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Research Article

Multiplicity and Bifurcation of Solutions for a Class of Asymptotically Linear Elliptic Problems on the Unit Ball

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This paper mainly dealt with the exact number and global bifurcation of positive solutions for a class of semilinear elliptic equations with asymptotically linear function on a unit ball. As byproducts, some existence and multiplicity results are also obtained on a general bounded domain.

1. Introduction

In this paper, we are concerned with positive solutions of the following elliptic equation subject to homogeneous Dirichlet boundary condition

$$\begin{aligned} -\Delta u &= \lambda f(u), \quad \text{in } \Omega, \\ u &= 0, \quad \text{on } \partial\Omega, \end{aligned} \tag{P_λ}$$

where Ω is a smooth bounded domain in R^N, λ is a positive parameter, $u \in C^2(\Omega) \cap C(\overline{\Omega})$, and the function f satisfies the following.

(F1) $f : [0, +\infty) \to (0, +\infty)$ is a positive C^1 function, and f is strictly convex; that is, $f'(t)$ is strictly increasing in $t \in (0, \infty)$.

(F2) f is asymptotically linear, that is,

$$\lim_{t \to \infty} \frac{f(t)}{t} = a \in (0, +\infty). \tag{1}$$

For the past years, this problem attracted attentions of many authors. It was studied in [1–4] with f being strictly increasing and was studied in [5–7] with a specific function $f(u) = \sqrt{(u-b)^2 + \epsilon}$ which is not increasing.

The main goal of this paper is to study the exact number and bifurcation structure of the solutions of (P_λ) on a unit ball Ω, with a general asymptotically linear function f. Some results in this paper (see Section 3) can be viewed as an extension and improvement of that in [7], but the argument approach here is very different to that in [7]. As byproducts, we also get some new results which also hold for general domain Ω (see Section 2). The paper is organized as follows. In Section 2, we study the existence and multiplicity of solutions for problem (P_λ) on a general bounded domain, with some new results complementing those existing in the literature. In Section 3, we study the exact number and global bifurcation structure of positive solutions of (P_λ) on a unit ball.

2. Multiplicity of Positive Solutions on a General Domain

Throughout this section, we assume that Ω is a smooth bounded domain in R^N, and f satisfies (F1) and (F2). We also note that, by maximum principle, all solutions of (P_λ) are positive on Ω.

Before the statement of our main result, we derive some preliminary lemmas. Though some of them may be known, we provide their proofs for reader's convenience.

Lemma 1. *For any $\lambda \in (0, \lambda_1/a)$, (P_λ) is solvable.*

Proof. Consider the functional

$$J_\lambda(u) = \int_\Omega \left(\frac{|\nabla u|^2}{2} - \lambda F(u) \right) dx, \tag{2}$$

where $F(u) = \int_0^u f(t)dt$.

From (F1) and (F2), it is easy to see that

$$f'(t) < a, \tag{3}$$

so

$$F(u) \le \frac{au^2}{2} + f(0)u. \tag{4}$$

Poincàre's inequality $\int_\Omega u^2 \le (1/\lambda_1)\int_\Omega |\nabla u|^2$, and the imbedding theorem of $L^2(\Omega)$ to $L^1(\Omega)$ yield

$$\begin{aligned} J_\lambda(u) &\ge \int_\Omega \frac{|\nabla u|^2}{2} dx - \frac{a\lambda}{2}\int_\Omega u^2 dx - \lambda f(0)\int_\Omega u\,dx \\ &\ge \int_\Omega \frac{|\nabla u|^2}{2} dx - \frac{a\lambda}{2\lambda_1}\int_\Omega |\nabla u|^2 dx - \lambda f(0)\int_\Omega u\,dx \\ &\ge \frac{1}{2}\left(1 - \frac{a\lambda}{\lambda_1}\right)\int_\Omega |\nabla u|^2 dx - \lambda f(0) C\left(\int_\Omega |\nabla u|^2\right)^{1/2} dx, \end{aligned} \tag{5}$$

so $J_\lambda(u) \to \infty$ as $\|u\|_{H_0^1(\Omega)} \to \infty$, where $\|u\|_{H_0^1(\Omega)} = (\int_\Omega |\nabla u|^2)^{1/2}dx$, and then $J_\lambda(u)$ is coercive and bounded from below. It is also easy to see that $J_\lambda(u)$ is weakly lower semicontinuous [8, page 446, Theorem 1]. By applying direct variational methods [9, page 4, Theorem 1.2], we can get the desired result; that is, $\min_{u\in H_0^1(\Omega)} J_\lambda(u)$ is reached at some point $u(\lambda)$, and $u(\lambda)$ is a solution of (P_λ) when $\lambda \in (0, \lambda_1/a)$. □

Lemma 2. *For any $\lambda > \lambda_1/m$, (P_λ) has no solution, where $m = \inf_{t>0}(f(t)/t)$.*

Proof. If not, assume that u is a solution of (P_λ) for some $\lambda > \lambda_1/m$. Multiplying (P_λ) by $\varphi_1 > 0$, the normalized positive eigenfunction with respect to the first eigenvalue λ_1 of $-\Delta$ subject to homogenous Dirichlet boundary condition, and then integrating by parts, we get

$$\begin{aligned} \lambda_1 \int_\Omega u\varphi_1 dx &= \int_\Omega -\Delta u \varphi_1 dx \\ &= \lambda \int_\Omega f(u)\varphi_1 dx > \lambda_1 \int_\Omega u\varphi_1 dx, \end{aligned} \tag{6}$$

which is a contradiction. □

We begin by show the following.

Lemma 3. *There exists a number $\lambda_1/a \le \Lambda \le \lambda_1/m$, such that (P_λ) has at least a solution for $\lambda < \Lambda$ and has no solution for $\lambda > \Lambda$.*

Proof. Let

$$\Lambda = \{\lambda : (P_\lambda) \text{ has a solution}\}. \tag{7}$$

By Lemmas 1 and 2, $\lambda_1/a \le \Lambda \le \lambda_1/m$. We need just to prove that if (P_μ) has a solution, then (P_λ) also has a solution for all $0 < \lambda < \mu$. This can be done by a simple argument of sub-sup solution method, since it is easy to see that any solution of (P_μ) is a super solution of (P_λ) and $u \equiv 0$ a subsolution.

It is easy to see that $u_* \equiv 0$ is a subsolution of (P_λ), then a standard sub-super solution method's argument and comparison theorems give the following lemma. □

Lemma 4. *If (P_λ) is solvable, then one has a minimal solution u_λ, that is, for any solution v of (P_λ), $u_\lambda \le v$. Moreover, u_λ is increasing with respect to λ.*

Lemma 5. *If $\lambda \in (0, \lambda_1/a)$, then the solution of (P_λ) is unique.*

Proof. Suppose that v_1 and v_2 are solutions of (P_λ). Let $v = v_1 - v_2$, then

$$\begin{aligned} -\Delta v &= \lambda\left[f(v_1) - f(v_2)\right], \quad \text{in } \Omega, \\ v &= 0, \quad \text{on } \partial\Omega. \end{aligned} \tag{8}$$

By mean value theorem, v satisfies

$$-\Delta v = f'(\overline{v})v, \tag{9}$$

where $\overline{v}$ lies between v_1 and v_2. Multiplying v and integrating, we get

$$\begin{aligned} \int_\Omega |\nabla v|^2 dx &= \lambda \int_\Omega f'(\overline{v})v^2 dx \\ &\le a\lambda \int_\Omega v^2 dx \le \frac{a\lambda}{\lambda_1}\int_\Omega |\nabla v|^2 dx, \end{aligned} \tag{10}$$

which implies that $v \equiv 0$. The proof is complete. □

Lemma 6. *The minimal solution u_λ is stable, that is, $\lambda_1(-\Delta - \lambda f'(u_\lambda)) \ge 0$, where $\lambda_1(-\Delta - \lambda f'(u_\lambda))$ denotes the first eigenvalue of the following problem:*

$$\begin{aligned} -\Delta w - \lambda f'(u_\lambda)w &= \mu w, \quad \text{in } \Omega, \\ w &= 0, \quad \text{on } \partial\Omega. \end{aligned} \tag{11}$$

Proof. Suppose on the contrary that $\lambda_1(-\Delta - \lambda f'(u_\lambda)) = \mu < 0$, and $w > 0$ is the corresponding eigenvector. Let $v_\varepsilon = u_\lambda - \varepsilon\varphi$, then by (P_λ) and (11), we have

$$\begin{aligned} -\Delta v_\varepsilon - \lambda f(v_\lambda) &= \lambda f(u_\lambda) - \lambda\varepsilon f'(u_\lambda)\varphi \\ &\quad - \lambda f(u_\lambda - \varepsilon\varphi) - \mu\varepsilon\varphi \\ &= -\mu\varepsilon\varphi + o(\varepsilon\varphi) > 0, \end{aligned} \tag{12}$$

when ε is small enough, and hence $v_\varepsilon = u_\lambda - \varepsilon\varphi$ is a super solution of problem (P_λ). On the other hand, 0 is a subsolution of (P_λ), and Hopf's boundary lemma implies that

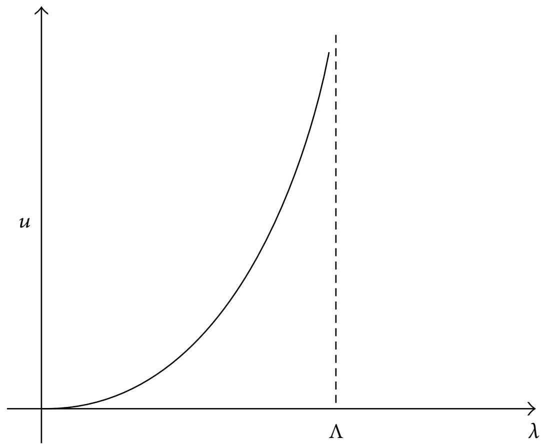

FIGURE 1: Diagram for $\Lambda = \lambda_1/a$.

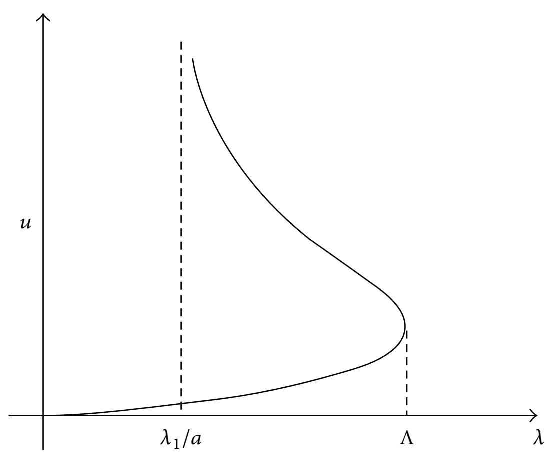

FIGURE 2: Minimal diagram for $\Lambda > \lambda_1/a$.

$0 < v_\varepsilon$ for $\varepsilon > 0$ small. An application of sub-sup solution method guarantees that there is a solution $\overline{u}$ of (P_λ) satisfying $0 < \overline{u} \le u_\lambda - \varepsilon\varphi$ in Ω, which is a contradiction with the minimality of u_λ. The proof is complete. □

Now we state our main result.

Theorem 7. *Suppose that f satisfies (F1) and (F2), then there exists $\Lambda \in [\lambda_1/a, \lambda_1/m]$ (where $m = \inf_{t>0}(f(t)/t)$) such that problem (P_λ)*

(i) *has at least one solution for $\lambda \in (0, \Lambda)$ and a unique solution for $\lambda \in (0, \lambda_1/a)$;*

(ii) *has no solution for $\lambda \in (\Lambda, +\infty)$;*

(iii) (a) *if $\Lambda = \lambda_1/a$, then problem (P_λ) has no solution at $\lambda = \Lambda$, and $\lim_{\lambda\to\Lambda-0} u_\lambda(x) = +\infty$ for all $x \in \Omega$, where u_λ denotes the unique solution of (P_λ) for $\lambda \in (0, \Lambda)$ (see Figure 1),*

(b) *if $\Lambda > \lambda_1/a$, then problem (P_λ) has a unique solution for $\lambda \in (0, \lambda_1/a]$ and $\lambda = \Lambda$, has at least two solutions for $\lambda \in (\lambda_1/a, \Lambda)$ (see Figure 2 for a minimal diagram).*

Proof. Statement (i) follows from Lemmas 3 and 5. Statement (ii) follows from Lemma 3. Now we give the proof of statement (iii).

(a) Suppose $\Lambda = \lambda_1/a$. The solution (P_λ) bifurcates at infinity near $\Lambda = \lambda_1/a$ (see [2, 10] for details). On the other hand, (P_λ) has a unique solution u_λ for $\lambda \in (0, \lambda_1/a)$, and no solution for $\lambda > \lambda_1/a$. Therefore the bifurcation curve from infinity is on the left of $\lambda = \lambda_1/a$, and hence $\lim_{\lambda\to\Lambda-0} u_\lambda(x) = +\infty$ for all $x \in \Omega$ by the expression of the bifurcation solution in Theorem 13 in Section 3.

If (P_Λ) has a solution, let u_Λ denote the minimal solution of (P_λ). By Lemma 4, $u_\lambda \le u_\Lambda$ for $\lambda \in (0, \Lambda)$, contradicting $\lim_{\lambda\to\Lambda-0} \| u_\lambda \|_\infty = \infty$.

(b) For clarity, the proof will be divided into 3 steps.

Step 1. The existence and uniqueness of solutions of (P_λ) for $\lambda = \lambda_1/a$.

The existence follows directly from Lemma 4. Note that $f' < a$, and the uniqueness can be proved in a similar way as in the proof of Lemma 5.

Step 2. The existence and uniqueness of solutions of (P_λ) for $\lambda = \Lambda$.

By Lemmas 3 and 4, (P_λ) has a minimal solution u_λ for any $\lambda \in (0, \Lambda)$, and u_λ is increasing in λ. Let $(\lambda_n) \subset (\lambda_1/a, \Lambda)$ be any sequence such that $\lim_{n\to\infty} \lambda_n = \Lambda$. Firstly we insure that case (u_{λ_n}) is $L^2(\Omega)$ bounded. Suppose the contrary that $\lim_{n\to\infty} \| u_{\lambda_n} \|_{L^2(\Omega)} = \infty$. Let $c_n = \| u_{\lambda_n} \|_{L^2(\Omega)}$ and $v_{\lambda_n} = u_{\lambda_n}/c_n$, then

$$\begin{gathered} -\Delta v_{\lambda_n} = \frac{\lambda_n}{c_n} f\left(c_n v_{\lambda_n}\right), \quad \text{in } \Omega, \\ v_{\lambda_n} = 0, \quad \text{on } \partial\Omega. \end{gathered} \tag{13}$$

Since $f(c_n v_{\lambda_n})/c_n$ is bounded in $L^2(\Omega)$, it follows from (13) that v_{λ_n} is bounded in $H_0^1(\Omega)$. Then subject to a subsequence, we may suppose that there exits v^*, such that

$$\begin{gathered} v_{\lambda_n} \rightharpoonup v^* \quad \text{weakly in } H_0^1(\Omega), \\ v_{\lambda_n} \longrightarrow v^* \quad \text{strongly in } L^2(\Omega), \\ v_{\lambda_n} \longrightarrow v^* \quad \text{a.e. in } \Omega. \end{gathered} \tag{14}$$

Then by letting $n \to \infty$, we get from (13) in the weak sense that

$$\begin{gathered} -\Delta v^* = a\Lambda v^*, \quad \text{in } \Omega, \\ v^* = 0, \quad \text{on } \partial\Omega, \end{gathered} \tag{15}$$

with $\| v^* \|_{L^2(\Omega)} = 1$, and $v^* > 0$ by strong maximum principle. Hence $a\Lambda = \lambda_1$, that is, $\Lambda = \lambda_1/a$, a desired contradiction.

Now in a similar way, the boundedness of (u_{λ_n}) in $L^2(\Omega)$ implies that (u_{λ_n}) is bounded in $H_0^1(\Omega)$. Then subject to a subsequence, we may suppose that there exits u^*, such that

$$\begin{gathered} u_{\lambda_n} \rightharpoonup u^* \quad \text{weakly in } H_0^1(\Omega), \\ u_{\lambda_n} \longrightarrow u^* \quad \text{strongly in } L^2(\Omega), \\ u_{\lambda_n} \longrightarrow u^* \quad \text{a.e. in } \Omega. \end{gathered} \tag{16}$$

Then by letting $n \to \infty$, we get

$$-\Delta u^* = \Lambda f(u^*), \quad \text{in } \Omega, \qquad u^* = 0, \quad \text{on } \partial\Omega, \tag{17}$$

and the existence is proved.

Now we prove the uniqueness. Let u_Λ be the minimal solution of (P_Λ) and $\overline{u}$ a different solution. Then $w := \overline{u} - u_\Lambda > 0$ satisfies

$$-\Delta v = \Lambda f'(u_\Lambda + \theta w) w, \quad \text{in } \Omega, \qquad v = 0, \quad \text{on } \partial\Omega, \tag{18}$$

where $\theta : \Omega \to \mathbb{R}$ satisfying $0 < \theta < 1$. It follows that $\lambda_1(-\Delta - \Lambda f'(u_\Lambda + \theta w)) = 0$, where $\lambda_1(-\Delta - \Lambda f'(u_\Lambda + \theta w))$ denotes the first eigenvalue of the operator $-\Delta - \Lambda f'(u_\Lambda + \theta w)$ subject to the Dirichlet boundary condition, as defined in Lemma 1. Since $f'(u_\Lambda) < f'(u_\Lambda + \theta w)$ in Ω, we have that $\lambda_1(-\Delta - \Lambda f'(u_\Lambda)) > \lambda_1(-\Delta - \Lambda f'(u_\lambda + \theta w)) = 0$, which implies that the operator $-\Delta - \Lambda f'(u_\Lambda)$ is nondegenerate. Then by the Implicit Function Theorem, the solution of (P_λ) forms a cure in a neighborhood of (Λ, u_Λ), which is clearly contradicted to the definition of Λ in (7).

Step 3. Prove that (P_λ) has at least two solutions for $\lambda \in (\lambda_1/a, \Lambda)$.

Following the argument in [5], we prove it by variational method of Nehari type (see [11]). As we have known (Lemma 5), there exists a minimal solution u_λ of (P_λ) when $\lambda \in (\lambda_1/a, \Lambda)$. Now we must look for another solution $u (> u_\lambda)$. Assuming that $u = v + u_\lambda$, with $v > 0$, then v satisfies

$$-\Delta v = \lambda \left[f(v + u_\lambda) - f(u_\lambda) \right], \quad \text{in } \Omega, \qquad v = 0, \quad \text{on } \partial\Omega. \tag{19}$$

For convenience, let $g(v) = f(v + u_\lambda) - f(u_\lambda)$ and $G(v) = \int_0^v g(t)dt$, then we have

$$-\Delta v = \lambda g(v), \quad \text{in } \Omega, \qquad v = 0, \quad \text{on } \partial\Omega. \tag{20}$$

Define

$$J_\lambda(v) = \int_\Omega \left(\frac{|\nabla v|^2}{2} - \lambda G(v) \right) dx, \qquad I_\lambda(v) = \int_\Omega \left(|\nabla v|^2 - \lambda v g(v) \right) dx, \tag{21}$$

and the solution manifold

$$M_\lambda = \left\{ v \in H_0^1(\Omega) : v > 0 \text{ in } \Omega, I_\lambda(v) = 0 \right\}. \tag{22}$$

Firstly we show that $M_\lambda \neq \phi$ for any $\lambda \in (\lambda_1/a, \Lambda)$. Let φ_1 be the first eigenfunction of $-\Delta$ in Ω subject to Dirichlet boundary condition and $\int_\Omega \varphi_1^2 dx = 1$, then

$$\begin{aligned} I_\lambda(t\varphi_1) &= \lambda_1 t^2 - \lambda \int_\Omega t\varphi_1 g(t\varphi_1) dx \\ &= t^2 \left(\lambda_1 - \lambda \int_\Omega \frac{\varphi_1 g(t\varphi_1)}{t} dx \right), \end{aligned} \tag{23}$$

$$\lim_{t\to\infty} \int_\Omega \frac{\varphi_1 g(t\varphi_1)}{t} dx = \lim_{t\to\infty} \int_\Omega \varphi_1^2 \cdot \frac{g(t\varphi_1)}{t\varphi_1} dx = a.$$

It follows from (23) that

$$I_\lambda(t\varphi_1) < 0, \tag{24}$$

for sufficiently large t if $\lambda \in (\lambda_1/a, \Lambda)$.

On the other hand, let ω_1 be the eigenfunction with $\int_\Omega \omega_1^2 dx = 1$ of the first eigenvalue μ_1 of the following equation:

$$-\Delta \omega_1 - \lambda f'(u_\lambda) \omega_1 = \mu_1 \omega_1, \quad \text{in } \Omega, \qquad \omega_1 = 0, \quad \text{on } \partial\Omega. \tag{25}$$

Since u_λ is the minimal solution, it follows from Lemmas 4 and 6 that $\mu_1 > 0$. Then

$$\begin{aligned} I_\lambda(s\omega_1) &= s^2 \int_\Omega |\nabla \omega_1|^2 dx - \lambda s \int_\Omega \omega_1 g(s\omega_1) dx \\ &= s^2 \int_\Omega |\nabla \omega_1|^2 dx - \lambda s \int_\Omega \left[f'(u_\lambda) s\omega_1^2 + o(s^2) \right] dx \\ &= s^2 \left[\int_\Omega \left(|\nabla \omega_1|^2 - \lambda f'(u_\lambda) \omega_1^2 \right) dx + o(1) \right] \\ &= s^2 (\mu_1 + o(1)). \end{aligned} \tag{26}$$

Hence $I_\lambda(s\omega_1) > 0$ when s is small enough. Now it is easy to see that M_λ is not empty. In fact, take $w_* = t\varphi_1$ for some large t, and $w^* = s\omega$ for some small $s > 0$, such that

$$I_\lambda(w_*) < 0, \qquad I_\lambda(w^*) > 0, \tag{27}$$

respectively. Define a continuous function G on $[0,1]$, namely,

$$G(\xi) = I_\lambda(\xi w_* + (1-\xi) w^*). \tag{28}$$

Then $G(0) > 0$, $G(1) < 0$, and hence there exist $\xi_0 \in (0,1)$ such that $G(\xi_0) = 0$, that is, $I_\lambda(\xi_0 w_* + (1-\xi_0)w^*) = 0$, and $M_\lambda \neq \phi$, a desired conclusion.

Since f is convex, $g(v)$ is convex with respect to $v > 0$ such that

$$g(v) = g(v) - g(0) \leq g'(v) v. \tag{29}$$

Integrating (29) with respect to v from 0 to v, we get

$$2G(v) \leq g(v) v. \tag{30}$$

Therefore, on M_λ

$$J_\lambda(v) = \frac{\lambda}{2}\int_\Omega [g(v)v - 2G(v)]\,dx \ge 0, \tag{31}$$

that is, $J_\lambda(v)$ is bounded from below.

And then we obtain a nonminimal positive solution of (P_λ) by using the *Nehari* variational method. The proof is complete. □

Remark 8. The solutions that we get from the above discussion are weak ones, but a standard elliptic regularity argument shows that they are indeed classical solutions.

In view of Theorem 7, we want to know what conditions ensure that $\Lambda = \lambda_1/a$ or $\Lambda > \lambda_1/a$. Following [4], we consider the function $L(t) = at - f(t)$. It is easy to see that $L(t)$ is strictly increasing, and hence $\lim_{t\to\infty} L(t) = L_\infty$ exists (may be $+\infty$). Also note that $L(0) = -f(0) < 0$.

Theorem 9. *If $L_\infty \le 0$, then $\Lambda = \lambda_1/a$; if $L_\infty > 0$, then $\Lambda > \lambda_1/a$.*

Proof. (i) If $L_\infty \le 0$, then $f(t) \ge at$ for all $t \ge 0$. We prove that (P_λ) has no solution and hence $\Lambda = \lambda_1/a$. Suppose the contrary that u is a solution (P_λ) for $\lambda = \lambda_1/a$, then

$$-\Delta u = \frac{\lambda_1}{a} f(u) \ge \lambda_1 u. \tag{32}$$

Let φ be a positive eigenfunction of the first eigenvalue λ of $-\Delta$ on Ω with Dirichlet boundary condition, that is

$$\begin{gathered}\Delta\varphi + \lambda_1\varphi = 0, \quad \text{in } \Omega,\\ \varphi = 0, \quad \text{on } \partial\Omega.\end{gathered} \tag{33}$$

Multiplying (32) by $\varphi > 0$, and integrating by parts, we get

$$\int_\Omega (f(u) - au)\varphi\,dx = 0, \tag{34}$$

which yields that $f(u) = au$, contradicting the fact that $f(0) > 0$.

(ii) If $L_\infty > 0$, we prove that $\Lambda > \lambda_1/a$.

Let $(\lambda(s), u(s))$ be the bifurcation curve as described in Theorem 13 in Section 3, then

$$\begin{gathered}\Delta u(s) + \lambda(s) f(u(s)) = 0, \quad \text{in } \Omega,\\ u(s) = 0, \quad \text{on } \partial\Omega.\end{gathered} \tag{35}$$

It follows from (33) and (35) that

$$\begin{aligned}\lambda(s)\int_\Omega f(u(s))\varphi\,dx &= \lambda_1\int_\Omega u(s)\varphi\,dx\\ &= \frac{\lambda_1}{a}\int_\Omega au(s)\varphi\,dx.\end{aligned} \tag{36}$$

By the fact that $u(s)(x) = s\varphi(x) + z(s)(x) \to \infty$ $(s \to \infty)$ a.e. in Ω, we have

$$\begin{aligned}&\int_\Omega au(s)\varphi\,dx - \int_\Omega f(u(s))\varphi\,dx\\ &\quad = \int_\Omega (au(s) - f(u(s)))\varphi\,dx > 0,\end{aligned} \tag{37}$$

for s sufficiently large. It follows from (36) that $\lambda(s) > \lambda_1/a$ when s is sufficiently large, which means that the bifurcation curve $(\lambda(s), u(s))$ from infinity is on the right of $\lambda = \lambda_1/a$, and hence $\Lambda > \lambda_1/a$ by the definition of Λ in (7). The proof is complete. □

Now we define another function which is also crucial in studying exact multiplicity in the next section. Let

$$K(t) = tf'(t) - f(t), \tag{38}$$

then $K'(t) = tf''(t) > 0$ a.e. in $(0, +\infty)$, and $K(t)$ is strictly increasing, and $K(0) = -f(0) < 0$. Denote

$$\lim_{t\to\infty} K(t) = K_\infty \in (-\infty, +\infty]. \tag{39}$$

Theorem 10. *If $K_\infty \le 0$, then $\Lambda = \lambda_1/a$; if $K_\infty > 0$, then $\Lambda > \lambda_1/a$.*

Proof. If $K_\infty \le 0$, then $(f(t)/t)' = K(t)/t^2 < 0$ for all $t > 0$. It follows that $f(t)/t$ is strictly decreasing and hence $f(t)/t > a$, which implies that $L_\infty \le 0$.

On the other hand, if $K_\infty > 0$, by

$$L(t) - K(t) = t\left(a - f'(t)\right) > 0, \quad \forall t > 0, \tag{40}$$

we get that $L_\infty > 0$. Then the conclusion follows for Theorem 9. □

3. Exact Number and Global Bifurcation of Solutions on a Unit Ball

From Theorem 7, the exact number of solutions (P_λ) is now clear in the case of $\Lambda = \lambda_1/a$; that is, the solution is unique if it exists. On the other hand, it is far from known in general exactly how may solutions of (P_λ) for $\lambda \in (\lambda_1/a, \Lambda)$ if $\Lambda > \lambda_1/a$. Using the bifurcation approach developed in [12–14], and also the idea and techniques developed in [7], we solve this problem on the unit ball under some conditions.

Throughout this section, we suppose that Ω is the unit ball in R^N centered with the origin.

The next remarkable results regarding (P_λ) are due to Gidas et al. [15] and Lin and Ni [16].

Lemma 11. *(1) If f is locally Lipschitz continuous in $[0,\infty)$, then all positive solutions of (P_λ) are radially symmetric, that is, $u(x) = u(r)$, $r = |x|$, and satisfies*

$$\begin{gathered}u'' + \frac{n-1}{r}u' + \lambda f(u) = 0, \quad r \in (0,1),\\ u'(0) = u(1) = 0.\end{gathered} \tag{41}$$

Moreover, $u'(r) < 0$ for all $r \in (0,1]$, and hence $u(0) = \max_{0\le r\le 1} u(r)$.

(2) Suppose $f \in C^1(R)$. If u is a positive solution to (P_λ), and w is a solution of the linearized problem (43) *(if it exists), then w is also radially symmetric and satisfies*

$$\begin{gathered}w'' + \frac{n-1}{r} + \lambda f'(u) w = 0, \quad r \in (0,1),\\ w'(0) = w(1) = 0.\end{gathered} \tag{42}$$

The next lemma also plays a key role in this section.

Lemma 12. *(1) For any $d > 0$, there is at most one $\lambda_d > 0$ such that (P_λ) have a positive solution $u(\cdot)$ with $\lambda = \lambda_d$ and $u(0) = d$.*

(2) Let $T = \{d > 0 : (P_\lambda)$ have a positive solution with $u(0) = d\}$, then T is open; $\lambda(d) = \lambda_d$ is a well-defined continuous function from T to R^+.

Lemma 12 is well known; see, for example, [13, 17, 18]. A simple proof of the first part of the lemma can be found in [14]. Because of Lemma 12, we call $R^+ \times R^+ = \{(\lambda, d) : \lambda > 0, d > 0\}$ the phase space, $\{(\lambda(d), d) : d \in T\}$ the bifurcation curve, and the phase space with bifurcation curve the bifurcation diagram.

We will also need the following theorem of bifurcation from infinity.

Theorem 13 (see [10, 19]). *Suppose $f \in C^1(R)$. Let $\lim_{u\to\infty} f(u)/u = a \in (0, \infty)$ and $\lambda_\infty = \lambda_1/a$. Then all positive solutions of (P_λ) near (λ_∞, ∞) have the form of $(\lambda(s), s\varphi + z(s))$ for $s \in (\delta, \infty)$ and some $\delta > 0$, where φ is a positive eigenfunction of the first eigenvalue λ_1 of $-\Delta$ on Ω subjected to Dirichlet boundary condition, $\lim_{s\to\infty} \lambda(s) = \lambda_\infty$, and $\| z(s)\|_{C^{2,\alpha}(\overline{B}^n)} = o(s)$ as $s \to \infty$.*

To make bifurcation argument work, a crucial thing is the following result.

Let u be a solution of problem (P_λ), then u is called a degenerate solution if the corresponding linearized equation

$$\begin{aligned} -\Delta w &= \lambda f'(u)\, w, \quad \text{in } \Omega, \\ w &= 0, \quad \text{on } \partial\Omega, \end{aligned} \tag{43}$$

has a nontrivial solution.

Now suppose that f satisfies (F1), (F2). As in the end of Section 2, let

$$\begin{aligned} K(t) &= t f'(t) - f(t) \\ K_\infty &= \lim_{t\to\infty} K(t). \end{aligned} \tag{44}$$

If $K_\infty > 0$, then there exists a unique real number $\beta > 0$, such that

$$\begin{aligned} K(t) < 0 \quad &\text{for } t \in [0, \beta); \\ K(t) > 0 \quad &\text{for } t \in (\beta, \infty); \ K(\beta) = 0. \end{aligned} \tag{45}$$

Lemma 14. *Suppose that $K_\infty > 0$. If u is a degenerate solution of (P_λ), then $u(0) > \beta$.*

Proof. Suppose the contrary that $u(0) \le \beta$, then

$$K(u) = u f'(u) - f(u) < 0, \quad \text{in } \Omega \setminus \{0\}. \tag{46}$$

Let w be a nontrivial solution of the corresponding linearized equation (43). From (P_λ) and (43), we get

$$0 = \int_\Omega (-\Delta w u + \Delta u w)\, dx = \lambda \int_\Omega \left(u f'(u) - f(u)\right) w\, dx. \tag{47}$$

It appears from (46) and (47) that w must change sign in Ω.

In view of Lemma 11(2), we suppose that $|x| = r_1$ is a maximal zero in $(0, 1)$. We may also suppose that $w(x) > 0$, for all $r_1 < |x| < 1$. Then

$$\begin{aligned} &\int_{\Omega\setminus B(r_1)} (-\Delta w u + \Delta u w)\, dx \\ &\quad = \lambda \int_\Omega \left(u f'(u) - f(u)\right) w\, dx < 0, \end{aligned} \tag{48}$$

where $B(r_1)$ denotes the ball of radius r_1 centered with the origin.

On the other hand, using integration by parts, we have

$$\int_{\Omega\setminus B(r_1)} (-\Delta w u + \Delta u w)\, dx = -\int_{\partial(\Omega\setminus B(r_1))} \frac{\partial w}{\partial \nu} u\, ds > 0. \tag{49}$$

a contradiction. □

Theorem 15. *Suppose that f satisfies (F1)-(F2) with $0 < K_\infty < a\beta$. If u is a degenerate solution of (P_λ), then any nontrivial solution of the corresponding linearized equation (43) does not change sign in Ω.*

Proof. By Lemma 14, $\max_{x\in\overline{\Omega}} u(x) = u(0) > \beta$. In view of Lemma 11, there exists $r^* \in (0, 1)$, such that $u(r^*) = \beta$. Let w be a non-trivial solution of the corresponding linearized equation (43), then $w(0) \ne 0$.

We assert that $w(r)$ has no zeroes on $[r^*, 1)$. Suppose the contrary and let r_1 be the largest zero of w on $[r^*, 1)$. We may suppose that $w > 0$ in $(r_1, 1)$. Note that $u(r) < \beta$ for $r \in (r_1, 1)$, a similar argument as in the proof of Lemma 14 yields a contradiction.

Now we prove that $w(r)$ has no zeroes on $(0, r^*)$. Suppose the contrary and let r_0 be the smallest zero of $w(r)$ on $(0, r^*)$. We may suppose that $w > 0$ in $B(r_0)$. Multiplying (P_λ) by $u - \beta$, (43) by w, subtracting, and integrating on $B(r_0)$, we get

$$\begin{aligned} &\int_{B(r_0)} \left[-\Delta w (u - \beta) + \Delta u w\right] dx \\ &\quad = \lambda \int_{B(r_0)} \left[(u - \beta) f'(u) - f(u)\right] w\, dx. \end{aligned} \tag{50}$$

Let $J(t) = (t - \beta) f'(t) - f(t)$, then $J(0) = -f(0) < 0$, $J(\infty) = \lim_{t\to\infty} J(u) = K_\infty - a\beta < 0$, and $J'(t) = (t-\beta) f''(t) > 0$ for $t > \beta$. Hence $J(u) = (u-\beta) f'(u) - f(u) < 0$ for $x \in B(r_0)$. Then

$$\int_{B(r_0)} \left[(u - \beta) f'(u) - f(u)\right] w\, dx < 0. \tag{51}$$

On the other hand, by Green formula,

$$\begin{aligned} &\int_{B(r_0)} \left[-\Delta w (u - \beta) + \Delta u w\right] dx \\ &\quad = -\int_{\partial(B(r_0))} \frac{\partial w}{\partial \nu} (u - \beta)\, dx > 0. \end{aligned} \tag{52}$$

A contradiction occurs from (50), (51), and (52). Hence $w(r)$ has no zeroes in $(0, 1)$, that is to say, w does not change sign in Ω. The proof is complete. □

Now define $F : C_0^{2,\alpha}(\overline{\Omega}) \to C^{\alpha}(\overline{\Omega})$, by

$$Fu = \Delta u + \lambda f(u), \quad (53)$$

then the linearized operator (Frechèt derivative) is

$$F_u(\lambda, u) w = \Delta w + \lambda f'(u) w. \quad (54)$$

From the maximum principle, all solutions of (P_λ) are positive on Ω. Moreover, if (λ^*, u^*) is degenerate solution of (P_λ), then by Theorem 15, the nontrivial solution w of (43) does not change sign in Ω, and hence w can be chosen to be positive. Then by Krein-Rutman's Theorem, $N(F_u(\lambda^*, u^*)) = \operatorname{span}\{w\}$, and it follows from Fredholm alternative theorem that $\operatorname{codim} R(F_u(\lambda^*, u^*)) = 1$. Now we prove that $F_\lambda(\lambda^*, u^*) \notin R(F_u(\lambda^*, u^*))$. If it is not the case, then there exists $v \in C_0^{2,\alpha}(\overline{\Omega})$, such that

$$\Delta v + \lambda^* f'(u^*) v = f(u^*). \quad (55)$$

We also have

$$\Delta w + \lambda^* f'(u^*) w = 0. \quad (56)$$

Multiplying (55) by w, (56) by v, subtracting, and integrating, we obtain

$$\int_\Omega f(u^*) w \, dx = 0, \quad (57)$$

a contradiction. As all the conditions of Crandall-Rabinowitz's bifurcation theorem [20] are satisfied, the solutions of (P_λ) near the degenerate solution (λ^*, u^*) form a smooth curve which is expressed in the form

$$(\lambda(s), u(s)) = (\lambda^* + \tau(s), u_0 + sw + z(s)), \quad (58)$$

where $s \to (\tau(s), z(s)) \in R \times Z$ is a smooth function near $s = 0$ with $\tau(0) = \tau'(0) = 0$, $z(0) = z'(0) = 0$, where Z is a complement of $\operatorname{span}\{w\}$ in X, and w is the positive solution of (43), which is unique if normalized.

Substituting u and λ by expression (58), then differentiating the equation (P_λ) twice, and evaluating at $s = 0$, we have

$$\Delta u_{ss} + \lambda f(u) u_{ss} + 2\lambda' f'(u) u_s + \lambda f''(u) u_s^2 + \lambda'' f(u) = 0,$$
$$\Delta u_{ss} + \lambda^* f'(u) u_{ss} + \lambda^* f''(u) w^2 + \lambda''(0) f(u) = 0. \quad (59)$$

Multiplying (59) by w, (43) by u_{ss}, subtracting, and integrating, we obtain

$$\tau''(0) = -\lambda^* \frac{\int_\Omega f''(u^*) w^3 dx}{\int_\Omega f(u^*) w \, dx} < 0. \quad (60)$$

By (60) and the Taylor expansion formula of $\tau(s)$ at $s = 0$, we conclude that at any degenerate solution (λ^*, u^*) of (P_λ), the solution curve turns left, that is to say, there is no any solution (λ, u) on the right near (λ^*, u^*). This observation is very important to our proof of the following theorem.

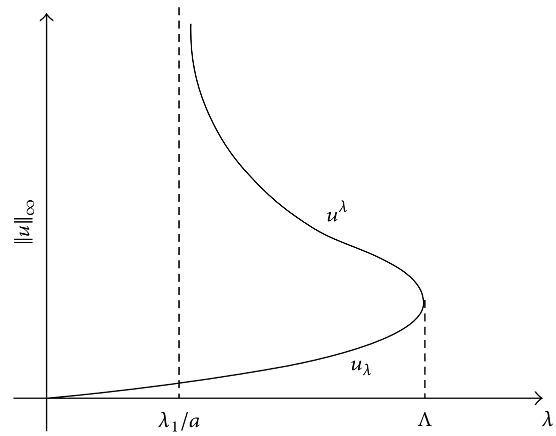

FIGURE 3: Precise bifurcation diagram on a unit ball.

Theorem 16. *Suppose that Ω is the unit ball in R^n, f satisfies (F1)-(F2), and $0 < K_\infty < a\beta$. Then for problem (P_λ),*

(1) *there exist no solutions for $\lambda > \Lambda$,*

(2) *there exists exactly one solution for $\lambda \in (0, \lambda_1/a] \cup \{\Lambda\}$,*

(3) *there exist exactly two solutions for $\lambda \in (\lambda_1/a, \Lambda)$.*

Moreover, the solution set $\{(\lambda, u)\}$ of (P_λ) forms a smooth curve in the space $R \times C(\overline{\Omega})$, which can be roughly described as in Figure 3.

Proof. By Theorem 10, $\Lambda > \lambda_1/a$, and Theorem 7 tells us that (P_λ) has a unique solution (Λ, u_Λ) for $\lambda = \Lambda$, and Implicit Function Theorem implies that (Λ, u_Λ) is a degenerate solution. By Theorem 15, non-trivial solution w of the corresponding linearized equation (43) does not change sign in Ω, and we may suppose that w is positive in Ω. Then Crandall-Rabinowitz's bifurcation theorem [20] and the discussion prior to this theorem imply that the solutions near (Λ, u_Λ) form a smooth curve which turns to the left in the phase space. We may call the part of the smooth solution curve $\{(\lambda, u)\}$ with $u(0) > u_\Lambda(0)$ the upper branch, and the rest the lower branch. We denote the upper branch by u^λ and the lower branch by u_λ.

For the upper branch, as long as (λ, u^λ) nondegenerate, the Implicit Function Theorem ensures that we can continue to extend this solution curve in the direction of decreasing λ. We still denote the extension by (λ, u^λ). This process of continuation towards smaller values of λ will not encounter any other degenerate solutions. This is because, if, say, (λ, u^λ) becomes degenerate at $\lambda = \lambda_0$, the discussion prior to this theorem implies that all the solutions near $(\lambda_0, u^{\lambda_0})$ must lie to the left side of it, which is a contradiction. Lemma 12 tells us that $\lambda \to u^\lambda(0)$ is decreasing. So in the progress of extension of (λ, u^λ) towards smaller values of λ, there are only the following two possibilities.

(i) The upper branch (λ, u^λ) stops at some $(0, u_0)$, and $u_0(0) > u_\Lambda(0)$.

(ii) $\| u_\lambda \|_\infty$ goes to infinity as $\lambda \to \widetilde{\lambda} + 0$, $0 \le \widetilde{\lambda} < \Lambda$.

But case (i) cannot happen, since $(0, u_0)$ is obviously not a solution of (P_λ). Hence case (ii) happens. We assert that $\tilde{\lambda} = \lambda_1/a$. In fact, let $\{\lambda_n\}$ be an arbitrary sequence such that $\lambda_n \rightarrow \tilde{\lambda}$. Denote $M_n = \| u_n\|_\infty$, $v_n = u_n/M_n$, then $M_n \rightarrow \infty$ and

$$\Delta v_n + \lambda_n \frac{f(M_n v_n)}{M_n} = 0, \quad \text{in } \Omega, \qquad v = 0, \quad \text{on } \partial\Omega. \tag{61}$$

Since $f(M_n v_n)/M_n$ is bounded, by Sobolev Imbedding Theorems and standard regularity of elliptic equation, it is easy to see that $\{v_n\}$ has a subsequence, still denoted by $\{v_n\}$, such that $v_n \rightarrow v$ in $C^{2,\alpha}(\Omega)$ $(n \rightarrow \infty)$, for some $v \in C^{2,\alpha}(\Omega)$, $v > 0$ in Ω. Letting $n \rightarrow \infty$ in (61), we get

$$\Delta v + \tilde{\lambda} a v = 0, \quad \text{in } \Omega, \qquad v = 0, \quad \text{on } \partial\Omega, \tag{62}$$

which implies that $\tilde{\lambda} = \lambda_1/a$.

Now we study the structure of the lower branch. As in the case of upper branch, as long as (λ, u_λ) nondegenerate, the Implicit Function Theorem ensures that we can continue to extend this solution curve in the direction of decreasing λ. We still denote the extension by (λ, u_λ). This process of continuation towards smaller values of λ will not encounter any other degenerate solutions. Lemma 12 implies that $\lambda \rightarrow u_\lambda(0)$ is increasing. So in the progress of extension of (λ, u_λ) towards smaller values of λ, there are only the following two possibilities.

(i) The lower branch (λ, u_λ) stops at some $(0, u_0)$ with $u_0(0) > 0$.

(ii) The lower branch (λ, u_λ) stops at some $(\lambda_0, 0)$ with $0 \leq \lambda_0 < \Lambda$.

As before, case (i) will not happen. Then case (ii) happens. By $f(0) > 0$, it is easy to see that $\lambda_0 = 0$. That is to say, the lower branch of solutions extends till the origin $(0, 0)$ in the phase plane.

By the above argument, we obtain a smooth positive solution curve which consists of an upper branch $\{(\lambda, u^\lambda)\}$ and a lower branch $\{(\lambda, u_\lambda)\}$. The lower branch starts from (Λ, u_Λ) and stops at $(0, 0)$, and $\lambda \rightarrow u_\lambda(0)$ is a strictly increasing function. The upper branch $\{(\lambda, u^\lambda)\}$ starts from (Λ, u_Λ) and stops at $(\lambda_1/a, \infty)$, and $\lambda \rightarrow u^\lambda(0)$ is a strictly decreasing function with $u^\lambda(0)$ blowing up as $\lambda \rightarrow \lambda_1/a + 0$. By Lemma 12, all solutions of (P_λ) are contained in this smooth solution curve, and the complete bifurcation diagram can be described as in Figure 3. The proof is complete. □

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