag{4}$$

Furthermore, Christ and Grafakos [4] also showed that the constant $p/(p-1)$ in (3) is the best possible.

In 2007, Fu et al. [5] considered the following commutator of fractional Hardy operator:

$$\mathscr{H}^b_\beta f(x) = b(x)\mathscr{H}_\beta f(x) - \mathscr{H}_\beta(fb)(x), \tag{5}$$

where $\mathscr{H}_\beta f(x)$ is defined by

$$\mathscr{H}_\beta f(x) = \frac{1}{|x|^{n-\beta}}\int_{|y|<|x|} f(y)\,dy \tag{6}$$

with $-n < \beta < n$. When $\beta = 0$, we simply denote $\mathscr{H}^b_0$ by $\mathscr{H}_b$ and $\mathscr{H}_0$ is just the n-dimensional Hardy operator proposed by Christ and Grafakos in [4] (without considering the constant ν_n).

In 2011, Fu et al. [6] studied the following n-dimensional fractional Hardy operator with a homogeneous kernel:

$$\mathscr{H}_{\Omega,\beta}f(x) = \frac{1}{|x|^{n-\beta}}\int_{|y|<|x|}\Omega(x-y)f(y)\,dy, \tag{7}$$

where $\Omega \in L^r(S^{n-1})$. Fu et al. [6] proved that $\mathscr{H}^b_{\Omega,\beta}$ is bounded on Herz type space and λ-central Morrey space. Here $\mathscr{H}^b_{\Omega,\beta}$

is just the commutator of fractional Hardy operator with a homogeneous kernel.

Recently, Zhao et al. [7] gave a counterexample to show that $\mathscr{H}_b$ is not bounded from H^1 to L^1, and they proved that $\mathscr{H}_b$ is bounded from H^1 to weak L^1 space where H^1 denotes the Hardy space.

On the other hand, in 1982, Cohen [8] studied the following generalized commutator:

$$T_A^2 f(x) = \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^{n+1}} \times (A(x) - A(y) - \nabla A(y)(x-y)) f(y)\, dy, \tag{8}$$

where $\Omega \in L^1(S^{n-1})$ is homogeneous of degree zero and satisfies the moment condition

$$\int_{S^{n-1}} \Omega(x) x^\alpha d\sigma(x) = 0 \tag{9}$$

for $|\alpha| = 1$. Cohen [8] proved that, if $\Omega \in \mathrm{Lip}_1(S^{n-1})$ and $\nabla A \in \mathrm{BMO}$, then T_A^2 is bounded on $L^p(\mathbb{R}^n)$ with $1 < p < \infty$.

Later, Cohen and Gosselin [9] considered another type of generalized commutator $T_A^m(f)$ as follows:

$$T_A^m f(x) = \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^{n+m-1}} R_m(A; x, y) f(y)\, dy, \tag{10}$$

where $R_m(A; x, y)$ ($m \in Z^+$) is defined by $R_m(A; x, y) = A(x) - \sum_{|\alpha|<m} (1/\alpha!) D^\alpha A(y)(x-y)^\alpha$, the mth remainder of Taylor series of the function A at x about y, and Ω satisfies the following moment conditions:

$$\int_{S^{n-1}} \Omega(x) x^\alpha d\sigma(x) = 0 \tag{11}$$

for $|\alpha| = m-1$. Obviously, if we choose $m = 1$, T_A^m becomes $[A, T]$, the commutator of T generalized by A and T.

Cohen and Gosselin [9] proved that T_A^m is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$ if $\Omega \in \mathrm{Lip}_1(S^{n-1})$ and the function A has derivatives of order $m-1$ in $\mathrm{BMO}(\mathbb{R}^n)$. Later, T_A^m was studied by many mathematicians; for example, see [10, 11] or [12] for more details. Particularly in [11], Lu and Wu studied the endpoint estimates of T_A^m on H^1 space.

It should be pointed out that the generalized commutators of some operators play an important role in the study of partial differential equation. Recently, by using the $W^{1,p}$ estimate for the elliptic equation of divergence form with partially BMO coefficients and the L^p boundedness of the Cohen-Gosselin type generalized commutators proved by Yan in [12], Wang and Zhang [13] gave a new proof of Wu's theorem in [14]. Here we would like to point out that the method used in [13] is much simpler than that in [14].

In this paper, we will consider the following generalized commutator of fractional Hardy operator with a homogeneous kernel:

$$\mathscr{H}_{\Omega,A,\beta}^m f(x) = \frac{1}{|x|^{n-\beta}} \int_{|y|<|x|} \frac{\Omega(x-y)}{|x-y|^{m-1}} f(y) R_m(A; x, y)\, dy, \tag{12}$$

where $R_m(A; x, y) = A(x) - \sum_{|\alpha|<m} (1/\alpha!) D^\alpha A(y)(x-y)^\alpha$, $0 \le \beta < n$, and $\Omega \in \mathrm{Lip}_1(S^{n-1})$.

As the Hardy operator is controlled by the Hardy-Littlewood maximal function, we have

$$\mathscr{H}_{\Omega,A,\beta}^m \le C M_{\Omega,A,\beta} f(x), \tag{13}$$

where $M_{\Omega,A,\beta} f(x)$ is defined by

$$M_{\Omega,A,\beta} f(x) = \sup_{r>0} \frac{1}{r^{n-\beta}} \int_{|x-y|<r} \frac{|\Omega(x-y)|}{|x-y|^{m-1}} |R_m(A; x, y) f(y)|\, dy. \tag{14}$$

By a simple computation or from [15, pp. 221-222], we have

$$\begin{aligned} M_{\Omega,A,\beta} f(x) &= \sup_{r>0} \frac{1}{r^{n-\beta}} \int_{|x-y|<r} \frac{|\Omega(x-y)|}{|x-y|^{m-1}} |R_m(A; x, y) f(y)|\, dy \\ &\le C \sup_{r>0} \frac{1}{r^{n-\beta}} \int_{(r/2)<|x-y|<r} \frac{|\Omega(x-y)|}{|x-y|^{m-1}} |R_m(A; x, y) f(y)|\, dy, \end{aligned} \tag{15}$$

and thus

$$\begin{aligned} \mathscr{H}_{\Omega,A,\beta}^m f(x) &\le C \sup_{r>0} \frac{1}{r^{n-\beta}} \int_{|x-y|<r} \frac{|\Omega(x-y)|}{|x-y|^{m-1}} |R_m(A; x, y) f(y)|\, dy \\ &\le C \sup_{r>0} \frac{1}{r^{n-\beta}} \int_{(r/2)<|x-y|<r} \frac{|\Omega(x-y)|}{|x-y|^{m-1}} |R_m(A; x, y) f(y)|\, dy \\ &\le C \sup_{r>0} r^{-(n-\beta+m-1)} \int_{(r/2)<|x-y|<r} |\Omega(x-y)| \times |R_m(A; x, y) f(y)|\, dy. \end{aligned} \tag{16}$$

So we have

$$\mathscr{H}_{\Omega,A,\beta}^m f(x) \le C \widetilde{M}_{\Omega,\beta}^A f(x), \tag{17}$$

where

$$\widetilde{M}_{\Omega,\beta}^A f(x) = \sup_{r>0} r^{-(n-\beta+m-1)} \times \int_{(r/2)<|x-y|<r} |\Omega(x-y) R_m(A; x, y) f(y)|\, dy. \tag{18}$$

From [15, p. 222], we have the following lemma.

Lemma 1 (see [15]). *Let $0 < \beta < n$ and $\Omega \in Lip_1(S^{n-1})$. If $1 < p$, $q < \infty$ with $1/p - 1/q = \beta/n$, and A has derivatives of order $m-1$ in $BMO(\mathbb{R}^n)$, then*

$$\left\| \widetilde{M}_{\Omega,\beta}^A f \right\|_{L^q} \le C \sum_{|\alpha|=m-1} \|D^\alpha A\|_{BMO} \|f\|_{L^p}, \tag{19}$$

where the constant C is independent of f and A.

By checking [15, p. 222] carefully, we deduce that (19) still holds if we take $\beta = 0$. So we have the following proposition.

Proposition 2. *Let $0 \le \beta < n$, $1/p - 1/q = \beta/n$ with $1 < p$, $q < \infty$. If $\Omega \in Lip_1(S^{n-1})$ and A has derivatives of order $m-1$ in $BMO(\mathbb{R}^n)$, then*

$$\left\|\mathscr{H}^m_{\Omega,A,\beta} f\right\|_{L^q} \le C \sum_{|\alpha|=m-1} \left\|D^\alpha A\right\|_{BMO} \|f\|_{L^p}, \tag{20}$$

where the constant C is independent of f and A.

Definition 3 (see [16]). One says a function $a(x)$ is an H^1 atom if a satisfies the following conditions:

$$\begin{aligned} &\text{(i)}\ \operatorname{supp}(a) \subset B(x_0, r), \\ &\text{(ii)}\ \|a\|_{L^\infty} \le \left|B(x_0, r)\right|^{-1}, \\ &\text{(iii)}\ \int a(x)\,dx = 0. \end{aligned} \tag{21}$$

It is well known that, if a function f belongs to H^1, then it can be written as $f = \sum_{i=1}^{\infty} \lambda_i a_i$ where each a_i is an H^1 atom. Moreover, one has

$$\|f\|_{H^1} \sim \inf\left\{\sum_{i=-\infty}^{+\infty} \left|\lambda_i\right|\right\}, \tag{22}$$

where the infimum is taken over all decompositions of f.

Definition 4 (see [17]). A function f is said to belong to $\mathrm{BMO}(\mathbb{R}^n)$ if the following sharp maximal function is bounded:

$$f^\sharp(x) = \sup_B \frac{1}{|B|} \int_B \left|f(y) - f_B\right| dy < \infty, \tag{23}$$

where the supreme is taken over all balls $B \subset \mathbb{R}^n$ and $f_B = (1/|B|)\int_B f(x)dx$ and $\|f\|_{\mathrm{BMO}} = \|f^\sharp\|_{L^\infty}$.

Proposition 5 (see [17]). *Let $1 < p < \infty$ and $f \in BMO(\mathbb{R}^n)$; then one has*

(a) $\|f\|_{BMO} \sim \sup_B((1/|B|)\int_B |f(x) - f_B|^p dx)^{1/p}$;

(b) $\|f\|_{BMO} \sim \sup_B \inf_{a\in R}(1/|B|)\int_B |f(x) - a| dx$.

Obviously, when $m = 1$, $\mathscr{H}^m_{\Omega,A,\beta}$ can be written as $\mathscr{H}^1_{\Omega,A,\beta} f(x) := \mathscr{H}_{\Omega,A,\beta} f(x) = A(x)\mathscr{H}_{\Omega,\beta} f(x) - \mathscr{H}_{\Omega,\beta}(fA)(x)$, just the commutator of fractional Hardy operator with a homogeneous kernel.

For the case $m = 2$, $\beta = 0$, and $\Omega \equiv 1$, Lu and Zhao [18] proved that $\mathscr{H}^2_{\Omega,A,0}$ is bounded on Herz type spaces and Morrey-Herz type spaces.

In this paper, we would like to show that $\mathscr{H}^m_{\Omega,A,\beta}$ is not bounded from H^1 to $L^{n/(n-\beta)}$ for all $m \in Z^+$. Furthermore, we will prove that $\mathscr{H}^m_{\Omega,A,\beta}$ is bounded from H^1 to $L^{n/(n-\beta),\infty}$, where $L^{n/(n-\beta),\infty}$ denotes the weak $L^{n/(n-\beta)}$ space. Some ideas of this paper come from Zhao et al. [7].

In this chapter, we would like to show that, if $A \in \mathrm{BMO}(\mathbb{R}^n)$, $\mathscr{H}_{\Omega,A,\beta}$ $(0 \le \beta < n)$ is not bounded from H^1 to $L^{n/(n-\beta)}$.

To show this, let $A(x) = \chi_{(4,\infty)}(x) \in \mathrm{BMO}$, $\Omega \equiv 1$, and $f_0(x) = \chi_{(0,4)}(x) - \chi_{(-4,0)}(x)$; then for $x > 8$ and $n = 1$, we have

$$\left|\mathscr{H}_{\Omega,A,\beta} f_0(x)\right| = \left|\frac{1}{x^{1-\beta}} \int_0^4 (1-0) \times 1 dy\right| = \frac{4}{x^{1-\beta}}, \tag{24}$$

and then

$$\int_{R^1} \left|\mathscr{H}_{\Omega,A,\beta} f_0(x)\right|^{1/(1-\beta)} dx \ge \int_8^\infty \frac{4^{1/(1-\beta)}}{x} dx = \infty, \tag{25}$$

which indicates that $\mathscr{H}_{\Omega,A,\beta}$ is not bounded from H^1 to $L^{n/(n-\beta)}$.

2. Endpoint Estimates for $\mathscr{H}^m_{\Omega,A,\beta}$ from H^1 to $L^{n/(n-\beta)}$

In Section 1, we know that, when $m = 1$, $\mathscr{H}^1_{\Omega,A,\beta}$ is not bounded from H^1 to $L^{n/(n-\beta)}$. In this section, we will prove that, when $m \ge 2$, $\mathscr{H}^m_{\Omega,A,\beta}$ is also not bounded from H^1 to $L^{n/(n-\beta)}$ unless $\mathscr{H}^m_{\Omega,A,\beta} \equiv 0$. We have the following conclusions.

Theorem 6. *Let $m \ge 2$, $0 \le \beta < n$, and $\Omega \in Lip_1(S^{n-1})$. Assume that A has derivatives of order $m-1$ in $BMO(\mathbb{R}^n)$; then the following two statements are equivalent;*

(i) *$\mathscr{H}^m_{\Omega,A,\beta}$ maps $H^1(\mathbb{R}^n)$ continuously into $L^{n/(n-\beta)}$;*

(ii) *for any H^1 atom supported on certain ball B and $u \in 3B \setminus 2B$, there is*

$$\begin{aligned} \int_{(4B)^c} &\left|\sum_{|\alpha|=m-1} \frac{1}{\alpha!} K_\alpha(x,u) \frac{1}{|x|^{n-\beta}} \int_B D^\alpha A(y)\, a(y)\, dy\right|^{n/(n-\beta)} \\ &\times dx \le C, \end{aligned} \tag{26}$$

where $K_\alpha(x,u) = \Omega(x-u)(x-u)^\alpha/|x-u|^{m-1}$ with $|\alpha| = m-1$.

In order to prove Theorem 6, we need the following lemma.

Lemma 7 (see [9]). *Let b be a function on $\mathbb{R}^n$ with mth order derivatives in $L^q_{loc}(\mathbb{R}^n)$ for some $q > n$; then*

$$\begin{aligned} \left|R_m(b; x, y)\right| \le C_{m,n}&|x-y|^m \\ &\times \sum_{|\alpha|=m} \left(\frac{1}{\left|\widetilde{Q}(x,y)\right|} \int_{\widetilde{Q}(x,y)} \left|D^\alpha b(z)\right|^q dz\right)^{1/q}, \end{aligned} \tag{27}$$

where $\widetilde{Q}(x,y)$ is the cube centered at x and having diameter $5\sqrt{n}|x-y|$.

Proof of Theorem 6. Suppose that $a(x)$ is an H^1 atom supported on $B(x_0, r)$ and satisfies (21). Now we take $\widetilde{a}(x) = a(x + x_0)$; then $\widetilde{a}$ is also an H^1 atom and satisfies

$$\begin{aligned}
&(\mathrm{i}')\ \operatorname{supp}(\widetilde{a}) \subset B(0,r),\\
&(\mathrm{ii}')\ \|\widetilde{a}\|_{L^\infty} \le |B(0,r)|^{-1},\\
&(\mathrm{iii}')\ \int_{\mathbb{R}^n} \widetilde{a}(x)\,dx = 0.
\end{aligned} \tag{28}$$

Thus by the main results in [19] and the atomic decomposition of the space $H^1(\mathbb{R}^n)$, it suffices to show that, for any H^1 atom $\widetilde{a}$, we have $\|\mathscr{H}^m_{\Omega,A,\beta}\widetilde{a}\|_{L^{n/(n-\beta)}} \le C$.

Let $B = B(0,r)$ and $\widetilde{A}(x) = A(x) - \sum_{|\alpha|=m-1}(1/\alpha!)m_B(D^\alpha A)x^\alpha$; then $R_m(A;x,y) = R_m(\widetilde{A};x,y)$. For each H^1 atom $\widetilde{a}$, we split each $\mathscr{H}^m_{\Omega,A,\beta}\widetilde{a}(x)$ as

$$\begin{aligned}
&\mathscr{H}^m_{\Omega,A,\beta}\widetilde{a}(x)\\
&\quad = \chi_{4B}(x)\,\mathscr{H}^m_{\Omega,A,\beta}\widetilde{a}(x) + \chi_{(4B)^c}(x)\,\mathscr{H}^m_{\Omega,A,\beta}\widetilde{a}(x)\\
&\quad := \mu_1(x) + \mu_2(x).
\end{aligned} \tag{29}$$

For $\mu_1(x)$, taking $n/(n-\beta) < q < \infty$ and p so that $1/p - 1/q = \beta/n$, it follows from Proposition 2 that

$$\begin{aligned}
\|\mu_1\|_{L^{n/(n-\beta)}} &\le |4B|^{(n-\beta)/n-1/q}\left\|\mathscr{H}^m_{\Omega,A,\beta}\widetilde{a}\right\|_{L^q}\\
&\le C\sum_{|\alpha|=m-1}\|D^\alpha A\|_{\mathrm{BMO}}|B|^{1-1/p}\|\widetilde{a}\|_{L^p}\\
&\le C\sum_{|\alpha|=m-1}\|D^\alpha A\|_{\mathrm{BMO}}.
\end{aligned} \tag{30}$$

For $\mu_2(x)$, as $x \in (4B)^c$ and $|y| < |x|$, we can deduce $\{y : |y| < |x|\} \cap \{y : y \in B(0,r)\} = \{y : y \in B(0,r)\}$; thus we have

$$\begin{aligned}
\mu_2(x) &= \chi_{(4B)^c}\frac{1}{|x|^{n-\beta}}\\
&\quad\times\int_{B(0,r)}\frac{\Omega(x-y)}{|x-y|^{m-1}}R_m\left(\widetilde{A};x,y\right)\widetilde{a}(y)\,dy.
\end{aligned} \tag{31}$$

Next we denote $B(0,r) = B$ and $B(0,kr) = kB$; then by the vanishing condition of $\widetilde{a}$, we decompose μ_2 as follows:

$$\begin{aligned}
&\mu_2(x)\\
&= \chi_{(4B)^c}(x)\frac{1}{|x|^{n-\beta}}\int_B\left(\frac{\Omega(x-y)}{|x-y|^{m-1}}R_{m-1}\left(\widetilde{A};x,y\right)\right.\\
&\qquad\left.-\frac{\Omega(x-u)}{|x-u|^{m-1}}R_{m-1}\left(\widetilde{A};x,u\right)\right)\\
&\quad\times\widetilde{a}(y)\,dy - \chi_{(4B)^c}(x)\frac{1}{|x|^{n-\beta}}\\
&\quad\times\sum_{|\alpha|=m-1}\frac{1}{\alpha!}\int_B\left[K_\alpha(x,y) - K_\alpha(x,u)\right]D^\alpha\widetilde{A}(y)\,\widetilde{a}(y)\,dy\\
&\quad-\chi_{(4B)^c}(x)\frac{1}{|x|^{n-\beta}}\\
&\quad\times\sum_{|\alpha|=m-1}\frac{1}{\alpha!}\int_B K_\alpha(x,u)\,D^\alpha\widetilde{A}(y)\,\widetilde{a}(y)\,dy\\
&= \mu_{21}(x,u) - \mu_{22}(x,u) - \mu_{23}(x,u),
\end{aligned} \tag{32}$$

where $u \in 3B \setminus 2B$.

For $\mu_{21}(x,u)$, as $u \in 3B \setminus 2B$, $y \in B$, and $x \in (4B)^c$, we have

$$|x-y| \sim |x-u| \sim |x|. \tag{33}$$

Thus we have

$$\begin{aligned}
&\left|\frac{\Omega(x-y)}{|x-y|^{m-1}}R_{m-1}\left(\widetilde{A};x,y\right) - \frac{\Omega(x-u)}{|x-u|^{m-1}}R_{m-1}\left(\widetilde{A};x,u\right)\right|\\
&\le\left|\frac{\Omega(x-y)}{|x-y|^{m-1}}R_{m-1}\left(\widetilde{A};x,y\right) - \frac{\Omega(x-y)}{|x-u|^{m-1}}R_{m-1}\left(\widetilde{A};x,y\right)\right|\\
&\quad+\left|\frac{\Omega(x-y)}{|x-u|^{m-1}}R_{m-1}\left(\widetilde{A};x,y\right) - \frac{\Omega(x-u)}{|x-u|^{m-1}}R_{m-1}\left(\widetilde{A};x,y\right)\right|\\
&\quad+\left|\frac{\Omega(x-u)}{|x-u|^{m-1}}\right|\left|R_{m-1}\left(\widetilde{A};x,y\right) - R_{m-1}\left(\widetilde{A};x,u\right)\right|\\
&:= I + II + III.
\end{aligned} \tag{34}$$

For the term I, by Lemma 7, we have

$$\begin{aligned}
I &\le C\frac{|\Omega(x-y)|}{|x-y|^m}|y-u|\left|R_{m-1}\left(\widetilde{A};x,y\right)\right|\\
&\le C\frac{|y-u|}{|x-y|^m}|x-y|^{m-1}k\sum_{|\alpha|=m-1}\|D^\alpha A\|_{\mathrm{BMO}}\\
&\le Ck2^{-k}\sum_{|\alpha|=m-1}\|D^\alpha A\|_{\mathrm{BMO}}.
\end{aligned} \tag{35}$$

For the term II, by the similar estimates of I, we obtain

$$II \le Ck2^{-k}\sum_{|\alpha|=m-1}\|D^\alpha A\|_{\mathrm{BMO}}. \tag{36}$$

For the term III, by the following formula (see [20]):

$$\begin{aligned}
&R_m(A;x,y) - R_m(A;x,z)\\
&\quad = \sum_{|\alpha|<m}\frac{1}{\beta!}R_{m-|\alpha|}\left(D^\alpha A;z,y\right)(x-z)^\alpha,
\end{aligned} \tag{37}$$

and then together with Lemma 7, we have

$$III \le C \sum_{|\alpha|=m-1} \|D^{\alpha}A\|_{\mathrm{BMO}} \sum_{i=0}^{m-2} |x-y|^{-(m-1)+l} |y-u|^{m-1-i} \le C \sum_{|\alpha|=m-1} \|D^{\alpha}A\|_{\mathrm{BMO}}\ k2^{-k}. \tag{38}$$

Thus for $\mu_{21}(x,u)$, by the size condition of $\tilde{a}$, we get the following estimates:

$$\begin{aligned} \|\mu_{21}(\cdot,u)\|_{L^{n/(n-\beta)}} &\le C \sum_{|\alpha|=m-1} \|D^{\alpha}A\|_{\mathrm{BMO}} \times \sum_{k=2}^{\infty} k2^{-k} \left[\int_{2^{k+1}B\setminus 2^kB} \left(\frac{1}{|x|^{n-\beta}} \times \int_B |\tilde{a}(y)|\,dy \right)^{n/(n-\beta)} dx \right]^{(n-\beta)/n} \\ &\le C \sum_{|\alpha|=m-1} \|D^{\alpha}A\|_{\mathrm{BMO}} \sum_{k=2}^{\infty} k2^{-k} \le C \sum_{|\alpha|=m-1} \|D^{\alpha}A\|_{\mathrm{BMO}}. \end{aligned} \tag{39}$$

For $\mu_{22}(x,u)$, since $\Omega \in \mathrm{Lip}_1(S^{n-1})$, we have the following estimates of $|K_{\alpha}(x-y) - K_{\alpha}(x-u)|$:

$$\begin{aligned} |K_{\alpha}(x-y) - K_{\alpha}(x-u)| &\le \left| \frac{\Omega(x-y)(x-y)^{\alpha}}{|x-y|^{m-1}} - \frac{\Omega(x-y)(x-u)^{\alpha}}{|x-y|^{m-1}} \right| \\ &+ \left| \frac{\Omega(x-y)(x-u)^{\alpha}}{|x-y|^{m-1}} - \frac{\Omega(x-u)(x-u)^{\alpha}}{|x-y|^{m-1}} \right| \\ &+ \left| \frac{\Omega(x-u)(x-u)^{\alpha}}{|x-y|^{m-1}} - \frac{\Omega(x-u)(x-u)^{\alpha}}{|x-u|^{m-1}} \right| \\ &\le C \frac{|y-u|}{|x-y|}. \end{aligned} \tag{40}$$

Thus by the size condition of $\tilde{a}$ and Lemma 7, we have

$$\begin{aligned} \|\mu_{22}(\cdot,u)\|_{L^{n/(n-\beta)}} &\le C \sum_{|\alpha|=m-1} \sum_{k=2}^{\infty} \left[\int_{2^{k+1}B\setminus 2^kB} \left(\int_B \frac{|y-u|}{|x-y|\,|x|^{n-\beta}}\ |\tilde{a}(y)| \times |D^{\alpha}(\tilde{A})(y)|\,dy \right)^{n/(n-\beta)} dx \right]^{(n-\beta)/n} \\ &\le C \sum_{|\alpha|=m-1} \|D^{\alpha}A\|_{\mathrm{BMO}} \times \sum_{k=2}^{\infty} \left|2^kB\right|^{(\beta-n)/n} |B|^{-1}\,|B|\,\left|2^kB\right|^{(n-\beta)/n} 2^{-k} \\ &\le C \sum_{|\alpha|=m-1} \|D^{\alpha}A\|_{\mathrm{BMO}}. \end{aligned} \tag{41}$$

Now we can deduce that $\|\mathscr{H}^m_{\Omega,A,\beta}\tilde{a}\|_{L^{n/(n-\beta)}} \le C$ is equivalent to $\|\mu_{23}(\cdot,u)\|_{L^{n/(n-\beta)}} \le C$. By the vanishing condition of $\tilde{a}$, we can easily get

$$\int_{(4B)^c} \left| \sum_{|\alpha|=m-1} \frac{1}{\alpha!} K_{\alpha}(x,u) \frac{1}{|x|^{n-\beta}} \int_B D^{\alpha}A(y)\,\tilde{a}(y)\,dy \right|^{n/(n-\beta)} \times dx \le C. \tag{42}$$

Consequently, we have finished the proof of Theorem 6. □

Next we would like to show that $\mathscr{H}^m_{\Omega,A,\beta}$ is not bounded from H^1 to $L^{n/(n-\beta)}$ unless $\mathscr{H}^m_{\Omega,A,\beta} \equiv 0$. We have the following theorem.

Theorem 8. *Let $m \ge 2$, $0 \le \beta < n$, and $\Omega \in Lip_1(S^{n-1})$, and assume that A has derivatives of order $m-1$ in $BMO(\mathbb{R}^n)$. Then the following two statements are equivalent:*

(i) *$\mathscr{H}^m_{\Omega,A,\beta}$ maps H^1 continuously into $L^{n/(n-\beta)}$,*

(ii) *A is a polynomial of degree no more than $m-1$ or $\Omega \equiv 0$.*

Remark 9. From Theorem 8, we can draw the conclusion that, when $m \ge 2$, $\mathscr{H}^m_{\Omega,A,\beta}$ is not bounded from H^1 to $L^{n/(n-\beta)}$ unless $\mathscr{H}^m_{\Omega,A,\beta} \equiv 0$.

Proof of Theorem 8. It is clear that (ii) ⇒ (i) is obvious. We only need to prove (i) ⇒ (ii).

Let $\tilde{a}$ be an H^1 atom supported on the ball $B = B(0,r)$, and denote $C_{\alpha} = (1/\alpha!)\int_B D^{\alpha}A(y)\tilde{a}(y)dy$ with $|\alpha| = m-1$. By Theorem 6, for any $u \in 3B \setminus 2B$ and $N > 8$ with $N \in Z^+$, we have

$$\begin{aligned} C &\ge \int_{(4B)^c} \left| \frac{1}{|x|^{n-\beta}} \sum_{|\alpha|=m-1} C_{\alpha} \frac{\Omega(x-u)(x-u)^{\alpha}}{|x-u|^{m-1}} \right|^{n/(n-\beta)} dx \\ &\ge C_1 \int_{(4B)^c} \left| \frac{1}{|x-u|^{n-\beta}} \sum_{|\alpha|=m-1} C_{\alpha} \frac{\Omega(x-u)(x-u)^{\alpha}}{|x-u|^{m-1}} \right|^{n/(n-\beta)} dx \end{aligned}$$

$$\geq C_1 \int_{8r\leq|x|\leq Nr} \left| \frac{1}{|x|^{n-\beta}} \sum_{|\alpha|=m-1} C_\alpha \Omega(x) \frac{x^\alpha}{|x|^{m-1}} \right|^{n/(n-\beta)} dx$$
$$= C_1 \log\left(\frac{N}{8}\right) \int_{S^{n-1}} \left| \sum_{|\alpha|=m-1} C_\alpha \Omega(x')(x')^\alpha \right|^{n/(n-\beta)} d\sigma(x'). \tag{43}$$

Let $N \to +\infty$, we know $\log(N/8) \to +\infty$. Thus we have

$$\int_{S^{n-1}} \left| \sum_{|\alpha|=m-1} C_\alpha \Omega(x')(x')^\alpha \right|^{n/(n-\beta)} d\sigma(x') = 0. \tag{44}$$

From (44) we can deduce

$$\sum_{|\alpha|=m-1} C_\alpha \Omega(x')(x')^\alpha = 0. \tag{45}$$

If $\Omega \equiv 0$, (45) is obviously true. Otherwise, we can easily obtain

$$C_\alpha = \frac{1}{\alpha!} \int_B D^\alpha A(y)\, \tilde{a}(y)\, dy = 0. \tag{46}$$

Since $\tilde{a}$ is arbitrary, $D^\alpha A$ must be a constant for each α with $|\alpha| = m-1$. So we can deduce that A is a polynomial with degree no more than $m-1$.

Consequently, we have finished the proof of Theorem 8. □

3. Boundedness of $\mathscr{H}^m_{\Omega,A,\beta}$ from H^1 to $L^{n/(n-\beta),\infty}$

In Section 2, we prove that, when $m \geq 2$, $\mathscr{H}^m_{\Omega,A,\beta}$ is not bounded from H^1 to $L^{n/(n-\beta)}$ unless $\mathscr{H}^m_{\Omega,A,\beta} \equiv 0$. In this section, we will prove that $\mathscr{H}^m_{\Omega,A,\beta}$ is bounded from H^1 to $L^{n/(n-\beta),\infty}$ with $m \geq 1$. Here $L^{n/(n-\beta),\infty}$ is defined by

$$\|f\|_{L^{n/(n-\beta),\infty}} = \sup_{\lambda>0} \lambda \left|\{x \in \mathbb{R}^n : |f(x)| > \lambda\}\right|^{(n-\beta)/n} < \infty. \tag{47}$$

Our results can be stated as follows.

Theorem 10. *Suppose that $m \geq 2$, $0 \leq \beta < n$, and $\Omega \in Lip_1(S^{n-1})$. If A has derivatives of order $m-1$ in $BMO(\mathbb{R}^n)$, then there exists a constant C independent of f and A, such that*

$$\left|\left\{x \in \mathbb{R}^n : \left|\mathscr{H}^m_{\Omega,A,\beta} f(x)\right| > \lambda\right\}\right|^{(n-\beta)/n} \leq C \sum_{|\alpha|=m-1} \|D^\alpha A\|_{BMO} \frac{\|f\|_{H^1}}{\lambda} \tag{48}$$

for any $\lambda > 0$.

For the case $m = 1$, we have the following.

Theorem 11. *Let $0 \leq \beta < n$, $A \in BMO(\mathbb{R}^n)$, and $\Omega \in Lip_1(S^{n-1})$; then there exists a constant C independent of f and A, such that*

$$\left|\left\{x \in \mathbb{R}^n : \left|\mathscr{H}^1_{\Omega,A,\beta} f(x)\right| > \lambda\right\}\right|^{(n-\beta)/n} \leq C \sum_{|\alpha|=m-1} \|D^\alpha A\|_{BMO} \frac{\|f\|_{H^1}}{\lambda} \tag{49}$$

for any $\lambda > 0$.

Before the proof of Theorems 10 and 11, we need the following lemma.

Lemma 12. *Let $\widetilde{\mathscr{H}}^\alpha_\beta$ be defined by*

$$\widetilde{\mathscr{H}}^\alpha_\beta h(x) = \frac{1}{|x|^{n-\beta}} \int_{|y|<|x|} h(y)(x-y)^\alpha \frac{\Omega(x-y)}{|x-y|^{m-1}} dy, \tag{50}$$

where $0 \leq \beta < n$, $\Omega \in Lip_1(S^{n-1})$, and $|\alpha| = m-1$; then $\widetilde{\mathscr{H}}^\alpha_\beta$ is bounded from L^1 to $L^{n/(n-\beta),\infty}$.

Proof. For any $\lambda > 0$, we have

$$\lambda \left|\left\{x \in \mathbb{R}^n : \left|\widetilde{\mathscr{H}}^\alpha_\beta h(x)\right| > \lambda\right\}\right|^{(n-\beta)/n}$$
$$= \lambda \left|\left\{x \in \mathbb{R}^n : \left| \frac{1}{|x|^{n-\beta}} \int_{|y|<|x|} h(y)(x-y)^\alpha \times \frac{\Omega(x-y)}{|x-y|^{m-1}} dy \right| > \lambda\right\}\right|^{(n-\beta)/n}$$
$$\leq \lambda \left|\left\{x \in \mathbb{R}^n : \left| \frac{1}{|x|^{n-\beta}} \int_{|y|<|x|} |h(y)| \frac{1}{|x-y|^{m-1-|\alpha|}} dy \right| > \lambda\right\}\right|^{(n-\beta)/n}$$
$$\leq \lambda \left|\left\{x \in \mathbb{R}^n : \left| \frac{1}{|x|^{n-\beta}} \int_{|y|<|x|} |h(y)|\, dy \right| > \lambda\right\}\right|^{(n-\beta)/n}$$
$$\leq \lambda \left|\left\{x \in \mathbb{R}^n : |x| < \frac{\|h\|_{L^1}^{1/(n-\beta)}}{\lambda^{1/(n-\beta)}}\right\}\right|^{(n-\beta)/n}$$
$$= \lambda \left(\int_{S^{n-1}} \int_0^{(\|h\|_{L^1}^{1/(n-\beta)})/(\lambda^{1/(n-\beta)})} r^{n-1}\, dr\, d\sigma(x') \right)^{(n-\beta)/n}$$
$$\leq C\lambda \left(\frac{\|h\|_{L^1}^{n/(n-\beta)}}{\lambda^{n/(n-\beta)}} \right)^{(n-\beta)/n} = C\|h\|_{L^1}. \tag{51}$$

□

Proposition 13. *By the proof of Lemma 12 with minor changes, one can draw the conclusion that $\mathscr{H}_{\Omega,\beta}$ is bounded from L^1 to $L^{n/(n-\beta),\infty}$ with $\Omega \in Lip_1(S^{n-1})$.*

Proof of Theorem 10. Before giving the proof of Theorem 10, we introduce some notations that are very useful in this section.

For multi-indices $\alpha = (\alpha_1, \dots, \alpha_n)$, $\beta = (\beta_1, \dots, \beta_n)$, we denote $\alpha - \beta = (\alpha_1 - \beta_1, \dots, \alpha_n - \beta_n)$. Furthermore, $\beta < \alpha$ means that for each i, we have $\beta_i < \alpha_i$. Finally, we denote $C_\alpha^\beta = \prod_{j=1}^m C_{\alpha_j}^{\beta_j}$.

From [19] and by the atomic decomposition of H^1, it suffices to show

$$\left|\left\{x \in \mathbb{R}^n : \left|\mathscr{H}_{\Omega,A,\beta}^m \widetilde{a}(x)\right| > \lambda\right\}\right|^{(n-\beta)/n} \le \frac{C}{\lambda} \sum_{|\alpha|=m-1} \left\|D^\alpha A\right\|_{\mathrm{BMO}} \tag{52}$$

for any H^1 atom $\widetilde{a}$, where $\widetilde{a}$ is defined in Section 2. First we have the following decomposition:

$$\begin{aligned} &\left|\left\{x \in \mathbb{R}^n : \mathscr{H}_{\Omega,A,\beta}^m \widetilde{a}(x) > \lambda\right\}\right|^{(n-\beta)/n} \\ &\quad\le \left|\left\{x \in B(0,2r) : \mathscr{H}_{\Omega,A,\beta}^m \widetilde{a}(x) > \lambda\right\}\right|^{(n-\beta)/n} \\ &\quad\quad + \left|\left\{x \in \mathbb{R}^n \setminus B(0,2r) : \mathscr{H}_{\Omega,A,\beta}^m \widetilde{a}(x) > \lambda\right\}\right|^{(n-\beta)/n} \\ &\quad := L_1 + L_2. \end{aligned} \tag{53}$$

For the term L_1, choosing $B(0,r) = B$ and $B(0,kr) = kB$ with $k \in Z^+$, then by Proposition 2, Hölder inequality, and the size condition of $\widetilde{a}$, we have

$$\begin{aligned} &\lambda\left|\left\{x \in B(0,2r) : \mathscr{H}_{\Omega,A,\beta}^m \widetilde{a}(x) > \lambda\right\}\right|^{(n-\beta)/n} \\ &\quad\le C\left(\int_{B(0,2r)} \left|\mathscr{H}_{\Omega,A,\beta}^m \widetilde{a}(x)\right|^{n/(n-\beta)} dx\right)^{(n-\beta)/n} \\ &\quad\le C|2B|^{(n-\beta)/n-1/q}\left(\int_{B(0,2r)} \left|\mathscr{H}_{\Omega,A,\beta}^m \widetilde{a}(x)\right|^q dx\right)^{1/q} \\ &\quad\le C|2B|^{(n-\beta)/n-1/q} \sum_{|\alpha|=m-1} \left\|D^\alpha A\right\|_{\mathrm{BMO}} \|\widetilde{a}\|_{L^p} \\ &\quad\le C \sum_{|\alpha|=m-1} \left\|D^\alpha A\right\|_{\mathrm{BMO}}, \end{aligned} \tag{54}$$

where $1/p - 1/q = \beta/n$ and $p > 1$.

For the term L_2, as $x \in \mathbb{R}^n \setminus B(0,2r)$, we have

$$\{y : |y| < |x|\} \cap \{y : |y| < r\} = \{y : |y| < r\}. \tag{55}$$

Then for a fixed $B = B(0,r)$, we set $A_B(x) = A(x) - \sum_{|\alpha|=m-1}(1/\alpha!)(D^\alpha A)_B x^\alpha$. It is easy to check $R_m(A; x, y) = R_m(A_B; x, y)$.

Thus by the vanishing condition of $\widetilde{a}$ and the fact $\operatorname{supp}(\widetilde{a}) \subset B(0,r)$, we can decompose $\mathscr{H}_{\Omega,A,\beta}^m \widetilde{a}(x)$ as follows:

$$\begin{aligned} &\mathscr{H}_{\Omega,A,\beta}^m \widetilde{a}(x) \\ &= \frac{1}{|x|^{n-\beta}} \int_{\mathbb{R}^n} \left\{R_m(A_B; x, y) \frac{\Omega(x-y)}{|x-y|^{m-1}} \right. \\ &\quad - \left[R_{m-1}(A_B; x, 0) - \sum_{|\alpha|=m-1} \frac{1}{\alpha!} D^\alpha A_B(x) x^\alpha\right] \\ &\quad \left. \times \frac{\Omega(x)}{|x|^{m-1}}\right\} \widetilde{a}(y)\, dy \\ &= -\frac{1}{|x|^{n-\beta}} \int_{\mathbb{R}^n} \left[R_{m-1}(A_B; x, 0) - \sum_{|\alpha|=m-1} \frac{1}{\alpha!} D^\alpha A_B(x) x^\alpha\right] \\ &\quad \times \left[\frac{\Omega(x)}{|x|^{m-1}} - \frac{\Omega(x-y)}{|x-y|^{m-1}}\right] \widetilde{a}(y)\, dy + \frac{1}{|x|^{n-\beta}} \\ &\quad \times \int_{\mathbb{R}^n} \left[\sum_{|\alpha|<m-1} \frac{1}{\alpha!} D^\alpha A_B(y)(x-y)^\alpha \right. \\ &\quad \left. - \sum_{|\alpha|<m-1} D^\alpha A_B(0) x^\alpha\right] \frac{\Omega(x-y)}{|x-y|^{m-1}} \widetilde{a}(y)\, dy + \frac{1}{|x|^{n-\beta}} \\ &\quad \times \int_{\mathbb{R}^n} \left[\sum_{|\alpha|=m-1} \frac{1}{\alpha!} D^\alpha A_B(y) x^\alpha \right. \\ &\quad \left. - \sum_{|\alpha|=m-1} \frac{1}{\alpha!} D^\alpha A_B(x) x^\alpha\right] \frac{\Omega(x-y)}{|x-y|^{m-1}} \widetilde{a}(y)\, dy \\ &\quad + \frac{1}{|x|^{n-\beta}} \int_{\mathbb{R}^n} \sum_{|\alpha|=m-1} \frac{1}{\alpha!} D^\alpha A_B(y) \\ &\quad \times \sum_{\gamma<\alpha} C_\alpha^\gamma x^\gamma (-y)^{\alpha-\gamma} \frac{\Omega(x-y)}{|x-y|^{m-1}} \widetilde{a}(y)\, dy \\ &:= I_1 + I_2 + I_3 + I_4. \end{aligned} \tag{56}$$

Here we can simply denote each I_i by

$$I_i = \frac{1}{|x|^{n-\beta}} \int_{\mathbb{R}^n} K_i(A_B; x, y)\, \widetilde{a}(y)\, dy. \tag{57}$$

For K_1, by the fact that $\Omega \in \mathrm{Lip}_1(S^{n-1})$ and $|x-y| \sim |x|$, we have

$$\begin{aligned} \left|\frac{\Omega(x)}{|x|^{m-1}} - \frac{\Omega(x-y)}{|x-y|^{m-1}}\right| &\le \left|\frac{\Omega(x)}{|x|^{m-1}} - \frac{\Omega(x-y)}{|x|^{m-1}}\right| \\ &\quad + \left|\frac{\Omega(x-y)}{|x|^{m-1}} - \frac{\Omega(x-y)}{|x-y|^{m-1}}\right| \end{aligned}$$

$$= \left| \frac{\Omega\left((x/|x|)\left(|x|/|x-y|\right)\right)}{|x|^{m-1}} - \frac{\Omega\left((x-y)/|x-y|\right)}{|x|^{m-1}} \right| + \left| \frac{\Omega(x-y)}{|x|^{m-1}} - \frac{\Omega(x-y)}{|x-y|^{m-1}} \right| \le C\frac{|y|}{|x|^{m}}. \tag{58}$$

As $|x| > r$ for $x \in \mathbb{R}^n \setminus 2B$, Lemma 7 in this paper and Lemma 2.2 in [8] tell us that

$$\begin{aligned} &\left|K_1\left(A_B; x, y\right)\right| \\ &\le C|x|^{m-1}\left[\sum_{|\alpha|=m-1}\left(\frac{1}{|\widetilde{Q}|}\int_{\widetilde{Q}}\left|D^{\alpha}A(z)-\left(D^{\alpha}A\right)_B\right|^q dz\right)^{q} + \sum_{|\alpha|=m-1}\left|D^{\alpha}A(x)-\left(D^{\alpha}A\right)_B\right|\right]\frac{|y|}{|x|^{m}} \\ &\le C\frac{|y|}{|x|}\left(\sum_{|\alpha|=m-1}\left\|D^{\alpha}A\right\|_{\mathrm{BMO}}\left(C_1+\log\frac{|x|}{r}\right) + \sum_{|\alpha|=m-1}\left|D^{\alpha}A(x)-\left(D^{\alpha}A\right)_B\right|\right), \end{aligned} \tag{59}$$

where C_1 is a constant only depending on n, and $\widetilde{Q}$ is a cube centered at x and having diameter $5\sqrt{n}|x|$.

Thus we obtain

$$\begin{aligned} &\lambda\left|\left\{x\in\mathbb{R}^n\setminus B(0,2r) : \left|I_1\right|>\lambda\right\}\right|^{(n-\beta)/n} \\ &\le \left(\int_{\mathbb{R}^n\setminus B(0,2r)}\left|I_1\right|^{n/(n-\beta)}dx\right)^{(n-\beta)/n} \\ &\le C\left(\int_{\mathbb{R}^n\setminus B(0,2r)}|x|^{(\beta-n)(n/(n-\beta))} \times\left(\int_{\mathbb{R}^n}\frac{|y|}{|x|}\left(\sum_{|\alpha|=m-1}\left\|D^{\alpha}A\right\|_{\mathrm{BMO}}\left(C_1+\log\frac{|x|}{r}\right) + \sum_{|\alpha|=m-1}\left(D^{\alpha}A(x)-\left(D^{\alpha}A\right)_B\right)\right) \times\widetilde{a}(y)\,dy\right)^{n/(n-\beta)}dx\right)^{(n-\beta)/n} \\ &\le C\left(\sum_{k=1}^{+\infty}\int_{2^{k+1}B\setminus 2^kB}|x|^{-n}|B|^{1/(n-\beta)}|x|^{n/(\beta-n)} \times\left(\sum_{|\alpha|=m-1}\left\|D^{\alpha}A\right\|_{\mathrm{BMO}}\left(C_1+\log\frac{|x|}{r}\right) + \sum_{|\alpha|=m-1}\left(D^{\alpha}A(x)-\left(D^{\alpha}A\right)_B\right)\right)^{n/(n-\beta)}dx\right)^{(n-\beta)/n} \\ &\le C\left(\sum_{k=1}^{+\infty}|B|^{1/(n-\beta)}\left|2^kB\right|^{(1/(\beta-n))-1}\int_{2^{k+1}B\setminus 2^kB} \times\left(\sum_{|\alpha|=m-1}\left\|D^{\alpha}A\right\|_{\mathrm{BMO}}\left(C_1+\log\frac{|x|}{r}\right) + \sum_{|\alpha|=m-1}\left(D^{\alpha}A(x)-\left(D^{\alpha}A\right)_B\right)\right)^{n/(n-\beta)}dx\right)^{(n-\beta)/n} \\ &\le C\left(\sum_{|\alpha|=m-1}\left\|D^{\alpha}A\right\|_{\mathrm{BMO}}^{n/(n-\beta)}\sum_{k=1}^{+\infty}\left(C_1+\log\frac{\left|2^{k+1}r\right|}{r}\right)^{n/(n-\beta)} \times|B|^{1/(n-\beta)}\left|2^{k+1}B\right|^{(1/(\beta-n))-1}\left|2^{k+1}B\right|\right)^{(n-\beta)/n} \\ &\quad+\left(\sum_{k=1}^{+\infty}|B|^{1/(n-\beta)}\left|2^kB\right|^{(1/(n-\beta))-1}\int_{2^{k+1}B\setminus 2^kB} \times\sum_{|\alpha|=m-1}\left(D^{\alpha}A(x)-\left(D^{\alpha}A\right)_B\right)^{n/(n-\beta)}dx\right)^{(n-\beta)/n} \\ &\le C\left(\sum_{|\alpha|=m-1}\left\|D^{\alpha}A\right\|_{\mathrm{BMO}}^{n/(n-\beta)}\sum_{k=1}^{\infty}\left(k+C_1\right)^{n/(n-\beta)}|B|^{1/(n-\beta)} \times\left|2^{k+1}B\right|^{(1/(\beta-n))-1}\left|2^{k+1}B\right|\right)^{(n-\beta)/n} \\ &\le C\sum_{|\alpha|=m-1}\left\|D^{\alpha}A\right\|_{\mathrm{BMO}}. \end{aligned} \tag{60}$$

Next we will give the estimates of I_2. First by a cumbersome but straightforward computation, we have

$$\left|K_2\left(A_B, x, y\right)\right| \le \sum_{k=0}^{m-2}\sum_{|\alpha|=k}\left|R_{m-k-1}\left(D^{\alpha}A_B;\ 0, y\right)\right||x|^{-m+k+1}. \tag{61}$$

Also noting the fact that

$$\begin{aligned} &\left|R_{m-k-1}\left(D^{\alpha}A_B;\ 0, y\right)\right| \\ &\le C|y|^{m-k-1} \times\sum_{|\alpha|=m-1}\left(\frac{1}{|\widetilde{Q}_0|}\int_{\widetilde{Q}_0}\left|D^{\alpha}A(z)-\left(D^{\alpha}A\right)_B\right|^q dz\right)^{1/q} \\ &\le Cr^{m-k-1}\sum_{|\alpha|=m-1}\left\|D^{\alpha}A\right\|_{\mathrm{BMO}}, \end{aligned} \tag{62}$$

where $q > n$, and $\widetilde{Q}_0$ is a cube centered at 0 and having diameter $5\sqrt{n}|y|$.

Thus we obtain

$$\left|K_2\left(A_B, x, y\right)\right| \le C\sum_{|\alpha|=m-1}\left\|D^{\alpha}A\right\|_{\mathrm{BMO}}\sum_{k=0}^{m-2}r^{m-k-1}|x|^{-m+k+1}. \tag{63}$$

So we have the following estimates of I_2:

$$\begin{aligned} I_2 &\le C\frac{1}{|x|^{n-\beta}}\int_{\mathbb{R}^n}\sum_{k=0}^{m-2} r^{m-k-1}|x|^{-m+k+1}\left|\widetilde{a}(y)\right|dy \\ &= C\sum_{k=0}^{m-2} r^{m-k-1}|x|^{-m+k+1-n+\beta}\int_B \left|\widetilde{a}(y)\right|dy \\ &\le C\sum_{k=0}^{m-2} r^{m-k-1}|x|^{-m+k+1-n+\beta}. \end{aligned} \tag{64}$$

Now we get

$$\begin{aligned} &\lambda\left|\left\{x\in\mathbb{R}^n \setminus B(0,2r) : \left|I_2\right| > \lambda\right\}\right|^{(n-\beta)/n} \\ &\le C\sum_{|\alpha|=m-1}\left\|D^\alpha A\right\|_{\mathrm{BMO}}\left(\int_{\mathbb{R}^n\setminus B(0,2r)}\left|I_2\right|^{n/(n-\beta)}dx\right)^{(n-\beta)/n} \\ &\le C\sum_{|\alpha|=m-1}\left\|D^\alpha A\right\|_{\mathrm{BMO}} \\ &\qquad\times\left(\sum_{k=0}^{m-2} r^{(m-k-1)(n/(n-\beta))}\right. \\ &\qquad\qquad\left.\times\sum_{j=1}^{+\infty}\int_{2^{j+1}B\setminus 2^jB}|x|^{(n(1+k-m)/(n-\beta))-n}dx\right)^{(n-\beta)/n} \\ &\le C\sum_{|\alpha|=m-1}\left\|D^\alpha A\right\|_{\mathrm{BMO}} \\ &\qquad\times\left(\sum_{k=0}^{m-2}|B|^{(m-k-1)/(n-\beta)}\right. \\ &\qquad\qquad\left.\times\sum_{j=1}^{\infty}\left|2^{j+1}B\right|^{((1+k-m)/(n-\beta))-1}\left|2^{j+1}B\right|\right)^{(n-\beta)/n} \\ &\le C\sum_{|\alpha|=m-1}\left\|D^\alpha A\right\|_{\mathrm{BMO}} \\ &\quad\times\left(\sum_{k=0}^{m-2}|B|^{((m-k-1)/(n-\beta))+((1+k-m)/(n-\beta))-1+1}\right. \\ &\qquad\qquad\left.\times\sum_{j=1}^{\infty}2^{jn(((1+k-m)/(n-\beta))-1+1)}\right)^{(n-\beta)/n} \\ &\le C\sum_{|\alpha|=m-1}\left\|D^\alpha A\right\|_{\mathrm{BMO}}. \end{aligned} \tag{65}$$

For I_3, by the vanishing condition of $\widetilde{a}(y)$, we can split I_3 as follows:

$$\begin{aligned} I_3 &= \frac{1}{|x|^{n-\beta}}\int_{\mathbb{R}^n}\left[\sum_{|\alpha|=m-1}\frac{1}{\alpha!}D^\alpha A_B(y)\,x^\alpha\right. \\ &\qquad\left.-\sum_{|\alpha|=m-1}\frac{1}{\alpha!}D^\alpha A_B(x)\,x^\alpha\right] \\ &\qquad\times\frac{\Omega(x-y)}{|x-y|^{m-1}}\widetilde{a}(y)\,dy \\ &\le \frac{1}{|x|^{n-\beta}}\sum_{|\alpha|=m-1}\frac{1}{\alpha!}\int_{\mathbb{R}^n}\left[D^\alpha A(y)-(D^\alpha A)_B\right] \\ &\qquad\times(x-y)^\alpha\frac{\Omega(x-y)}{|x-y|^{m-1}}\widetilde{a}(y)\,dy \\ &\quad+\frac{1}{|x|^{n-\beta}}\sum_{|\alpha|=m-1}\frac{1}{\alpha!}\int_{\mathbb{R}^n}D^\alpha A_B(y) \\ &\qquad\times\sum_{\gamma<\alpha}C_\alpha^\gamma(x-y)^\gamma y^{\alpha-\gamma} \\ &\qquad\times\frac{\Omega(x-y)}{|x-y|^{m-1}}\widetilde{a}(y)\,dy \\ &\quad+\frac{1}{|x|^{n-\beta}}\sum_{|\alpha|=m-1}\frac{1}{\alpha!}\int_{\mathbb{R}^n}x^\alpha\left[D^\alpha A(x)-(D^\alpha A)_B\right] \\ &\qquad\times\frac{\Omega(x-y)}{|x-y|^{m-1}}\widetilde{a}(y)\,dy \\ &= I_{31}+I_{32}+I_{33}. \end{aligned} \tag{66}$$

For the term I_{31}, by Lemma 12 and the size condition of $\widetilde{a}$, we obtain

$$\begin{aligned} \lambda\left|\left\{x\in\mathbb{R}^n\setminus B(0,2r) : \left|I_{31}\right|>\lambda\right\}\right| &\le \sum_{|\alpha|=m-1}\left\|\left(D^\alpha A-(D^\alpha A)_B\right)\widetilde{a}\right\|_{L^1} \\ &\le \sum_{|\alpha|=m-1}\left\|D^\alpha A\right\|_{\mathrm{BMO}}. \end{aligned} \tag{67}$$

For the term I_{32}, by the vanishing condition of $\widetilde{a}$ and the fact $\Omega\in\mathrm{Lip}_1(S^{n-1})$, we have

$$\begin{aligned} I_{32} &= \frac{1}{|x|^{n-\beta}}\sum_{|\alpha|=m-1}\frac{1}{\alpha!}\int_{\mathbb{R}^n}\left[D^\alpha A_B(y)-(D^\alpha A)_B\right] \\ &\qquad\times\sum_{|\gamma|<\alpha}C_\alpha^\gamma(x-y)^\gamma y^{\alpha-\gamma} \\ &\qquad\times\frac{\Omega(x-y)}{|x-y|^{m-1}}\times\widetilde{a}(y)\,dy \end{aligned}$$

$$\leq C\frac{1}{|x|^{n-\beta}}\sum_{|\alpha|=m-1}\frac{1}{\alpha!}\int_{\mathbb{R}^n}\left|D^{\alpha}A_B(y)-(D^{\alpha}A)_B\right| \times\sum_{|\gamma|<m-1}r^{m-1-|\gamma|}|x-y|^{1-m+|\gamma|} \times|\tilde{a}(y)|\,dy$$
$$\leq C\sum_{|\gamma|<m-1}\frac{1}{|x|^{n-\beta}}|B|^{(m-1-|\gamma|)/n} \times\int_{\mathbb{R}^n}\sum_{|\alpha|=m-1}\left|D^{\alpha}A_B(y)-(D^{\alpha}A)_B\right| \times|x-y|^{1-m+|\gamma|}\,|\tilde{a}(y)|\,dy. \tag{68}$$

Thus we get

$$\lambda\left|\{x\in\mathbb{R}^n\setminus B(0,2r):|I_{32}|>\lambda\}\right|^{(n-\beta)/n}$$
$$\leq\left(\int_{\mathbb{R}^n\setminus B(0,2r)}|I_{32}|^{n/(n-\beta)}dx\right)^{(n-\beta)/n}$$
$$\leq C\left(\sum_{|\gamma|<m-1}|B|^{((m-1-|\gamma|)/n)(n/(n-\beta))}\sum_{k=1}^{\infty}\int_{2^{k+1}B\setminus2^kB}\left(\frac{1}{|x|^{n-\beta}}\int_{\mathbb{R}^n}\sum_{|\alpha|=m-1}|D^{\alpha}A_B(y)-(D^{\alpha}A)_B| \times|x-y|^{1-m+|\gamma|}\,|\tilde{a}(y)|\,dy\right)^{n/(n-\beta)}dx\right)^{(n-\beta)/n}$$
$$\leq C\left(\sum_{|\gamma|<m-1}|B|^{(m-1-|\gamma|)/(n-\beta)}\sum_{k=1}^{\infty}\left|2^kB\right|^{(1-m+|\gamma|+\beta-n)/(n-\beta)}\left|2^kB\right| \times\left(\int_B\sum_{|\alpha|=m-1}|D^{\alpha}A_B(y)-(D^{\alpha}A)_B|\,|\tilde{a}(y)|dy\right)^{n/(n-\beta)}\right)^{(n-\beta)/n}$$
$$\leq C\sum_{|\alpha|=m-1}\|D^{\alpha}A\|_{\mathrm{BMO}}\left(\sum_{|\gamma|<m-1}|B|^{(m-1-|\gamma|)/(n-\beta)}|B|^{(1-m+|\gamma|+\beta-n)/(n-\beta)} \times|B|^{1-1+1}\sum_{k=1}^{\infty}k2^{kn((1-m+|\gamma|+\beta-n)/(n-\beta))}\right)^{(n-\beta)/n}$$
$$\leq C\sum_{|\alpha|=m-1}\|D^{\alpha}A\|_{\mathrm{BMO}}. \tag{69}$$

For the term I_{33}, by the vanishing condition of $\tilde{a}$, $\Omega\in\mathrm{Lip}_1(S^{n-1})$, and (58), we have

$$I_{33}=\frac{1}{|x|^{n-\beta}}\sum_{|\alpha|=m-1}\frac{1}{\alpha!}x^{\alpha}\left[D^{\alpha}A(x)-(D^{\alpha}A)_B\right] \times\int_{\mathbb{R}^n}\left(\frac{\Omega(x-y)}{|x-y|^{m-1}}-\frac{\Omega(x)}{|x|^{m-1}}\right)\tilde{a}(y)\,dy$$
$$\leq\frac{1}{|x|^{n-\beta}}\sum_{|\alpha|=m-1}\frac{1}{\alpha!}x^{\alpha}\left[D^{\alpha}A(x)-(D^{\alpha}A)_B\right] \times\int_{\mathbb{R}^n}\left|\frac{\Omega(x-y)}{|x-y|^{m-1}}-\frac{\Omega(x)}{|x-y|^{m-1}}\right||\tilde{a}(y)|\,dy$$
$$+\frac{1}{|x|^{n-\beta}}\sum_{|\alpha|=m-1}\frac{1}{\alpha!}x^{\alpha}\left[D^{\alpha}A(x)-(D^{\alpha}A)_B\right] \times\int_{\mathbb{R}^n}\left|\frac{\Omega(x)}{|x-y|^{m-1}}-\frac{\Omega(x)}{|x|^{m-1}}\right||\tilde{a}(y)|\,dy$$
$$\leq C\frac{1}{|x|^{n-\beta}}\sum_{|\alpha|=m-1}\frac{1}{\alpha!}|x|^{m-1}\left|D^{\alpha}A(x)-(D^{\alpha}A)_B\right| \times\int_B\frac{|y|}{|x-y|^m}\,|\tilde{a}(y)|\,dy. \tag{70}$$

Thus by Lemma 2.2 in [8], we obtain

$$\lambda\left|\{x\in\mathbb{R}^n\setminus B(0,2r):|I_{33}|>\lambda\}\right|^{(n-\beta)/n}$$
$$\leq\left(\int_{\mathbb{R}^n\setminus B(0,2r)}|I_{33}|^{n/(n-\beta)}dx\right)^{(n-\beta)/n}$$
$$\leq C\left(\int_{\mathbb{R}^n\setminus B(0,2r)}\left|\frac{1}{|x|^{n-\beta}}\sum_{|\alpha|=m-1}\frac{1}{\alpha!}|x|^{m-1}\left|D^{\alpha}A(x)-(D^{\alpha}A)_B\right| \times\int_B\frac{|y|}{|x-y|^m}\,|\tilde{a}(y)|\,dy\right|^{n/(n-\beta)}dx\right)^{(n-\beta)/n}$$
$$\leq C\left(\sum_{k=1}^{\infty}\int_{2^{k+1}\setminus2^kB}|x|^{(\beta-n-1)(n/(n-\beta))}|B|^{1/(n-\beta)} \times\sum_{|\alpha|=m-1}\left|D^{\alpha}A(x)-(D^{\alpha}A)_B\right|^{n/(n-\beta)}dx\right)^{(n-\beta)/n}$$
$$\leq C\sum_{|\alpha|=m-1}\|D^{\alpha}A\|_{\mathrm{BMO}} \times\left(\sum_{k=1}^{\infty}(k+1)\left|2^kB\right|^{(\beta-n-1)/(n-\beta)}\left|2^kB\right| \times|B|^{1/(n-\beta)}\right)^{(n-\beta)/n}$$

$$\leq C \sum_{|\alpha|=m-1} \left\| D^{\alpha} A \right\|_{\mathrm{BMO}}. \tag{71}$$

Finally, we will give the estimates of I_4. By a similar argument as in the estimates of I_{32}, we can easily get

$$\lambda \left| \left\{ x \in \mathbb{R}^n \setminus B(0, 2r) : \left| I_4 \right| > \lambda \right\} \right|^{(n-\beta)/n} \leq C \sum_{|\alpha|=m-1} \left\| D^{\alpha} A \right\|_{\mathrm{BMO}}. \tag{72}$$

Combining the estimates of I_1, I_2, I_3, and I_4, we finish the proof of Theorem 10. □

Proof of Theorem 11. Theorem 11 was proved in [7] in the case $\beta = 0$ and $\Omega \equiv 1$. For the case $0 \leq \beta < n$ and $\Omega \equiv 1$, we can easily prove Theorem 11 by the proof of Theorem 5.3 in [7] with minor changes. Then for the case $0 \leq \beta < n$ and $\Omega \in \mathrm{Lip}_1(S^{n-1})$, by the main idea used in the proof of Theorem 10, we can prove Theorem 11 easily and we omit the details here. □

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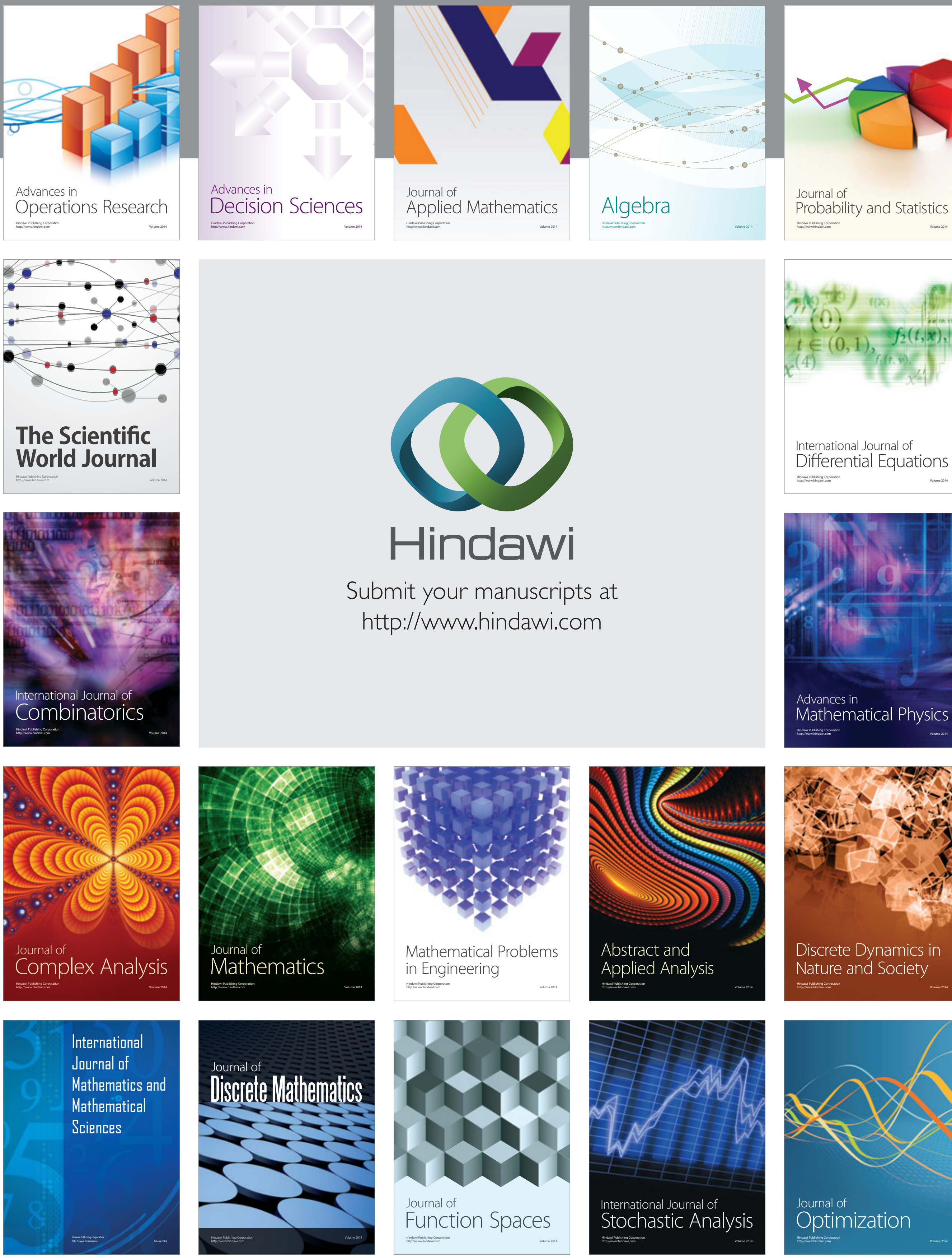
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