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Research Article

Multiplicative Perturbations of Convoluted C-Cosine Functions and Convoluted C-Semigroups

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We obtain the multiplicative perturbation theorems for convoluted C-cosine functions (resp., convoluted C-semigroups) and n-times integrated C-cosine functions (resp., n-times integrated C-semigroups) for $n \in \mathbb{N}$. Moreover, we obtain some new results for perturbations on C-cosine functions (resp., C-semigroups). Some examples are presented.

1. Introduction and Preliminaries

The α-times integrated C-semigroups, α-times integrated C-cosine functions ($\alpha > 0$) [1–6], 0-times integrated semigroups (i.e., C-semigroups), and 0-times integrated C-cosine functions (i.e., C-cosine functions) [5, 7–11] are powerful tools in studying ill-posed abstract Cauchy problems. The convoluted C-cosine functions (resp., convoluted C-semigroups) are the extension of α-times integrated C-cosine functions (resp., α-times integrated C-semigroups), they can be used to deal with more complicated ill-posed abstract Cauchy problems of evolution equations [5, 12–16].

Many researchers studied the perturbations on C-cosine functions and C-semigroups [17–22]. In [16], Kostić studied the additive perturbations of convoluted C-cosine functions and convoluted C-semigroups. However, to the authors' knowledge, few papers can be found in the literature for the multiplicative perturbations on the convoluted C-cosine functions (resp., convoluted C-semigroups).

In this paper, based on the previously mentioned works we study the multiplicative perturbations on the convoluted C-cosine functions and convoluted C-semigroups. Moreover, we obtain the corresponding new results for n-times integrated C-semigroups (resp., n-times integrated C-cosine functions) ($n \in \mathbb{N}_0$, $\mathbb{N}_0$ denotes the nonnegative integers).

Throughout this paper, $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{R}$, and $\mathbb{C}$ denote the positive integers, the nonnegative integers, the real numbers, the complex plane, respectively. X denotes a nontrivial complex Banach space, $L(X)$ denotes the space of bounded linear operators from X into X. In the sequel, we assume that $C \in L(X)$ is an injective operator. $\mathbf{C}([a,b],X)$ denotes the space of all continuous functions from $[a,b]$ to X. For a closed linear operator A on X, its domain, range, resolvent set, and the C-resolvent set are denoted by $D(A)$, $R(A)$, $\rho(A)$, and $\rho_c(A)$, respectively, where $\rho_c(A)$ is defined by

$$\rho_c(A) := \{\lambda \in \mathbb{C} : R(C) \subset R(\lambda - A) \text{ and } \lambda - A \text{ is injective}\}. \tag{1}$$

$K \in \mathbf{C}([0,\infty),\mathbb{C})$ is an exponentially bounded function and for $\beta \in \mathbb{R}$, $\widehat{K}(\lambda) \neq 0$ ($\operatorname{Re}\lambda > \beta$), where $\widehat{K}(\lambda)$ is the Laplace transform of $K(t)$. We define $\vartheta(t) := \int_0^t K(s)ds$.

The next definition is the convoluted version of Definition 4.1 in Chapter 1 of [5].

Definition 1 (see [5, 13, 15]). Let $\omega \geq 0$. If $\{\lambda^2 : \operatorname{Re}\lambda > \max(\omega,\beta)\} \subset \rho_c(A)$ and there exists a strongly continuous operator family $\{C_K(t)\}_{t\geq 0}$($C_K(t) \in L(X), t \geq 0$) such that $\|C_K(t)\| \leq Me^{\omega t}, t \geq 0$ for some $M > 0$, and

$$\lambda\left(\lambda^2 - A\right)^{-1}Cx = \frac{1}{\widehat{K}(\lambda)}\int_0^\infty e^{-\lambda t}C_K(t)\,x\,dt, \qquad \operatorname{Re}\lambda > \max(\omega,\beta),\ x \in X, \tag{2}$$

then it is said that A is a subgenerator of an exponentially bounded K-convoluted C-cosine function $\{C_K(t)\}_{t\geq 0}$. The operator $\overline{A} := C^{-1}AC$ is called the generator of $\{C_K(t)\}_{t\geq 0}$.

Theorem 2 (see [13–15]). *Let $\{C_K(t)\}_{t\geq 0}$ be a strongly continuous, exponentially bounded operator family, and let A be a closed operator. Then the statements (i) and (ii) are equivalent, where*

(i) *A is the subgenerator of a K-convoluted C-cosine function $\{C_K(t)\}_{t\geq 0}$,*

(ii) (1) $C_K(t)C = CC_K(t)$, $t \geq 0$,

(2) $C_K(t)A \subset AC_K(t)$, $t \geq 0$ *and*

$$A\int_0^t\int_0^s C_K(\sigma)\,x\,d\sigma\,ds = C_K(t)\,x - \vartheta(t)\,Cx, \quad t \geq 0,\ x \in X. \tag{3}$$

Definition 3. Let $0 \leq \omega < \infty$. If $\{\lambda : \operatorname{Re}\lambda > \max(\omega, \beta)\} \subset \rho_c(A)$ and there exists a strongly continuous operator family $\{T_K(t)\}_{t\geq 0}$ such that $\|T_K(t)\| \leq Me^{\omega t}$, $t \geq 0$ for some $M > 0$, and

$$(\lambda - A)^{-1}Cx = \frac{1}{\widehat{K}(\lambda)}\int_0^\infty e^{-\lambda t}T_K(t)\,x\,dt, \qquad \operatorname{Re}\lambda > \max(\omega,\beta), \quad x \in X, \tag{4}$$

then it is said that A is a subgenerator of an exponentially bounded K-convoluted C-semigroup $\{T_K(t)\}_{t\geq 0}$. The operator $\overline{A} := C^{-1}AC$ is called the generator of $\{T_K(t)\}_{t\geq 0}$.

Theorem 4. *Let $\{T_K(t)\}_{t\geq 0}$ be a strongly continuous, exponentially bounded operator family, and let A be a closed operator. Then the assertions (i) and (ii) are equivalent, where*

(i) *A is the subgenerator of a K-convoluted C-semigroup $\{T_K(t)\}_{t\geq 0}$,*

(ii) (1) $T_K(t)C = CT_K(t)$, $t \geq 0$,

(2) $T_K(t)A \subset AT_K(t)$, $t \geq 0$ *and*

$$A\int_0^t T_K(s)\,x\,ds = T_K(t)\,x - \vartheta(t)\,Cx, \quad t \geq 0,\ x \in X. \tag{5}$$

Remark 5 (see [16]). In Theorems 2 and 4, putting $K(t) = t^{r-1}/\Gamma(r)$, where $\Gamma(\cdot)$ denotes the Gamma function, one obtains the classes of r-times integrated C-cosine functions and r-times integrated C-semigroups; a 0-times integrated C-cosine function (resp., 0-times integrated C-semigroup) is defined to be a C-cosine function (resp., C-semigroup). More knowledge for them, we refer the reader to, for example, [1–3, 5, 7–11, 18] and references there in.

Next, we recall the definitions of r-times integrated C-semigroup and r-times integrated C-cosine functions ($r \geq 0$).

Definition 6 (see [5]). Let $0 \leq \omega < \infty$ and let $r \in [0,\infty)$. If $(\omega^2,\infty) \subset \rho_c(A)$ (resp., $(\omega,\infty) \subset \rho_c(A)$) and there exists a strongly continuous operator family $\{C_r(t)\}_{t\geq 0}$ (resp., $\{T_r(t)\}_{t\geq 0}$) such that $\|C_r(t)\| \leq Me^{\omega t}$, $t \geq 0$ (resp., $\|T_r(t)\| \leq Me^{\omega t}$, $t \geq 0$) for some $M > 0$, and

$$\begin{aligned}\lambda\left(\lambda^2 - A\right)^{-1}Cx &= \lambda^r\int_0^\infty e^{-\lambda t}C_r(t)\,x\,dt, \quad \lambda > \omega,\ x \in X,\\ \Bigg(\text{resp. } (\lambda - A)^{-1}Cx &= \lambda^r\int_0^\infty e^{-\lambda t}T_r(t)\,x\,dt,\ \lambda > \omega,\ x \in X,\Bigg),\end{aligned} \tag{6}$$

then it is said that A is a subgenerator of an exponentially bounded r-times integrated C-cosine function $\{C_r(t)\}_{t\geq 0}$ (resp., r-times integrated C-semigroup $\{T_r(t)\}_{t\geq 0}$) on X. If $r = 0$, then $\{C_r(t)\}_{t\geq 0}$ (resp., $\{T_r(t)\}_{t\geq 0}$) is called an exponentially bounded 0-times integrated C-cosine function (resp., 0-times integrated C-semigroup).

We present the definition of C-cosine functions which will be used in the proof of Theorem 12.

Definition 7 (see [1, 5]). A strongly continuous family $\{C(t)\}_{t\geq 0}$ of bounded linear operators on X is called a C-cosine function on X, if $CC(\cdot) = C(\cdot)C$, $C(0) = C$ and $C(t+s)C + C(|t-s|)C = 2C(t)C(s)$, for all $t, s \geq 0$.

2. Main Results

Suppose that A is a subgenerator of an exponentially bounded K-convoluted C-cosine function $\{C_K(t)\}_{t\geq 0}$ on X, $S_K(t) = \int_0^t C_K(s)ds$, for any $\Psi \in \mathbf{C}([0,\infty), L(X))$ with $\|\Psi(t)\| = O(e^{\omega t})$, we set

$$\begin{aligned}L(\lambda) := \sup\Bigg\{&\int_0^a e^{-\lambda t}\left\|\int_0^t \delta(\lambda)\,\Psi(s)\,C^{-1}PAC_K(t-s)\,x\,ds\right\|dt,\\ &x \in D(A),\ \|x\| \leq 1\Bigg\} < \infty,\end{aligned} \tag{7}$$

for some $a \in (0,+\infty]$ and $\lambda > \max(\omega,\beta)$, where $\delta(\lambda)$ is some function and $P = B/\widehat{K}(\lambda)$, $B \in L(X)$ with $R(B) \subset R(C)$.

We have the following multiplicative perturbation theorem.

Theorem 8. *Suppose that A is a subgenerator of an exponentially bounded K-convoluted C-cosine function $\{C_K(t)\}_{t\geq 0}$ on X. Let $BC = CB$, and $D(A)$ is dense in X,*

$$\left\{\lambda^2 : \lambda > \max(\omega,\beta)\right\} \subset \rho\left((I + \delta(\lambda)B)A\right). \tag{8}$$

If $\lim_{\lambda\to\infty}L(\lambda)e^{\lambda t} = 0$ for all $t \geq 0$, then $(I + \delta(\lambda)B)A$ subgenerates an exponentially bounded K-convoluted C-cosine function on X.

Proof. For all $x \in D(A)$, $\|x\| \le 1$, λ is large enough and ε is small enough, we have

$$\begin{aligned} &\left\| \int_0^t \delta(\lambda) \Psi(s) C^{-1} PAS_K(t-s) x ds \right\| \\ &\quad = \left\| \int_0^t \int_0^s \delta(\lambda) \Psi(\sigma) C^{-1} PAC_K(s-\sigma) x d\sigma ds \right\| \\ &\quad \le e^{\lambda t} \int_0^t e^{-\lambda s} \left\| \int_0^s \delta(\lambda) \Psi(\sigma) C^{-1} PAC_K(s-\sigma) x d\sigma \right\| ds \\ &\quad \le e^{\lambda t} L(\lambda) < \varepsilon < 1, \quad t \ge 0. \end{aligned} \tag{9}$$

Let $\mathscr{V} : [0, \infty) \to L(X)$ be any strongly continuous function with $\|\mathscr{V}(t)\| = O(e^{\omega t})$; we define

$$(\mathscr{M}\mathscr{V})(t) x = \int_0^t \delta(\lambda) \mathscr{V}(s) C^{-1} PAS_K(t-s) x ds, \quad x \in D(A),\ t \ge 0. \tag{10}$$

Obviously, $(\mathscr{M}\mathscr{V})(t)x$ is continuous on $t \ge 0$, from (9) and the denseness of $D(A)$, $\mathscr{M}$ maps $\mathbf{C}([0,\infty), L(X))$ into $\mathbf{C}([0,\infty), L(X))$.

It follows from (9) that $(I - \mathscr{M})^{-1}$ is bounded. For each $t \ge 0$, set

$$\widehat{C}_K(t) x := (I - \mathscr{M})^{-1} [C_K(\cdot) x](t), \quad x \in X. \tag{11}$$

Then, $\widehat{C}_K(t) C = C \widehat{C}_K(t)$, and there exists a constant $\widehat{M}$ such that $\|\widehat{C}_K(t)\| \le \widehat{M} e^{\omega t}$,

$$\widehat{C}_K(t) x = C_K(t) x + \delta(\lambda) \int_0^t \widehat{C}_K(s) C^{-1} PAS_K(t-s) x ds. \tag{12}$$

For sufficiently large λ, we set

$$\mathscr{L}(\lambda) x = \int_0^\infty e^{-\lambda t} \widehat{C}_K(t) x dt, \quad x \in X. \tag{13}$$

Taking Laplace transform of (12), we have

$$\begin{aligned} \mathscr{L}(\lambda) x &= \lambda \widehat{K}(\lambda) (\lambda^2 - A)^{-1} Cx \\ &\quad + \delta(\lambda) \mathscr{L}(\lambda) C^{-1} BA (\lambda^2 - A)^{-1} Cx, \quad x \in X. \end{aligned} \tag{14}$$

Therefore for $x \in D(A)$,

$$\mathscr{L}(\lambda) (\lambda^2 - (I + \delta(\lambda) B) A) x = \lambda \widehat{K}(\lambda) Cx. \tag{15}$$

Noting (8), for $x \in X$, we have

$$\begin{aligned} &\mathscr{L}(\lambda) (\lambda^2 - (I + \delta(\lambda) B) A) (\lambda^2 - (I + \delta(\lambda) B) A)^{-1} x \\ &\quad = \lambda \widehat{K}(\lambda) (\lambda^2 - (I + \delta(\lambda) B) A)^{-1} Cx, \end{aligned} \tag{16}$$

that is

$$\begin{aligned} \frac{1}{\widehat{K}(\lambda)} \int_0^\infty e^{-\lambda t} \widehat{C}_K(t) x dt &= \frac{1}{\widehat{K}(\lambda)} \mathscr{L}(\lambda) x \\ &= \lambda (\lambda^2 - (I + \delta(\lambda) B) A)^{-1} Cx. \end{aligned} \tag{17}$$

Then from Definition 1, $(I + \delta(\lambda) B) A$ subgenerates an exponentially bounded K-convoluted C-cosine function $\{\widehat{C}_K(t)\}_{t \ge 0}$. □

Theorem 9. *Suppose A is a subgenerator of an exponentially bounded K-convoluted C-cosine function $\{C_K(t)\}_{t \ge 0}$ on X, $S_K(t) = \int_0^t C_K(s) ds$. Let $B \in L(X)$ with $BC = CB$ and let $R(B) \subset R(C)$, and $D(A)$ is dense in X. If for any $\Phi \in \mathbf{C}([0,\infty), L(X))$,*

$$\begin{aligned} &\left\| \int_0^t \Phi(s) C^{-1} BAS_K(t-s) x ds \right\| \\ &\quad \le \widetilde{M} \int_0^t e^{\omega(t-s)} \|\Phi(s)\| ds \cdot \|x\|, \\ &\quad x \in D(A),\ t \ge 0, \end{aligned} \tag{18}$$

where $\widetilde{M}$ is a constant, then for some (and all) λ, $\operatorname{Re}\lambda > \max(\omega, \beta)$, $(I + \widehat{K}(\lambda) B) A$ subgenerates an exponentially bounded K-convoluted C-cosine function on X.

Proof. Define the operator functions $\{\overline{C}_n(t)\}_{t \ge 0}$ as follows:

$$\begin{aligned} \overline{C}_0(t) x &= C_K(t) x, \\ \overline{C}_n(t) x &= \int_0^t \overline{C}_{n-1}(s) C^{-1} BAS_K(t-s) x ds, \\ &x \in D(A), \quad t \ge 0,\ n = 1, 2, \ldots. \end{aligned} \tag{19}$$

By induction, we obtain

(i) for any $x \in X$, $\overline{C}_n(t) x \in \mathbf{C}([0,\infty), X)$,

(ii) $\|\overline{C}_n(t) x\| \le (M \widetilde{M}^n t^n / n!) e^{\omega t} \|x\|$, $t \ge 0$, $x \in X$, for all $n \ge 0$.

Define the operator function

$$h(t) = \sum_{n=0}^\infty \overline{C}_n(t), \quad t \ge 0. \tag{20}$$

Noting that the series $\sum_{n=0}^\infty (M \widetilde{M}^n t^n / n!) e^{\omega t}$ is uniformly converge on every compact interval in t, we can see that the series (20) is uniformly converge on every compact interval in t, so does the operator $h(t)$. It is obvious that $\|h(t)\| \le M e^{(\omega + \widetilde{M}) t}$ and $t \to h(t)x$ is continuous on $[0,\infty)$ for any $x \in X$. Moreover,

$$h(t) x = C_K(t) x + \int_0^t h(s) C^{-1} BAS_K(t-s) x ds, \quad x \in X,\ t \ge 0. \tag{21}$$

For $\operatorname{Re}\lambda$ sufficiently large, we set

$$\mathscr{L}(\lambda)x = \int_0^\infty e^{-\lambda t} h(t)\, x\, dt, \quad x \in X. \tag{22}$$

Next, we show that the following equalities hold:

$$\mathscr{L}(\lambda)\left[\lambda^2 - \left(I + \widehat{K}(\lambda)B\right)A\right]x = \lambda\widehat{K}(\lambda)Cx, \quad x \in D(A), \tag{23}$$

$$\left[\lambda^2 - \left(I + \widehat{K}(\lambda)B\right)A\right]\mathscr{L}(\lambda)x = \lambda\widehat{K}(\lambda)Cx, \quad x \in X. \tag{24}$$

By induction, it is not difficult to see that

$$\begin{aligned}&\int_0^\infty e^{-\lambda t}\overline{C}_n(t)\, x\, dt \\ &\quad = \lambda\widehat{K}(\lambda)\left(\lambda^2 - A\right)^{-1}\left[\widehat{K}(\lambda)BA\left(\lambda^2 - A\right)^{-1}\right]^n Cx, \\ &\qquad x \in X,\ n \geq 0.\end{aligned} \tag{25}$$

Let

$$Q(t)x = \int_0^t C^{-1}BAS_K(t-s)\, x\, ds, \quad x \in D(A). \tag{26}$$

By hypothesis, $Q(t)$ can be extended to X and satisfies

$$\|Q(t)\| \leq \frac{\widetilde{M}}{\omega}\left(e^{\omega t} - 1\right), \quad t \geq 0. \tag{27}$$

Set

$$\widehat{Q}(\lambda)x = \int_0^\infty e^{-\lambda t} Q(t)\, x\, dt, \quad x \in X. \tag{28}$$

Then from (25) and (27), $\|\lambda\widehat{Q}(\lambda)\| = \|C^{-1}\widehat{K}(\lambda)BA(\lambda^2 - A)^{-1}C\| < 1$ for $|\lambda|$ sufficiently large. Therefore, the series

$$\begin{aligned}&\sum_{n=0}^{\infty}\left[\widehat{K}(\lambda)BA\left(\lambda^2 - A\right)^{-1}\right]^n C \\ &\quad = \sum_{n=0}^{\infty} C\left[C^{-1}\widehat{K}(\lambda)BA\left(\lambda^2 - A\right)^{-1}C\right]^n\end{aligned} \tag{29}$$

converges.

For $x \in D(A)$ and $\operatorname{Re}\lambda > \max(\omega, \beta)$, from (25), we have

$$\begin{aligned}&\mathscr{L}(\lambda)\left[\lambda^2 - \left(I + \widehat{K}(\lambda)B\right)A\right]x \\ &\quad = \int_0^\infty e^{-\lambda t}\sum_{n=0}^{\infty}\overline{C}_n(t)\left[\lambda^2 - \left(I + \widehat{K}(\lambda)B\right)A\right]x\, dt \\ &\quad = \widehat{K}(\lambda)\sum_{n=0}^{\infty}\lambda\left(\lambda^2 - A\right)^{-1}\left[\widehat{K}(\lambda)BA\left(\lambda^2 - A\right)^{-1}\right]^n \\ &\qquad \times C\left[\lambda^2 - \left(I + \widehat{K}(\lambda)B\right)A\right]x \\ &\quad = \lambda\widehat{K}(\lambda)Cx - \lambda\left(\widehat{K}(\lambda)\right)^2\left(\lambda^2 - A\right)^{-1}CBAx \\ &\qquad + \sum_{n=1}^{\infty}\lambda\widehat{K}(\lambda)\left(\lambda^2 - A\right)^{-1}\left[\widehat{K}(\lambda)BA\left(\lambda^2 - A\right)^{-1}\right]^n \\ &\qquad \times C\left(\lambda^2 - A\right)x \\ &\qquad - \sum_{n=1}^{\infty}\lambda\widehat{K}(\lambda)\left(\lambda^2 - A\right)^{-1}\left[\widehat{K}(\lambda)BA\left(\lambda^2 - A\right)^{-1}\right]^n \\ &\qquad \times C\widehat{K}(\lambda)BAx \\ &\quad = \lambda\widehat{K}(\lambda)Cx + \sum_{n=2}^{\infty}\lambda\widehat{K}(\lambda)\left(\lambda^2 - A\right)^{-1} \\ &\qquad \times\left[\widehat{K}(\lambda)BA\left(\lambda^2 - A\right)^{-1}\right]^n C\left(\lambda^2 - A\right)x \\ &\qquad - \sum_{n=1}^{\infty}\lambda\widehat{K}(\lambda)\left(\lambda^2 - A\right)^{-1}\left[\widehat{K}(\lambda)BA\left(\lambda^2 - A\right)^{-1}\right]^n \\ &\qquad \times C\widehat{K}(\lambda)BAx \\ &\quad = \lambda\widehat{K}(\lambda)Cx.\end{aligned} \tag{30}$$

Similarly, we can prove (24). Now, from Definition 1, we conclude that $(I + \widehat{K}(\lambda)B)A$ subgenerates an exponentially bounded K-convoluted C-cosine function on X. □

By the proof of Theorems 8 and 9, we immediately obtain the following results for K-convoluted C-semigroups.

Theorem 10. *Suppose that A is a subgenerator of an exponentially bounded K-convoluted C-semigroup $\{T_K(t)\}_{t\geq 0}$ on X. $D(A)$ is dense in X. Let $B \in L(X)$ with $BC = CB$ and let $R(B) \subset R(C)$.*

(i) *One sets*

$$\begin{aligned}L(\lambda) := \sup\Bigg\{&\int_0^a e^{-\lambda s}\left\|\delta(\lambda)C^{-1}PAT_K(s)x\right\| ds, \\ &x \in D(A), \|x\| \leq 1\Bigg\} < \infty,\end{aligned} \tag{31}$$

for some $a \in (0, +\infty]$ and $\lambda > \max(\omega, \beta)$, where $\delta(\lambda)$ is a function and $P = B/\widehat{K}(\lambda)$. If $\{\lambda : \lambda > \max(\omega, \beta)\} \subset \rho((I + \delta(\lambda)B)A)$, then $(I + \delta(\lambda)B)A$ subgenerates an *exponentially bounded K-convoluted C-semigroup on X provided that $\lim_{\lambda\to\infty} L(\lambda)e^{\lambda t} = 0$ for all $t \geq 0$.*

(ii) *If for any $\Phi \in \mathbf{C}([0,\infty), L(X))$,*

$$\left\| \int_0^t \Phi(s) C^{-1} BA\mathrm{T}_K(t-s)x\,ds \right\| \leq \widetilde{M} \int_0^t e^{\omega(t-s)} \|\Phi(s)\|\,ds \cdot \|x\|, \quad x \in D(A), t \geq 0, \tag{32}$$

where $\widetilde{M}$ is a constant, then for some (and all) λ, $\operatorname{Re}\lambda > \max(\omega, \beta)$, $(I+\widehat{K}(\lambda)B)A$ subgenerates an exponentially bounded K-convoluted C-semigroup on X.

Proof. (i) For any $\Psi \in \mathbf{C}([\,0,\infty), L(X))$ with $\|\Psi(t)\| = O(e^{\omega t})$, sufficiently large λ and sufficiently small ε, we have

$$\begin{aligned} &\left\| \int_0^t \delta(\lambda)\Psi(s) C^{-1} PA\mathrm{T}_K(t-s)x\,ds \right\| \\ &\quad\leq M^* e^{(\lambda+\omega)t} \int_0^t e^{-\lambda s} \left\| \delta(\lambda) C^{-1} PA\mathrm{T}_K(s)x \right\| ds \\ &\quad\leq M^* e^{(\lambda+\omega)t} L(\lambda) < \varepsilon < 1, \quad t \geq 0,\ x \in D(A), \\ &\qquad \|x\| \leq 1, \end{aligned} \tag{33}$$

where M^* is a constant. The rest part of the proof is exactly the same as the corresponding part of the proof of Theorem 8.

The proof of (ii) is similar to the one of Theorem 9. □

In Theorems 8–10, take $K(t) = t^{n-1}/\Gamma(n)$, we have the following result for n-times integrated C-cosine function (resp., n-times integrated C-semigroup).

Theorem 11. *Suppose A is a subgenerator of an exponentially bounded n-times integrated C-cosine function $\{\mathrm{C}_n(t)\}_{t\geq 0}$ (resp., n-times integrated C-semigroup $\{\mathrm{T}_n(t)\}_{t\geq 0}$) on X. Let $B \in L(X)$ with $BC = CB$ and let $R(B) \subset R(C)$, and $D(A^{n+1})$ is dense in X.*

(i) *One sets*

$$\begin{aligned} L(\lambda) := \sup \Bigg\{ &\int_0^a e^{-\lambda t} \left\| \int_0^t \delta(\lambda)\Psi(s) C^{-1} BA \right. \\ &\left. \times \left(\frac{d^n}{dt^n} \mathrm{C}_n(t-s)x \right) ds \right\| dt, \\ &x \in D\left(A^{n+1}\right), \|x\| \leq 1 \Bigg\} < \infty, \end{aligned} \tag{34}$$

for any $\Psi \in \mathbf{C}([0,\infty), L(X))$ with $\|\Psi(t)\| = O(e^{\omega t})$,

$$\begin{aligned} \Bigg(\text{resp. } &L(\lambda) \\ &:= \sup \left\{ \int_0^a e^{-\lambda s} \left\| \delta(\lambda) C^{-1} BA \left(\frac{d^n}{ds^n} \mathrm{T}_n(s)x \right) \right\| ds, \right. \\ &\left. x \in D\left(A^{n+1}\right), \|x\| \leq 1 \right\} < \infty \Bigg) \end{aligned} \tag{35}$$

for some $a \in (0,+\infty]$ and $\lambda > \omega$, where $\delta(\lambda)$ is a function. If $(\omega^2,\infty) \subset \rho((I+\delta(\lambda)B)A)$ (resp., $(\omega,\infty) \subset \rho((I+\delta(\lambda)B)A)$), then $(I+\delta(\lambda)B)A$ subgenerates an exponentially bounded n-times integrated C-cosine function (resp., n-times integrated C-semigroup) on X provided that $\lim_{\lambda\to\infty} L(\lambda)e^{\lambda t} = 0$ for all $t \geq 0$.

(ii) *If for any $\Phi \in \mathbf{C}([0,\infty), L(X))$,*

$$\begin{aligned} &\left\| \int_0^t \Phi(s) C^{-1} BA \left(\frac{d^n}{dt^n} \mathrm{S}_n(t-s)x \right) ds \right\| \\ &\quad\leq \widetilde{M} \int_0^t e^{\omega(t-s)} \|\Phi(s)\|\,ds \cdot \|x\|, \\ &\qquad x \in D\left(A^{n+1}\right),\ t \geq 0, \end{aligned} \tag{36}$$

where $\mathrm{S}_n(t) = \int_0^t \mathrm{C}_n(s)ds$ and $\widetilde{M}$ is a constant,

$$\begin{aligned} \Bigg(\text{resp. } &\left\| \int_0^t \Phi(s) C^{-1} BA \left(\frac{d^n}{dt^n} \mathrm{T}_n(t-s)x \right) ds \right\| \\ &\leq \widetilde{M} \int_0^t e^{\omega(t-s)} \|\Phi(s)\|\,ds \cdot \|x\|, \\ &x \in D\left(A^{n+1}\right), t \geq 0 \Bigg) \end{aligned} \tag{37}$$

then for some (and all) λ, $\lambda > \omega$, $(I+\widehat{K}(\lambda)B)A$ subgenerates an exponentially bounded n-times integrated C-cosine function (resp., n-times integrated C-semigroup) on X.

When $n = 0$, from Theorem 11(ii), we immediately obtain the result of 0-times integrated C-cosine function (resp., 0-times integrated C-semigroup).

Theorem 12. *Let $B \in L(X)$ with $BC = CB$ and let $R(B) \subset R(C)$, and $D(A)$ is dense in X. Suppose that A is an exponentially bounded generator of a C-cosine function $\{\mathrm{C}(t)\}_{t\geq 0}$ (resp., C-semigroup $\{\mathrm{T}(t)\}_{t\geq 0}$) on X. If for any $\Phi \in \mathbf{C}([0,\infty), L(X))$,*

$$\left\| \int_0^t \Phi(s) C^{-1} BA\mathrm{S}(t-s)x\,ds \right\| \leq M \int_0^t e^{\omega(t-s)} \|\Phi(s)\|\,ds \cdot \|x\|, \quad x \in D(A),\ t \geq 0, \tag{38}$$

where $\mathrm{S}(t) = \int_0^t \mathrm{C}(s)ds$.

$$\left(\text{resp. } \left\| \int_0^t \Phi(s) C^{-1} BA\mathrm{T}(t-s)x\,ds \right\| \leq M \int_0^t e^{\omega(t-s)} \|\Phi(s)\|\,ds \cdot \|x\|, \quad x \in D(A), t \geq 0 \right) \tag{39}$$

for some $a \in (0, +\infty]$ *and* $\lambda > \omega$*, then* $(I+B)A$ *subgenerates an exponentially bounded* C*-cosine function (resp.,* C*-semigroup) on* X*.*

Noting the Definition 7 and the special properties of C-cosine functions (resp., C-semigroups), we obtain a different result from Theorem 11(i) (when $n = 0$).

Theorem 13. *Let* $B \in L(X)$ *with* $BC = CB$ *and let* $R(B) \subset R(C)$*, and* $D(A)$ *is dense in* X*,* $(\omega^2, \infty) \subset \rho((I + B)A)$ *(resp.,* $(\omega, \infty) \subset \rho((I + B)A)$*). Suppose that* A *is an exponentially bounded generator of a* C*-cosine function* $\{\mathrm{C}(t)\}_{t\ge0}$ *(resp.,* C*-semigroup* $\{\mathrm{T}(t)\}_{t\ge0}$*) on* X*. If*

$$L(\lambda) := \sup\left\{\int_0^a e^{-\lambda t}\left\|\int_0^t C^{-1}BA\mathrm{C}(t-s)\,x\,ds\right\|dt,\right.\qquad \left. x \in D(A), \|x\| \le 1\right\} < \infty, \tag{40}$$

$$\left(\text{resp. } L(\lambda) := \sup\left\{\int_0^a e^{-\lambda s}\left\|C^{-1}BA\mathrm{T}(s)\,x\right\|ds,\right.\qquad \left. x \in D(A), \|x\| \le 1\right\} < \infty\right) \tag{41}$$

for some $a \in (0, +\infty]$ *and* $\lambda > \omega$*, letting* $L(\infty) = \lim_{\lambda\to\infty} L(\lambda)$*, then for any* $\varepsilon < (L(\infty))^{-1}$*,* $(I + \varepsilon B)A$ *subgenerates an exponentially bounded* C*-cosine function (resp.,* C*-semigroup) on* X*.*

Proof. We prove only for C-cosine functions. Choose $0 < \mu < \mu_1 < \mu_2 < 1$ such that $|\varepsilon| = \mu(L(\infty))^{-1}$. For any $\Psi \in \mathbf{C}([0, t], L(X))$, pick a λ large enough such that $L(\lambda)/L(\infty) < \mu_1/\mu$, and then pick a $\tau \in (0, a)$ small enough such that $e^{\lambda\tau}\sup_{s\in[0,\tau]}\|\Psi(s)\| \le \mu_2/\mu_1$, then for all $x \in D(A)$, $\|x\| \le 1$, we have

$$\begin{aligned}
&\left\|\int_0^t \varepsilon\Psi(s)\,C^{-1}BA\mathrm{S}(t-s)\,x\,ds\right\| \\
&\quad= \left\|\int_0^t\int_0^s \varepsilon\Psi(\sigma)\,C^{-1}BA\mathrm{C}(s-\sigma)\,x\,d\sigma\, ds\right\| \\
&\quad\le e^{\lambda t}\int_0^t e^{-\lambda s}\left\|\int_0^s \varepsilon C^{-1}BA\mathrm{C}(s-\sigma)\,x\,d\sigma\right\|ds\cdot \sup_{s\in[0,\tau]}\|\Psi(s)\| \\
&\quad\le e^{\lambda\tau}|\varepsilon|\,L(\lambda)\cdot\sup_{s\in[0,\tau]}\|\Psi(s)\| < \mu_2 < 1,\quad t \in [0,\tau],
\end{aligned} \tag{42}$$

where $\mathrm{S}(t) = \int_0^t \mathrm{C}(s)ds$.

Let $\mathscr{V} : [0, \tau] \to L(X)$ be any strongly continuous function; we define

$$\begin{aligned}
&(\mathscr{M}\mathscr{V})(t)\,x \\
&\quad= \int_0^t \varepsilon\mathscr{V}(s)\,C^{-1}BA\mathrm{S}(t-s)\,x\,ds,\quad x \in D(A),\ t \in [0,\tau].
\end{aligned} \tag{43}$$

Obviously, $(\mathscr{M}\mathscr{V})(t)x$ is continuous on $t \ge 0$, from (42) and the denseness of $D(A)$, $\mathscr{M}$ maps $\mathbf{C}([0, \tau], L(X))$ into $\mathbf{C}([0, \tau], L(X))$.

It follows from (42) that $(I - \mathscr{M})^{-1}$ is bounded. For each $t \in [0, \tau]$, set

$$V(t)\,x := (I-\mathscr{M})^{-1}[\mathrm{C}(\cdot)\,x](t),\quad x \in X. \tag{44}$$

Then, $V(t)C = CV(t)$, and there exists a constant $\overline{M}$ such that $\|V(t)\| \le \overline{M}e^{\omega t}$:

$$V(t)\,x = \mathrm{C}(t)\,x + \int_0^t \varepsilon V(s)\,C^{-1}BA\mathrm{S}(t-s)\,x\,ds,\quad t \in [0,\tau]. \tag{45}$$

For $t \in ((n-1)\tau, n\tau]$, $n = 2, 3, \ldots$, we define inductively

$$\begin{aligned}
V(t) :=\ &-V(2(n-1)\tau - t) \\
&+ 2C^{-1}V(t-(n-1)\tau)\,V((n-1)\tau).
\end{aligned} \tag{46}$$

Next, we will prove by induction that for any $n \in \mathbb{N}$, $R(V(\sigma)V((n-1)\tau)) \subset R(C)$, for $\sigma \in [0, \tau]$, and that for every $n \in \mathbb{N}$, $V(\cdot)$ is strongly continuous in $[0, n\tau]$ and

$$V(t)\,x = \mathrm{C}(t)\,x + \int_0^t \varepsilon V(s)\,C^{-1}BA\mathrm{S}(t-s)\,x\,ds,\qquad x \in X,\ t \in [0,n\tau]. \tag{47}$$

Indeed for $n = 1$, this is true. Assume that (47) holds for n. Then for $x \in X$, $\sigma \in [0, \tau]$ we get

$$\begin{aligned}
&2V(\sigma)\,V(n\tau)\,x \\
&\quad= 2\mathrm{C}(\sigma)\,\mathrm{C}(n\tau)\,x \\
&\qquad+ 2\int_0^{n\tau}\varepsilon V(\sigma)\,V(s)\,C^{-1}BA\mathrm{S}(n\tau-s)\,x\,ds \\
&\qquad+ 2\int_0^{\sigma}\varepsilon V(s)\,C^{-1}BA\mathrm{S}(\sigma-s)\,\mathrm{C}(n\tau)\,x\,ds \\
&\quad= C\left[\mathrm{C}(\sigma+n\tau)\,x + \mathrm{C}(n\tau-\sigma)\,x\right] \\
&\qquad+ 2\int_0^{n\tau}\varepsilon V(\sigma)\,V(s)\,C^{-1}BA\mathrm{S}(n\tau-s)\,x\,ds \\
&\qquad+ 2\int_0^{\sigma}\varepsilon V(s)\,C^{-1}BA\mathrm{S}(\sigma-s)\ \mathrm{C}(n\tau)\,x\,ds \\
&\quad= 2\mathscr{M}\left[V(\sigma)\,V(\cdot)\right](n\tau)\,x + C\left[\mathrm{C}(\sigma+n\tau)x + \mathrm{C}(n\tau-\sigma)\,x\right] \\
&\qquad+ C\int_0^{\sigma}\varepsilon V(s)\,C^{-1}BA \\
&\qquad\quad\times\left[\mathrm{S}(n\tau+\sigma-s)\,x - \mathrm{S}(n\tau-\sigma+s)\,x\right]ds.
\end{aligned} \tag{48}$$

Then for $x \in X$, $\sigma \in [0, \tau]$,

$$\begin{aligned}
&2V(\sigma)V(n\tau)x \\
&\quad = C(I-\mathscr{M})^{-1}\{\mathrm{C}(\sigma+n\tau)x+\mathrm{C}(n\tau-\sigma)x \\
&\qquad + \int_0^\sigma \varepsilon V(s)C^{-1}BA[\mathrm{S}(n\tau+\sigma-s)x \\
&\qquad -\mathrm{S}(n\tau-\sigma+s)x]\,ds\}.
\end{aligned} \tag{49}$$

Hence, $V(\sigma)V(n\tau) \subset R(C)$, $\sigma \in [0,\tau]$, and $\sigma \to C^{-1}V(\sigma)V(n\tau)x$ is continuous in $[0,\tau]$ for each $x \in X$. From (48), we have

$$\begin{aligned}
&2V(\sigma)V(n\tau)x \\
&\quad = C[\mathrm{C}(\sigma+n\tau)x+\mathrm{C}(n\tau-\sigma)x] \\
&\qquad + 2\int_0^{n\tau} \varepsilon V(\sigma)V(s)C^{-1}BAS(n\tau-s)x\,ds \\
&\qquad + C\int_0^\sigma \varepsilon V(s)C^{-1}BA[\mathrm{S}(n\tau+\sigma-s)x \\
&\qquad -\mathrm{S}(n\tau-\sigma+s)x]\,ds \\
&\quad = C[\mathrm{C}(\sigma+n\tau)x+\mathrm{C}(n\tau-\sigma)x] + C\int_0^{n\tau} \varepsilon[V(\sigma+s) \\
&\qquad +V(|s-\sigma|)]C^{-1}BAS(n\tau-s)x\,ds \\
&\qquad + C\int_0^\sigma \varepsilon V(s)C^{-1}BA[\mathrm{S}(n\tau+\sigma-s)x \\
&\qquad -\mathrm{S}(n\tau-\sigma+s)x]\,ds \\
&\quad = C[\mathrm{C}(\sigma+n\tau)x+\mathrm{C}(n\tau-\sigma)x] \\
&\qquad + C\int_\sigma^{\sigma+n\tau} \varepsilon V(s)C^{-1}BAS(n\tau+\sigma-s)x\,ds \\
&\qquad + C\int_0^\sigma \varepsilon V(s)C^{-1}BAS(n\tau-\sigma+s)x\,ds \\
&\qquad + C\int_0^{n\tau-\sigma} \varepsilon V(s)C^{-1}BAS(n\tau-\sigma-s)x\,ds \\
&\qquad + C\int_0^\sigma \varepsilon V(s)C^{-1}BA[\mathrm{S}(n\tau+\sigma-s)x \\
&\qquad -\mathrm{S}(n\tau-\sigma+s)x]\,ds \\
&\quad = C[\mathrm{C}(\sigma+n\tau)x+\mathrm{C}(n\tau-\sigma)x] \\
&\qquad + C\int_0^{\sigma+n\tau} \varepsilon V(s)C^{-1}BAS(n\tau+\sigma-s)x\,ds \\
&\qquad + C\int_0^{n\tau-\sigma} \varepsilon V(s)C^{-1}BAS(n\tau-\sigma-s)x\,ds \\
&\quad = CV(\sigma+n\tau)x+CV(n\tau-\sigma)x.
\end{aligned} \tag{50}$$

Therefore, $V(\cdot)$ is strongly continuous in $[0,\infty)$ and (47) holds for all $t \ge 0$. Taking Laplace transform of (47), then the conclusion can be proved in a similar way in the proof of Theorem 8.

We can prove the case of C-semigroups in a similar way. □

3. Examples

Example 14. Let

$$\begin{aligned}
X &:= \left\{ f \in C^\infty[0,1] : \|f\| := \sup_{p\ge0}\frac{\|f^{(p)}\|_\infty}{p!^2} < \infty \right\}, \\
A &:= -\frac{d}{dx}, \qquad D(A) := \left\{ f \in X : f' \in X, f(0)=0 \right\}.
\end{aligned} \tag{51}$$

It is well known that there exist positive real numbers m and M such that

$$\begin{aligned}
\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \ge 0\} \subset \rho(A), \qquad \|R(\lambda,A)\| \le Me^{m\sqrt{|\lambda|}}, \\
\operatorname{Re}\lambda \ge 0.
\end{aligned} \tag{52}$$

Moreover, A generates an exponentially bounded K_a-convoluted semigroup $\{\mathrm{T}_K(t)\}_{t\ge0}$ for some $a > \sqrt{2}m$, where $K(t) = (a/(2\sqrt{\pi t^3}))e^{-a^2/(4t)}$, $t \ge 0$, then $\widehat{K}_a(\lambda) = e^{-a\sqrt{\lambda}}$, $\operatorname{Re}\lambda > 0$ [14, 23]. We set

$$Bf(x) := \sum_{n=1}^{j}\int_0^x \frac{(x-s)^{n-1}}{(n-1)!}f(s)\,ds, \quad j \in \mathbb{N},\ x \in [0,1], \quad f \in X. \tag{53}$$

Obviously, $B \in L(X)$ and $BA \subset AB$. Then from Theorem 10(ii) ($C = I$), $(I + e^{-a\sqrt{\lambda}}B)A$ subgenerates an exponentially bounded K_a-convoluted semigroup $\{\widetilde{\mathrm{T}}_K(t)\}_{t\ge0}$ on X.

Example 15. Let $X := C_0(\mathbb{R}) \oplus C_0(\mathbb{R}) \oplus C_0(\mathbb{R})$,

$$\begin{aligned}
&A(f,g,h)(\cdot) := \left(f', g', (\chi_{[0,\infty)} - \chi_{(-\infty,0]})h\right), \\
&(f,g,h) \in D(A) \\
&\quad = \left\{(f,g,h) \in X : f' \in C_0(\mathbb{R}), g' \in C_0(\mathbb{R}), h(0)=0\right\}
\end{aligned} \tag{54}$$

and $C(f,g,h) := (f, g, \sin(\cdot)h(\cdot))$, $f, g, h \in C_0(\mathbb{R})$. Arguing as in [3, Examples 8.1 and 8.2], one gets that A is a generator of an exponentially bounded once integrated C-semigroup [16].

For $f, g, h \in C_0(\mathbb{R})$ and $t \in \mathbb{R}$, we set

$$\begin{aligned}
&B(f,g,h)(t) \\
&\quad = \left(e^{-t}\int_0^t f(s)\,ds,\ e^{-2t}\int_0^t g(s)\,ds,\ te^{-3t}\sin t\cdot h(t)\right).
\end{aligned} \tag{55}$$

Then one can simply verify that $B \in L(X), R(B) \subset C(D(A))$, and $BC(f,g,h) = CB(f,g,h)$, $(f,g,h) \in X$. Then from Theorem 11(i), $(I + e^{-\lambda^2}B)A$ subgenerates an exponentially bounded once integrated C-semigroup on X.

Example 16. Let $X_1 = L^2(\mathbb{R}^3)$, $X_2 = L^p(\mathbb{R}^3)$ $(1 \le p \le \infty)$,

$$\begin{gathered} A_1 = \Delta, \qquad D(A_1) = H^2(\mathbb{R}^3), \\ A_2 = a\Delta + \sum_{i=1}^{3} c_i \frac{\partial}{\partial x_1} + c_4 \;\; (a > 0,\; c_i \in \mathbb{R},\; i = 1,2,3,4), \\ D(A_2) = W^{2,p}(\mathbb{R}^3). \end{gathered} \tag{56}$$

Then A_1 generates a strongly continuous cosine function $\mathrm{C}_1(\cdot)$ on X_1. It follows from [5] that A_2 generates an exponentially bounded C_2-cosine function $\mathrm{C}_2(\cdot)$ on X_2, where $C_2 = (1-\Delta)^{-1}$.

Set $r_1(\cdot) \in H^2(\mathbb{R}^3)$, $r_2(\cdot) \in W^{2,p}(\mathbb{R}^3)$, $q_1(\cdot) \in C_c^2(\mathbb{R}^3)$, $q_2(\cdot) \in C_c(\mathbb{R}^3)$. Define bounded linear operators $B_1 : X_2 \to X_1$, $B_2 : X_1 \to X_2$ as follows:

$$\begin{aligned} (B_1\phi)(\xi) &= r_1(\xi) \int_{\mathbb{R}^3} q_1(\sigma)\phi(\sigma)\, d\sigma, \\ (B_2\phi)(\xi) &= r_2(\xi) \int_{\mathbb{R}^3} q_2(\sigma)\phi(\sigma)\, d\sigma. \end{aligned} \tag{57}$$

Let $X = X_1 \times X_2$,

$$\begin{gathered} A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}, \qquad D(A) := D(A_1) \times D(A_2), \\ B = \begin{pmatrix} 0 & B_1 \\ B_2 & 0 \end{pmatrix}, \qquad D(B) := X. \end{gathered} \tag{58}$$

Taking $\lambda_0 \in \rho(A)$ and putting $C = (\lambda_0 - A)^{-1}$, then A generates an exponentially bounded C-cosine function $\mathrm{C}(\cdot)$ on X, where

$$\mathrm{C}(t) = \begin{pmatrix} \mathrm{C}_1(t)(\lambda_0 - A_1)^{-1} & 0 \\ 0 & \mathrm{C}_2(t) C_2^{-1}(\lambda_0 - A_2)^{-1} \end{pmatrix}. \tag{59}$$

We denote $\mathrm{S}_1(t) := \int_0^t \mathrm{C}_1(s)ds$, $\mathrm{S}_2(t) := \int_0^t \mathrm{C}_2(s)ds$, $\mathrm{S}(t) := \int_0^t \mathrm{C}(s)ds$, then

$$\mathrm{S}(t) = \begin{pmatrix} \mathrm{S}_1(t)(\lambda_0 - A_1)^{-1} & 0 \\ 0 & \mathrm{S}_2(t) C_2^{-1}(\lambda_0 - A_2)^{-1} \end{pmatrix}, \tag{60}$$

and for any $x = \binom{x_1}{x_2} \in D(A)$, $0 \le s \le t < \infty$,

$$\begin{aligned} &C^{-1}B A \mathrm{S}(t-s)x \\ &\quad = \begin{pmatrix} (\lambda_0 - A_1) B_1 A_2 \mathrm{S}_2(t-s) C_2^{-1}(\lambda_0 - A_2)^{-1} x_2 \\ (\lambda_0 - A_2) B_2 A_1 \mathrm{S}_1(t-s)(\lambda_0 - A_1)^{-1} x_1 \end{pmatrix}. \end{aligned} \tag{61}$$

It follows from $R(B_1) \subset D(A_1)$ and $R(B_2) \subset D(A_2)$ that there exist $M, \omega > 0$ such that

$$\begin{aligned} &e^{-\lambda t} \left\| \int_0^t C^{-1}BAC(t-s)\, x ds \right\| \\ &\quad \le \frac{M}{\omega} e^{-\lambda t} \left(e^{\omega t} - 1\right) \|x\|, x \in D(A), \end{aligned} \tag{62}$$

then

$$\begin{aligned} L(\lambda) := \sup \Bigg\{ &\int_0^a e^{-\lambda t} \left\| \int_0^t C^{-1}BAC(t-s)\, x ds \right\| dt, \\ &x \in D(A), \|x\| \le 1 \Bigg\} < \infty, \end{aligned} \tag{63}$$

and then (40) is satisfied.

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