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*Research Article*

# Relative Infinite Determinacy for Map-Germs

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The infinite determinacy for smooth map-germs with respect to two equivalence relations will be investigated. We treat the space of smooth map-germs with a constraint, and the constraint is that a fixed submanifold in the source space is mapped into another fixed submanifold in the target space. We study the infinite determinacy for such map-germs with respect to a subgroup of right-left equivalence group and finite and infinite determinacy with respect to a subgroup of contact group and give necessary and sufficient conditions for the corresponding determinacy.

## 1. Introduction

This work is concerned with the singularity theory of differentiable maps. Singularity theory of differentiable maps is a wide-ranging generalization of the theory of the maxima and minima of functions of one variable, and it is now an essential part of nonlinear analysis. This theory has numerous applications in mathematics and the natural sciences; see, for example, [1, 2].

In differentiable analysis, the local behavior of a differentiable map can be determined by the derivatives of the map at a point. Hence we have the theories of finitely and infinitely determined map-germs. We know that every finitely determined map-germ is equivalent to its Taylor polynomial of some degree, and infinite determinacy is a way to express the stability of smooth map-germs under flat perturbations. The analysis of the conditions for a map-germ to be finitely or infinitely determined involves the most important local aspects of singularity theory. Therefore, the study of finite and infinite determinacy of smooth map-germs is a very important subject in singularity theory.

Now there are several articles treating the question of infinite determinacy with respect to the most frequently encountered and naturally occurring equivalence groups, for instance, the right-equivalence group $\mathscr{R}$, left-equivalence group $\mathscr{L}$, group $\mathscr{C}$, contact group $\mathscr{K}$, and right-left-equivalence group $\mathscr{A}$. In [3, 4], Wilson characterized the infinite determinacy of smooth map-germs with respect to one of the groups $\mathscr{R}$, $\mathscr{L}$, $\mathscr{C}$, and $\mathscr{K}$ and the infinite determinacy for finitely $\mathscr{K}$-determined map-germs with respect to $\mathscr{A}$. Besides, for the case of $\mathscr{A}$, Brodersen [5] showed that the results of [4] hold for map-germs without assuming $\mathscr{K}$-finiteness. We can see [6] for detailed survey of all of these results. Recently, there appeared the notion of relative infinite determinacy for smooth function-germs with nonisolated singularities. Sun and Wilson [7] treated the smooth function-germs with real isolated line singularities, and this work was later generalized and modified in the case where germs have a nonisolated singular set containing a more general set; for instance, see [8, 9]. In addition, [10] studied the relative versality for map-germs and these map-germs with the constraint that a fixed submanifold in the source space is mapped into another fixed submanifold in the target space.

Inspired by the aforementioned papers, we will study the relative infinite determinacy for map-germs with respect to a subgroup of the group $\mathscr{A}$ and relative finite and infinite determinacy with respect to a subgroup of the group $\mathscr{K}$ by means of some algebraic ideas and tools, and our main results extend the part of the works in [3, 4].

Now, we consider the following map-germs.

Let $N$ and $P$ be submanifolds without boundary of $\mathbb{R}^n$ and $\mathbb{R}^p$, respectively, both containing the origin. Since this paper

is concerned with a local study, without loss of generality, we may assume that

$$N = \{0\} \times \mathbb{R}^{n-n_0} \subset \mathbb{R}^n, \quad P = \{0\} \times \mathbb{R}^{p-p_0} \subset \mathbb{R}^p \quad (n_0, p_0 \geqslant 1). \tag{1}$$

Denote by $C_n^p$ the space of map-germs $f : (\mathbb{R}^n, 0) \to \mathbb{R}^p$, with $f(N) \subset P$. Such map-germs are quite common in singularity theory and geometry.

*Example 1.* Let $f : (\mathbb{R}^2, 0) \to \mathbb{R}^3$ be given by (right helicoid)

$$f(u, v) = (u \cos v, u \sin v, cv). \tag{2}$$

It is clear that $f(N) \subset P$ for $N = \{0\} \times \mathbb{R}^1 \subset \mathbb{R}^2$ and $P = \{0\} \times \mathbb{R}^1 \subset \mathbb{R}^3$.

*Example 2.* Let $f : (\mathbb{R}^4, 0) \to \mathbb{R}^4$ be given by

$$f(x_1, x_2, x_3, x_4) = \left(x_1, x_2, x_3^2 + x_1 x_4, x_4^2 + x_2 x_3\right). \tag{3}$$

Then $f(N) \subset P$ for $N = P = \{0\} \times \mathbb{R}^2 \subset \mathbb{R}^4$.

In this paper we want to characterize the relative infinite determinacy for such map-germs. To formulate the main results we need to introduce some notations and definitions.

Let $E_n$ denote the ring of smooth function-germs at the origin in $\mathbb{R}^n$, and let $m_n$ denote its unique maximal ideal. For a germ $f$, let $j^k f(x)$ denote the Taylor expansion of $f$ of order $k$ at $x$. In the case $k = \infty$, $j^\infty f(x)$ can be identified with the Taylor series of $f$ at $x$.

Let $\mathscr{R}$ denote the group of germs at the origin of local diffeomorphisms of $\mathbb{R}^n$, and let

$$\mathscr{R}_N = \{\phi \in \mathscr{R} : \phi|_N \equiv \mathrm{id}_N\}, \tag{4}$$

where id denotes the identity. Then $\mathscr{R}_N$ is a subgroup of $\mathscr{R}$.

Let $C_N$ be the local ring $\{f \in E_n : f|_N \equiv \text{constant}\}$, and let $m_N$ denote the maximal ideal of $C_N$.

Similarly, we can define the corresponding notation for $(\mathbb{R}^p, P)$.

Let $\mathscr{M}_N = \{M \mid M : (\mathbb{R}^n, 0) \to \mathrm{GL}(p, \mathbb{R})$ a $C^\infty$ map-germ and $M|_N = I_p\}$, where $\mathrm{GL}(p, \mathbb{R})$ denotes the general linear group, and $I_p$ denotes the $(p \times p)$ identity matrix.

Now, we define two groups

$$\mathscr{A}_{N,P} = \mathscr{R}_N \times \mathscr{R}_P, \qquad \mathscr{K}_N = \mathscr{M}_N \times \mathscr{R}_N. \tag{5}$$

Obviously, $\mathscr{A}_{N,P}$ and $\mathscr{K}_N$ are subgroups of the groups $\mathscr{A}$ and $\mathscr{K}$, respectively. In particular, when $N = P = \{0\}$, then $\mathscr{A}_{N,P} = \mathscr{A}$. The two groups act on the space $C_n^p$ in the following way.

If $f \in C_n^p$, let $(\phi, \psi) \in \mathscr{A}_{N,P}$ and $(M, h) \in \mathscr{K}_N$; then $(\phi, \psi) \cdot f$ and $(M, h) \cdot f$ are defined by

$$(\phi, \psi) \cdot f = \psi \circ f \circ \phi^{-1}, \qquad (M, h) \cdot f = M \cdot (f \circ h). \tag{6}$$

*Remark 3.* (1) Let $f, g \in C_n^p$. If $f$ and $g$ are $\mathscr{A}_{N,P}$-equivalent or $\mathscr{K}_N$-equivalent, then $f|_N \equiv g|_N$. Set $\mathscr{E}(f, N) = \{g \in C_n^p : f|_N \equiv g|_N\}$.

(2) In general, if $f$ and $g$ are $\mathscr{A}$-equivalent, then $f$ and $g$ also are $\mathscr{K}$-equivalent. However, this result does not hold for $\mathscr{A}_{N,P}$ and $\mathscr{K}_N$. For example, let $f(x_1, x_2) = (x_1, x_2^2)$ and $g(x_1, x_2) = (x_1, x_1 + x_2^2)$; where $N = P = \{0\} \times \mathbb{R}^1 \subset \mathbb{R}^2$. Set $\phi = \mathrm{id}$ and $\psi(y_1, y_2) = (y_1, y_1 + y_2)$, then $(\phi, \psi) \in \mathscr{A}_{N,P}$ and $(\phi, \psi) \cdot f = g$. So $f$ is $\mathscr{A}_{N,P}$-equivalent to $g$. But $f$ and $g$ are not $\mathscr{K}_N$-equivalent.

Let $e_1, \ldots, e_p$ be the canonical basis of the vector space $\mathbb{R}^p$, and they define a system of generators of $C_N$-module

$$(C_N)^{\times p} = C_N \{e_1, \ldots, e_p\}. \tag{7}$$

For any $f \in C_n^p$, then the germ $f$ induces a ring homomorphism

$$f^* : C_P \longrightarrow C_N \tag{8}$$

defined by $f^*(h) = h \circ f$, for any $h \in C_P$.

This allows us to consider every $C_N$-module as an $C_P$-module via $f^*$.

Let $f^* m_P = \langle f_1, \ldots, f_{p_0} \rangle$ be the ideal generated by the components $f_1, \ldots, f_{p_0}$, and let $f^*(m_P)$ denote the image of $m_P$ under $f$, which is not (in general) an ideal of $C_N$.

For a map-germ $f \in C_n^p$, define

$$\begin{aligned} T\mathscr{A}_{N,P} f &= m_N \left\langle \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right\rangle + f^*(C_P) \{e_1, \ldots, e_p\}, \\ T\mathscr{K}_N f &= m_N \left\langle \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right\rangle + f^* m_p \cdot (m_N)^{\times p}. \end{aligned} \tag{9}$$

The notation $T\mathscr{A}_{N,P} f$ and $T\mathscr{K}_N f$ are very nearly the tangent spaces to the orbit of germ $f$ under $\mathscr{A}_{N,P}$-equivalence and $\mathscr{K}_N$-equivalence, respectively.

*Definition 4.* Let $f \in C_n^p$; let $G$ be a group acting on $C_n^p$. We say that the map-germ $f$ is $k$-$G$-determined if for any germ $g \in \mathscr{E}(f, N)$ with $j^k f(0) = j^k g(0)$, $g$ is $G$-equivalent to $f$. If $f$ is $k$-$G$-determined for some $k < \infty$, then it is finitely $G$-determined. If $k = \infty$, we say that $f$ is $\infty$-$G$-determined.

The purpose of this paper is to characterize the $\infty$-$\mathscr{A}_{N,P}$-determinacy and $\infty$-$\mathscr{K}_N$-determinacy for map-germs. The main results in this paper are stated in Theorems 5 and 10.

The rest of this paper is organized as follows. In Section 2, we will give a suffcient condition for $\infty$-$\mathscr{A}_{N,P}$-determined map-germs. In Section 3, we study the $\mathscr{K}_N$-determinacy of map-germs. We will give necessary and sufficient conditions for a map-germ to be finitely $\mathscr{K}_N$-determined or $\infty$-$\mathscr{K}_N$-determined.

Throughout the paper, all map-germs will be assumed smooth.

## 2. The $\infty$-$\mathscr{A}_{N,P}$-Determinacy of Map-Germs

In this section, the main result is the following theorem.

**Theorem 5.** *Let $f \in C_n^p$ be a map-germ. Suppose that $f$ satisfies the following conditions.*

(1) *For some $r < \infty$, $m_n^r m_N \subset m_N J(f) + f^* m_P \cdot m_N$, where $J(f)$ denotes the ideal in $E_n$ generated by the determinants of $(p \times p)$ minors of the Jacobian matrix $df$ of $f$.*

(2) $(m_n^\infty m_N)^{\times p} \subset T\mathscr{A}_{N,P} f$.

*Then $f$ is $\infty$-$\mathscr{A}_{N,P}$-determined.*

In order to prove this theorem, we need the following results.

**Lemma 6** (see [10]). *Let $f \in C_n^p$ and $M$ be a finitely generated $C_N$-module. Then $M$ is finitely generated as a $f^*(C_P)$-module if and only if $\dim_{\mathbb{R}}(M/f^* m_P \cdot M) < \infty$.*

**Lemma 7.** *Let $f, g \in C_n^p$. Let $M$ be a finitely generated $(f,g)^*(C_{P\times P})$-module. Then $M$ is finitely generated as a $g^*(C_P)$-module if and only if*

$$\dim_{\mathbb{R}} \frac{M}{g^* m_P \cdot M} < \infty. \tag{10}$$

*Proof.* The proof is essentially the same as that of Corollary 1.17 in [11]. □

**Lemma 8.** *Let $f, g \in C_n^p$ and $g - f \in (m_n^\infty m_N)^{\times p}$. Then*

$$(f,g)^*(C_{P\times P}) \subset f^*(C_P) + m_n^\infty m_N. \tag{11}$$

*Proof.* For any $h \in C_{P\times P}$, then $(f,g)^*(h) - (f,f)^*(h)$ is in $m_n^\infty m_N$. So,

$$(f,g)^*(C_{P\times P}) \subset (f,f)^*(C_{P\times P}) + m_n^\infty m_N. \tag{12}$$

Let $\phi : (\mathbb{R}^p, P) \to (\mathbb{R}^p \times \mathbb{R}^p, P \times P)$ be given by $\phi(y) = (y, y)$.

Since $(f,f)^*(h) = f^*(\phi^*(h))$, it follows that

$$(f,f)^*(C_{P\times P}) \subset f^*(C_P). \tag{13}$$

Thus, (11) holds. □

*Proof of Theorem 5.* Suppose that $\tilde{f} \in \mathscr{E}(f,N)$ and $j^\infty \tilde{f}(0) = j^\infty f(0)$. Let $u = \tilde{f} - f$; then $u \in (m_n^\infty m_N)^{\times p}$. For any $t_0 \in [0,1]$, define

$$g : (\mathbb{R}^n \times \mathbb{R}, (0, t_0)) \longrightarrow \mathbb{R}^p \tag{14}$$

by $g(x,t) = f(x) + tu(x)$ and $g_t(x) = g(x,t)$.

Let $F(x,t) = (f(x), t)$ and $G(x,t) = (g(x,t), t)$ denote the map-germs at $(0, t_0)$; then $F$ (or $G$) induces a ring homomorphism:

$$F^* : C_{P\times\{t_0\}} \longrightarrow C_{N\times\{t_0\}}, \qquad h \longmapsto h \circ F = F^*(h), \tag{15}$$

where $C_{N\times\{t_0\}}$ (resp., $C_{P\times\{t_0\}}$) denotes the ring of function-germs at $(0, t_0)$ which are constant when restricted to $N \times \{t_0\}$ (resp., $P \times \{t_0\}$).

Set $m_N^k C_{N\times\{t_0\}} = m_T^k$ and $m_P^k C_{P\times\{t_0\}} = m_{T'}^k$.

Let $\overline{T}\mathscr{A}_{N,P} F = tF + \omega F$, where $tF$ denotes $m_T\langle \partial f/\partial x_1, \ldots, \partial f/\partial x_n\rangle$ and $\omega F$ denotes $F^*(C_{P\times\{t_0\}})\{e_1, \ldots, e_p\}$.

We are trying to show that $\tilde{f}$ is $\mathscr{A}_{N,P}$-equivalent to $f$. It suffices to show that there exist germs $\phi : (\mathbb{R}^n \times \mathbb{R}, (0, t_0)) \to (\mathbb{R}^n, 0)$ and $\psi : (\mathbb{R}^p \times \mathbb{R}, (0, t_0)) \to (\mathbb{R}^P, 0)$ satisfying the following conditions.

(1) $\phi(x,t) = x$ and $\psi(y,t) = y$, for any $t$ sufficiently close to $t_0$, $x \in N$ and $y \in P$.

(2) $\phi_{t_0} = \mathrm{id}_{\mathbb{R}^n}$ and $\psi_{t_0} = \mathrm{id}_{\mathbb{R}^P}$.

(3) $g_{t_0} = \psi_t \circ g_t \circ \phi_t^{-1}$, for any $t$ sufficiently close to $t_0$.

By the method initiated by Mather in [12], it suffices to show that

$$(m_n^\infty m_T)^{\times p} \subset m_T \left\langle \frac{\partial g}{\partial x_1}, \ldots, \frac{\partial g}{\partial x_n} \right\rangle + G^*(m_{T'}) \{e_1, \ldots, e_p\}. \tag{16}$$

To see this, it remains to show that

(i) $(m_n^\infty m_T)^{\times p} \subset \overline{T}\mathscr{A}_{N,P} F$.

(ii) $\overline{T}\mathscr{A}_{N,P} F = \overline{T}\mathscr{A}_{N,P} G$.

If (i) and (ii) hold, then

$$(m_n^\infty m_T)^{\times p} \subset \overline{T}\mathscr{A}_{N,P} G. \tag{17}$$

Multiplying (17) by $G^* m_{T'}$, we get

$$G^* m_{T'} \cdot (m_n^\infty m_T)^{\times p} \subset m_T \left\langle \frac{\partial g}{\partial x_1}, \ldots, \frac{\partial g}{\partial x_n} \right\rangle + G^*(m_{T'}) \{e_1, \ldots, e_p\}. \tag{18}$$

On the other hand, using condition (1) in Theorem 5, and

$$J(f) \cdot (m_N)^{\times p} \subset m_N \left\langle \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right\rangle, \tag{19}$$

we get

$$(m_n^r m_N)^{\times p} \subset m_N \left\langle \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right\rangle + f^* m_P \cdot (m_N)^{\times p}. \tag{20}$$

Obviously, we have

$$(m_n^\infty m_N)^{\times p} \subset m_N \left\langle \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right\rangle + f^* m_P \cdot (m_N)^{\times p}. \tag{21}$$

Since $g_t - f \in (m_n^\infty m_N)^{\times p}$, for each $t \in \mathbb{R}$, this gives that

$$\begin{aligned} & m_N \left\langle \frac{\partial g}{\partial x_1}, \ldots, \frac{\partial g}{\partial x_n} \right\rangle + g_t^* m_P \cdot (m_N)^{\times p} \\ & \quad \subset m_N \left\langle \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right\rangle + f^* m_P \cdot (m_N)^{\times p}, \\ & m_N \left\langle \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right\rangle + f^* m_P \cdot (m_N)^{\times p} \\ & \quad \subset m_N \left\langle \frac{\partial g}{\partial x_1}, \ldots, \frac{\partial g}{\partial x_n} \right\rangle + g_t^* m_P \cdot (m_N)^{\times p} \\ & \quad + m_n (m_n^\infty m_N)^{\times p}. \end{aligned} \tag{22}$$

Hence, by Nakayama's lemma we have

$$\begin{aligned} & m_N \left\langle \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right\rangle + f^* m_P \cdot (m_N)^{\times p} \\ & \quad = m_N \left\langle \frac{\partial g}{\partial x_1}, \ldots, \frac{\partial g}{\partial x_n} \right\rangle + g_t^* m_P \cdot (m_N)^{\times p}. \end{aligned} \tag{23}$$

From (21), it follows that

$$(m_n^\infty m_N)^{\times p} \subset m_N \left\langle \frac{\partial g}{\partial x_1}, \ldots, \frac{\partial g}{\partial x_n} \right\rangle + g_t^* m_P \cdot (m_N)^{\times p}, \tag{24}$$

for any given $t \in \mathbb{R}$.

Multiplying (24) by $m_n^\infty C_{N\times\{t_0\}}$, we get

$$(m_n^\infty m_T)^{\times p} \subset m_T \left\langle \frac{\partial g}{\partial x_1}, \ldots, \frac{\partial g}{\partial x_n} \right\rangle + G^* m_{T'} \cdot (m_n^\infty m_T)^{\times p}. \tag{25}$$

Thus, (16) follows by substituting (18) in (25).

Now we prove the assertion (i).

By hypothesis, we have

$$m_n^r m_N \subset m_N J(f) + f^* m_P \cdot C_N, \tag{26}$$

for some $r < \infty$.

This means that ideal $m_N J(f) + f^* m_P \cdot C_N$ has finite codimension in $C_N$. Let

$$\frac{C_N}{(m_N J(f) + f^* m_P \cdot C_N)} = \mathbb{R}\{\overline{a}_1, \ldots, \overline{a}_s\}, \tag{27}$$

where $a_i \in C_N$ and $\overline{a}_i$ is the projection of $a_i$ in the quotient space, $i = 1, \ldots, s$.

Now, set $M = C_{N\times\{t_0\}}/m_T J(f)$; then $M$ is a finitely generated $C_{N\times\{t_0\}}$-module. Since $F^* m_{P\times\{t_0\}} \cdot C_{N\times\{t_0\}} = f^* m_P \cdot C_{N\times\{t_0\}} + (t - t_0) C_{N\times\{t_0\}}$;

$$\frac{M}{F^* m_{P\times\{t_0\}} \cdot M} \cong \frac{C_N}{m_N J(f) + f^* m_P \cdot C_N}. \tag{28}$$

Thus, by Lemma 6, $M$ is finitely generated $F^*(C_{P\times\{t_0\}})$-module; that is,

$$M = F^*\left(C_{P\times\{t_0\}}\right)\{a_1, \ldots, a_s\}. \tag{29}$$

In particular,

$$C_{N\times\{t_0\}} = m_T J(f) + F^*\left(C_{P\times\{t_0\}}\right) C_N. \tag{30}$$

Since

$$\begin{aligned} \overline{T}\mathscr{A}_{N,P} F \supset & \left( m_N \left\langle \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right\rangle \right. \\ & \left. + F^*\left(C_{P\times\{t_0\}}\right)\{e_1, \ldots, e_p\} \right) \cdot F^*\left(C_{P\times\{t_0\}}\right) \\ \supset & \, T\mathscr{A}_{N,P} f \cdot F^*\left(C_{P\times\{t_0\}}\right) \\ \supset & \, (m_n^\infty m_N)^{\times p} \cdot F^*\left(C_{P\times\{t_0\}}\right), \end{aligned} \tag{31}$$

and $m_N J(f) \cdot (C_{N\times\{t_0\}})^{\times p} \subset m_T \langle \partial f/\partial x_1, \ldots, \partial f/\partial x_n \rangle$, it follows that

$$\begin{aligned} \overline{T}\mathscr{A}_{N,P} F \supset & (m_n^\infty m_N)^{\times p} \\ & \cdot \left[ m_T J(f) + F^*\left(C_{P\times\{t_0\}}\right) C_N \right] \\ = & (m_n^\infty m_T)^{\times p}. \end{aligned} \tag{32}$$

So (i) holds.

*Proof of Assertion (ii).* Since $g - f \in (m_n^\infty m_T)^{\times p}$, by (i) we get

$$\overline{T}\mathscr{A}_{N,P} G \subseteq \overline{T}\mathscr{A}_{N,P} F = \overline{T}\mathscr{A}_{N,P} G + (m_n^\infty m_T)^{\times p}. \tag{33}$$

Applying (25) and (i), we have

$$\begin{aligned} (m_n^\infty m_T)^{\times p} \subset & \, m_T \left\langle \frac{\partial g}{\partial x_1}, \ldots, \frac{\partial g}{\partial x_n} \right\rangle \\ & + G^* m_{P\times\{t_0\}} \cdot (m_n^\infty m_T)^{\times p} \\ \subset & \, m_T \left\langle \frac{\partial g}{\partial x_1}, \ldots, \frac{\partial g}{\partial x_n} \right\rangle \\ & + G^* m_{P\times\{t_0\}} \cdot \overline{T}\mathscr{A}_{N,P} F. \end{aligned} \tag{34}$$

By (33) and (34), we have

$$\overline{T}\mathscr{A}_{N,P} G \subseteq \overline{T}\mathscr{A}_{N,P} F = \overline{T}\mathscr{A}_{N,P} G + G^* m_{P\times\{t_0\}} \cdot \overline{T}\mathscr{A}_{N,P} F. \tag{35}$$

Set $E = \overline{T}\mathscr{A}_{N,P} F/tG$, and (35) implies

$$E = \frac{\overline{T}\mathscr{A}_{N,P} G}{tG} + G^* m_{P\times\{t_0\}} \cdot E. \tag{36}$$

Now it remains to show that

(a) $E$ is a finitely generated $G^*(C_{P\times\{t_0\}})$-module.

For if this holds, then by Nakayama's lemma,

$$E = \frac{\overline{T}\mathscr{A}_{N,P} G}{tG}. \tag{37}$$

Hence, $\overline{T}\mathscr{A}_{N,P}F = \overline{T}\mathscr{A}_{N,P}G$.

To prove (a), by Lemma 7, it suffices to show that

(b) $E$ is a finitely generated $(F,G)^*(C_{P\times\{t_0\}\times P\times\{t_0\}})$-module.

(c) $\dim_{\mathbb{R}}(E/G^* m_{P\times\{t_0\}}\cdot E) < \infty$.

Set $R = (F,G)^*(C_{P\times\{t_0\}\times P\times\{t_0\}})$. By Lemma 8 and (i), we get

$$R\cdot\overline{T}\mathscr{A}_{N,P}F \subset \left[F^*\left(C_{P\times\{t_0\}}\right) + m_n^{\infty} m_T\right]\cdot\overline{T}\mathscr{A}_{N,P}F \subset \overline{T}\mathscr{A}_{N,P}F. \tag{38}$$

Thus, $\overline{T}\mathscr{A}_{N,P}F$ is a $R$-module. Hence, $E$ is a $R$-module.

Obviously, $\omega F$ is a finitely generated $F^*(C_{P\times\{t_0\}})$-module; hence $\omega F$ is also finitely generated as a $R$-module.

Moreover, since $tF/tG\cap tF$ is a finitely generated $C_{N\times\{t_0\}}$-module and annihilated by $m_T J(g_t)$. Thus, $tF/tG\cap tF$ is a finitely generated $C_{N\times\{t_0\}}/m_T J(g_t)$-module. By the argument as equality (29), it follows that $C_{N\times\{t_0\}}/m_T J(g_t)$ is a finitely generated $G^*(C_{P\times\{t_0\}})$-module. Hence, $tF/tG\cap tF$ is a finitely generated $R$-module.

The earlier argument shows that $E$ is finitely generated as a $R$-module.

Besides, by (35), we have

$$\begin{aligned}
&\dim_{\mathbb{R}}\frac{E}{G^* m_{P\times\{t_0\}}\cdot E}\\
&\quad= \dim_{\mathbb{R}}\frac{\overline{T}\mathscr{A}_{N,P}G + G^* m_{P\times\{t_0\}}\cdot\overline{T}\mathscr{A}_{N,P}F}{tG + G^* m_{P\times\{t_0\}}\left(\overline{T}\mathscr{A}_{N,P}G + \overline{T}\mathscr{A}_{N,P}F\right)}\\
&\quad= \dim_{\mathbb{R}}\frac{\omega G + \left(tG + G^* m_{P\times\{t_0\}}\cdot\overline{T}\mathscr{A}_{N,P}F\right)}{G^* m_{P\times\{t_0\}}\cdot\omega G + \left(tG + G^* m_{P\times\{t_0\}}\cdot\overline{T}\mathscr{A}_{N,P}F\right)}\\
&\quad\le \dim_{\mathbb{R}}\frac{\omega G}{G^* m_{P\times\{t_0\}}\cdot\omega G} < \infty.
\end{aligned} \tag{39}$$

So (b) and (c) hold. This completes the proof. □

## 3. The $\mathscr{K}_N$-Determinacy of Map-Germs

In this section, by a similar way as [13], we will give necessary and sufficient conditions for finitely $\mathscr{K}_N$-determined map-germs and $\infty$-$\mathscr{K}_N$-determined map-germs.

**Theorem 9.** *Suppose that $f \in C_n^p$. The following conditions are equivalent.*

(1) *$f$ is finitely $\mathscr{K}_N$-determined.*

(2) *$(m_n^r m_N)^{\times p} \subset T\mathscr{K}_N f$, for some $r \in \mathbb{N}$.*

*Proof.* Let $J^{r+1}(n,p)$ denote the set of $(r+1)$-jets at 0 of elements in $\mathscr{E}(f,N)$. If $f$ is $r$-$\mathscr{K}_N$-determined, then for any $g \in \mathscr{E}(f,N)$ with the same $r$-jet as $f$, the $\mathscr{K}_N$-orbit of $f$ contains $g$; that is,

$$\begin{aligned}
&\{g\in\mathscr{E}(f,N) : j^r g(0) = j^r f(0)\}\\
&\quad\subset \{g\in\mathscr{E}(f,N) : g \text{ and } f\\
&\qquad\text{are } \mathscr{K}_N\text{-equivalent}\}.
\end{aligned} \tag{40}$$

Taking $(r+1)$-jets on both sides, we have

$$\begin{aligned}
&\left\{j^{r+1}g(0)\in J^{r+1}(n,p) : j^r g(0) = j^r f(0)\right\}\\
&\quad\subset \left\{j^{r+1}g(0)\in J^{r+1}(n,p) : g \text{ and } f\right.\\
&\qquad\left.\text{are } \mathscr{K}_N\text{-equivalent}\right\}.
\end{aligned} \tag{41}$$

Taking tangent spaces at $j^{r+1}f(0)$ on both sides, we have

$$\left(m_n^r m_N\right)^{\times p} \subset T\mathscr{K}_N f + \left(m_n^{r+1} m_N\right)^{\times p}. \tag{42}$$

By Nakayama's lemma, this implies

$$\left(m_n^r m_N\right)^{\times p} \subset T\mathscr{K}_N f. \tag{43}$$

Then (1) implies (2).

Conversely, let $t_0 \in [0,1]$ be fixed and $g \in \mathscr{E}(f,N)$ with $j^{r+1}f(0) = j^{r+1}g(0)$. Define

$$F : \left(\mathbb{R}^n\times\mathbb{R}, (0,t_0)\right) \longrightarrow \mathbb{R}^p \tag{44}$$

by $F(x,t) = f(x) + t(g(x) - f(x))$ and $F_t(x) = F(x,t)$.

We are trying to show that $g$ is $\mathscr{K}_N$-equivalent to $f$. Since $F_0 = f$ and $F_1 = g$, and $[0,1]$ is connected, it suffices to show that $F_t$ is $\mathscr{K}_N$-equivalent to $F_{t_0}$, for all $t$ sufficiently close to $t_0$.

Since $F_{t_0}(x) - f(x) = t_0(g(x) - f(x))$ and $g - f \in (m_n^{r+1} m_N)^{\times p}$, from condition (2), it follows that

$$T\mathscr{K}_N F_{t_0} \subset T\mathscr{K}_N f, \tag{45}$$

and $T\mathscr{K}_N f \subset T\mathscr{K}_N F_{t_0} + m_n\cdot(T\mathscr{K}_N f)$.

Since $T\mathscr{K}_N f$ is a finitely generated $E_n$-module, and $m_n$ is the maximal ideal of $E_n$, by Nakayama's lemma, we get

$$T\mathscr{K}_N F_{t_0} = T\mathscr{K}_N f. \tag{46}$$

Hence $T\mathscr{K}_N F_{t_0}$ also satisfies condition (2). So

$$\left(m_n^r m_N E_{n+1}^*\right)^{\times p} \subset T\mathscr{K}_N F_{t_0}\cdot E_{n+1}^*, \tag{47}$$

where $E_{n+1}^*$ denotes the ring of smooth function-germs in variables $(x,t)$ at the point $(0,t_0)$.

Let $m_{n+1}^*$ denote the maximal ideal of $E_{n+1}^*$.

Let $\overline{T}\mathscr{K}_N F = m_N E_{n+1}^*\langle\partial F/\partial x_1,\ldots,\partial F/\partial x_n\rangle + F^* m_p\cdot(m_N E_{n+1}^*)^{\times p}$. Obviously, $\overline{T}\mathscr{K}_N F$ is a finitely generated $E_{n+1}^*$-module.

By the same argument as (46), we have

$$\overline{T}\mathscr{K}_N F = T\mathscr{K}_N F_{t_0}\cdot E_{n+1}^*. \tag{48}$$

By (47) and (48), we get

$$\left(m_n^r m_N E_{n+1}^*\right)^{\times p} \subset \overline{T}\mathscr{K}_N F. \tag{49}$$

Since $\partial F/\partial t = g - f \in (m_n^{r+1} m_N)^{\times p}$, this means that there exist germs $X_i$ in $m_N E_{n+1}^*$, $i = 1,\ldots,n$, such that

$$(g - f) + \sum_{i=1}^{n} X_i(x,t)\frac{\partial F}{\partial x_i} \in F^* m_p\cdot\left(m_N E_{n+1}^*\right)^{\times p}. \tag{50}$$

Thus, we can find a germ of vector field $X$ in $\mathbb{R}^n \times \mathbb{R}$ of the following form:

$$\frac{\partial}{\partial t} + \sum_{i=1}^{n} X_i(x,t) \frac{\partial}{\partial x_i}, \tag{51}$$

such that $\mathrm{DF}(X) \in F^* m_p \cdot (m_N E_{n+1}^*)^{\times p}$.

That is, we can find a $(p \times p)$ matrix $A(x,t)$ with entries in $m_N E_{n+1}^*$ such that

$$\mathrm{DF}(X) = A(x,t) \cdot F(x,t). \tag{52}$$

By integrating the vector field $X$, we get a one-parameter family of local diffeomorphisms $\phi_t$ in $\mathscr{R}_N$. Thus

$$\begin{aligned} \frac{d}{dt} F(\phi_t(x), t) &= \frac{\partial}{\partial t} F(\phi_t(x), t) \\ &\quad + \sum_{i=1}^{n} X_i(\phi_t(x), t) \frac{\partial F}{\partial x_i}(\phi_t(x), t) \\ &= A(\phi_t(x), t) \cdot F(\phi_t(x), t). \end{aligned} \tag{53}$$

Hence, for fixed $x \in \mathbb{R}^n$, $F(\phi_t(x), t)$ is a solution of the differential equation $\dot{y} = A(\phi_t(x), t) y$ with initial condition $y(x, t_0) = F_{t_0}(x)$.

Since the solution of this differential equation is unique and of the form

$$y(x,t) = B(x,t) \cdot y(x, t_0), \tag{54}$$

with $B(x,t)$ as an invertible matrix, $B(x, t_0) = I_p$ and $B(x,t) = I_p$ for each $x \in N$. Thus

$$F_t(x) = B\left(\phi_t^{-1}(x), t\right) \cdot \left(F_{t_0} \circ \phi_t^{-1}(x)\right). \tag{55}$$

Thus, $F_t$ and $F_{t_0}$ are $\mathscr{K}_N$-equivalent for all $t$ sufficiently close to $t_0$, which completes the proof. □

**Theorem 10.** *Suppose that $f \in C_n^p$. Then $f$ is $\infty$-$\mathscr{K}_N$-determined if and only if*

$$\left(m_n^{\infty} m_N\right)^{\times p} \subset T\mathscr{K}_N f. \tag{56}$$

*Proof.* "Only if": if $f$ is $\infty$-$\mathscr{K}_N$-determined, by the definition, we get

$$f + \left(m_n^{\infty} m_N\right)^{\times p} \subset \mathscr{K}_N \cdot f. \tag{57}$$

Taking tangent spaces at $f$ on both sides, we have

$$T_f\left(f + \left(m_n^{\infty} m_N\right)^{\times p}\right) \subset T_f\left(\mathscr{K}_N \cdot f\right). \tag{58}$$

Note that $(m_n^{\infty} m_N)^{\times p} \subset T_f(f + (m_n^{\infty} m_N)^{\times p})$, and $T_f(\mathscr{K}_N \cdot f) \subset T\mathscr{K}_N f$. Hence,

$$\left(m_n^{\infty} m_N\right)^{\times p} \subset T\mathscr{K}_N f. \tag{59}$$

"If": the proof is the same as that of Theorem 9. □

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