Hindawi Publishing Corporation
Journal of Function Spaces and Applications
Volume 2013, Article ID 568202, 13 pages
http://dx.doi.org/10.1155/2013/568202

Research Article

Weighted Estimates for Commutators of n-Dimensional Rough Hardy Operators

Zhuanxi Ren and Shuangping Tao

College of Mathematics and Statistics, Northwest Normal University, Lanzhou, Gansu 730070, China

Correspondence should be addressed to Shuangping Tao; taosp@nwnu.edu.cn

Received 3 December 2012; Revised 21 January 2013; Accepted 5 February 2013

Academic Editor: Lars Diening

We establish the weighted estimates for the commutators $H_{\Omega,b}$ and $H^*_{\Omega,b}$ which are generated by the n-dimensional rough Hardy operators and central BMO functions on the weighted Lebesgue spaces, the weighted Herz spaces and the weighted Morrey-Herz spaces. Furthermore, the weighted Lipschitz estimates are also obtained.

1. Introduction

The classical Hardy operator and its adjoint operator are defined, respectively, by

$$hf(x) := \frac{1}{x}\int_0^x f(y)\,dy, \qquad h^* f(x) := \int_x^\infty \frac{f(y)}{y}dy \quad x > 0. \tag{1}$$

Hardy proved the following most celebrated inequality in [1]:

$$\|hf\|_{L^p(\mathbb{R}^+)} \le \frac{p}{p-1}\|f\|_{L^p(\mathbb{R}^+)}, \qquad \|h^* f\|_{L^q(\mathbb{R}^+)} \le \frac{p}{p-1}\|f\|_{L^q(\mathbb{R}^+)}, \quad 1 < p < \infty. \tag{2}$$

Moreover,

$$\|h\|_{L^p(\mathbb{R}^+)\to L^p(\mathbb{R}^+)} = \|h^*\|_{L^q(\mathbb{R}^+)\to L^q(\mathbb{R}^+)} = \frac{p}{p-1}, \tag{3}$$

where $1/p + 1/q = 1$. Hardy's inequality has received considerable attention. In 1995, Christ and Grafakos obtained it on $\mathbb{R}^n$. Firstly, we recall that the definitions of the n-dimensional Hardy operator and its adjoint operator given by Christ and Grafakos are as follows:

$$Hf(x) = \frac{1}{|x|^n}\int_{|y|<|x|} f(y)\,dy, \quad x \in \mathbb{R}^n \setminus \{0\}, \qquad H^* f(x) = \int_{|y|\ge|x|} \frac{f(y)}{|y|^n}dy. \tag{4}$$

Let $1 < p, q < \infty$. Then Christ and Grafakos's results in [2] are

$$\|Hf\|_{L^p(\mathbb{R}^n)} \le \frac{p\nu_n}{p-1}\|f\|_{L^p(\mathbb{R}^n)}, \qquad \|H^* f\|_{L^q(\mathbb{R}^n)} \le \frac{p\nu_n}{p-1}\|f\|_{L^q(\mathbb{R}^n)}, \tag{5}$$

where $\nu_n = \pi^{n/2}/\Gamma(1+n/2)$ and the constant $p\nu_n/(p-1)$ is the best.

Let f be a nonnegative integrable function on $\mathbb{R}^n$. The n-dimensional rough Hardy operator and its adjoint operator are defined, respectively, by

$$H_\Omega f(x) := \frac{1}{|x|^n}\int_{|y|<|x|} \Omega(x-y) f(y)\,dy, \quad x \in \mathbb{R}^n \setminus \{0\}, \tag{6}$$

$$H^*_\Omega f(x) := \int_{|y|\ge|x|} \Omega(x-y)\frac{f(y)}{|y|^n}dy, \tag{7}$$

where $\Omega \in L^s(S^{n-1})$, $1 < s \le \infty$, is homogeneous of degree zero. Then the commutators generated by H_Ω or H_Ω^* and a locally integrable function b are defined, respectively, as follows:

$$H_\Omega^b f(x) := \frac{1}{|x|^n} \int_{|y|<|x|} (b(x) - b(y)) \Omega(x-y) f(y)\, dy, \quad x \in \mathbb{R}^n \setminus \{0\}, \tag{8}$$

$$H_\Omega^{*,b} f(x) := \int_{|y|\ge|x|} (b(x) - b(y)) \Omega(x-y) \frac{f(y)}{|y|^n} dy. \tag{9}$$

In [3], it was proved that the commutators H_Ω^b were bounded on the Lebesgue spaces and Herz spaces if $b \in C\dot{M}O^{\max\{q,u\}}(\mathbb{R}^n)$. Recently, Gao obtained in [4] that H_Ω^b is also bounded from the Morrey-Herz spaces $M\dot{K}_{p_1,q_1}^{\alpha,\lambda}(\mathbb{R}^n)$ to $M\dot{K}_{p_2,q_2}^{\alpha,\lambda}(\mathbb{R}^n)$ if $b \in C\dot{M}O^{\max\{q_2,u\}}(\mathbb{R}^n)$. On the other hand, Gao and Wang in [5] had established the weighted estimates on weighted Lebesgue and Herz-type spaces for the commutators H_b and H_b^* which are generated by n-dimensional Hardy operator H and H^* and $b \in Lip_{\beta,\omega}$. It is easy to see that $H_\Omega^b = H_b$ and $H_\Omega^{*,b} = H_b^*$ when $\Omega = 1$. A natural question is whether commutators of n-dimensional standard rough Hardy operators H_Ω^b and $H_\Omega^{*,b}$ also have boundedness on these weighted spaces. The answer is affirmative. The main purpose of this paper is to generalize the above results on the weighted Lebesgue spaces, the weighted Herz spaces, and the weighted Morrey-Herz spaces.

First let us recall some standard definitions and notations before introducing our main results. The classical A_p weighted theory was first introduced by Muckenhoupt in the study of weighted L^p boundedness of the Hardy-Littlewood maximal functions in [6]. A weight ω is a locally integrable function on $\mathbb{R}^n$ which takes values in $(0,\infty)$ at almost everywhere. Let $B = B(y,r)$ denote the ball with the center y and radius r. C is a constant which may vary from line to line. For $k \in \mathbb{Z}$, let $B_k = \{x \in \mathbb{R}^n : |x| \le 2^k\}$ and $\Delta_k = B_k \setminus B_{k-1}$. We use χ_k to denote the characteristic functions of the set Δ_k. We also denote the weighted measure of E by $\omega(E)$; that is, $\omega(E) = \int_E \omega(x)dx$. And let q' be the conjugate index of q whenever $q \ge 1$, $1/q + 1/q' = 1$.

Definition 1. We say that $\omega \in A_p$, $1 < p < \infty$, if

$$\left(\frac{1}{|B|}\int_B \omega(x)\, dx\right)\left(\frac{1}{|B|}\int_B \omega(x)^{-1/(p-1)} dx\right)^{p-1} \le C \quad \text{for every ball } B \subset \mathbb{R}^n, \tag{10}$$

where C is a positive constant which is independent of the choice of B.

For the case $p = 1$, $\omega \in A_1$, if

$$\frac{1}{|B|}\int_B \omega(x)\, dx \le C \cdot \operatorname*{ess\,inf}_{x\in B} \omega(x) \quad \text{for every ball } B \subset \mathbb{R}^n. \tag{11}$$

A weight function $\omega \in A_\infty$ if it satisfies the A_p condition for some $1 < p < \infty$.

Definition 2 (see [7]). Let $\alpha \in \mathbb{R}$, $0 < p, q \le \infty$, and ω_1 and ω_2 a weighted function. Then the homogeneous weighted Herz space $\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)$ is defined by

$$\dot{K}_q^{\alpha,p}(\omega_1,\omega_2) = \left\{ f \in L_{\mathrm{loc}}^q(\mathbb{R}^n \setminus \{0\}, \omega_2) : \|f\|_{\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)} < \infty \right\}, \tag{12}$$

where

$$\|f\|_{\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)} = \left(\sum_{k=-\infty}^{\infty} \omega_1(B_k)^{\alpha p/n} \|f\chi_k\|_{L^q(\omega_2)}^p \right)^{1/p}, \tag{13}$$

with the usual modification made when $p = \infty$ or $q = \infty$.

Obviously, when $\omega_1 = \omega_2 = 1$, $\dot{K}_q^{\alpha,p}(\omega_1,\omega_2) = \dot{K}_q^{\alpha,p}(\mathbb{R}^n)$; when $\alpha = 0$, $p = q$, $\dot{K}_q^{\alpha,p}(\omega_1,\omega_2) = L^p(\omega_2)$.

Definition 3 (see [8]). Let $\alpha \in \mathbb{R}$, $\lambda \ge 0$, $0 < p, q \le \infty$, and ω_1 and ω_2 a weighted function. Then the homogeneous weighted Morrey-Herz space $M\dot{K}_{p,q}^{\alpha,\lambda}(\omega_1,\omega_2)$ is defined by

$$M\dot{K}_{p,q}^{\alpha,\lambda}(\omega_1,\omega_2) = \left\{ f \in L_{\mathrm{loc}}^q(\mathbb{R}^n \setminus \{0\}, \omega_2) : \|f\|_{M\dot{K}_{p,q}^{\alpha,\lambda}(\omega_1,\omega_2)} < \infty \right\}, \tag{14}$$

where

$$\|f\|_{M\dot{K}_{p,q}^{\alpha,\lambda}(\omega_1,\omega_2)} = \sup_{k_0\in Z} \omega_1(B_{k_0})^{-\lambda/n} \times \left(\sum_{k=-\infty}^{k_0} \omega_1(B_k)^{\alpha p/n} \|f\chi_k\|_{L^q(\omega_2)}^p \right)^{1/p}, \tag{15}$$

with the usual modification made when $p = \infty$ or $q = \infty$.

Obviously, when $\lambda = 0$, $M\dot{K}_{p,q}^{\alpha,\lambda}(\omega_1,\omega_2) = \dot{K}_q^{\alpha,p}(\omega_1,\omega_2)$; when $\omega_1 = \omega_2 = 1$, $M\dot{K}_{p,q}^{\alpha,\lambda}(\omega_1,\omega_2) = M\dot{K}_{p,q}^{\alpha,\lambda}(\mathbb{R}^n)$.

Definition 4 (see [9]). Let $1 \le p \le \infty$, $0 < \beta < 1$, and ω a weighted function. We say that a locally integrable function f belongs to the weighted Lipschitz space $Lip_{\beta,\omega}^p$ if

$$\sup_{B\subset\mathbb{R}^n} \frac{1}{\omega(B)^{\beta/n}} \left(\frac{1}{\omega(B)} \int_B |f(x) - f_B|^p \omega(x)^{1-p} dx \right)^{1/p} \le C < \infty, \tag{16}$$

where $f_B = (1/|B|)\int_B f(x)dx$.

The smallest bound C satisfying conditions above is then taken to be the norm of f in this space and is denoted by $\|f\|_{Lip^p_{\beta,\omega}}$. We also put $Lip_{\beta,\omega} = Lip^1_{\beta,\omega}$. Obviously, for the case $\omega = 1$, the space $Lip_{\beta,\omega}$ is the classical Lipschitz space Lip_β. If $\omega \in A_1(\mathbb{R}^n)$, then García-Cuerva in [9] proved that the space $Lip^p_{\beta,\omega}$ coincide for any $1 < p \leq \infty$ and the norms of $\|f\|_{Lip^p_{\beta,\omega}}$ are equivalent with respect to different values of p. That is $\|f\|_{Lip^p_{\beta,\omega}} \sim \|f\|_{Lip_{\beta,\omega}}$.

Definition 5 (see [5]). Let $1 \leq p < \infty$, and ω a weighted function. A function $f \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$ is said to belong to the weighted central BMO space $C\dot{M}O^p(\omega)$ if

$$\begin{aligned}\|f\|_{C\dot{M}O^p(\omega)} &= \sup_{r>0}\left(\frac{1}{\omega(B(0,r))}\int_{B(0,r)}\left|f(x)-f_B\right|^p\omega(x)^{1-p}dx\right)^{1/p} \\ &\leq C < \infty,\end{aligned} \tag{17}$$

where $f_B = (1/|B(0,r)|)\int_{B(0,r)} f(x)dx$.

Obviously, $C\dot{M}O^{pq}(\omega) \subseteq C\dot{M}O^p(\omega)$ $(1 \leq p < q < \infty)$. When $\omega = 1$, $C\dot{M}O^p(\omega) = C\dot{M}O^p(\mathbb{R}^n)$.

The organization of this paper is as follows. In Section 2, we shall present our main results. Finally, in Section 3, we shall give the proofs of theorems.

2. Main Results

Now, we present our main results as follows.

Theorem 6. *Let $1 < p, q < \infty$, $\omega \in A_1$, and $\Omega \in L^s(S^{n-1})$ with homogeneous of degree zero for some $1 < s \leq \infty$. Suppose that H^b_Ω and $H^{*,b}_\Omega$ are defined by (9) and $b \in C\dot{M}O^{p\max\{q,q'\}}(\omega)$; then H^b_Ω and $H^{*,b}_\Omega$ are bounded from $L^q(\omega)$ to $L^q(\omega^{1-q})$.*

Theorem 7. *Let $1 < p, q < \infty$, $0 < p_1 \leq p_2 \leq \infty$, $\omega \in A_1$, and $\Omega \in L^s(S^{n-1})$ with homogeneous of degree zero for some $1 < s \leq \infty$. Suppose that $b \in C\dot{M}O^{p\max\{q,q'\}}(\omega)$; then*

(a) *H^b_Ω is bounded from $\dot{K}^{\alpha,p_1}_q(\omega,\omega)$ to $\dot{K}^{\alpha,p_2}_q(\omega,\omega^{1-q})$ if $\alpha < n\delta/q'$;*

(b) *$H^{*,b}_\Omega$ is bounded from $\dot{K}^{\alpha,p_1}_q(\omega,\omega)$ to $\dot{K}^{\alpha,p_2}_q(\omega,\omega^{1-q})$ if $\alpha > -n\delta/q$.*

Remark 8. δ appearing in Theorem 7 and the following theorems is defined by the Lemma 16 in the next section.

Remark 9. Let $\alpha = 0$ and $p_1 = p_2 = q$ in Theorem 7; then Theorem 6 can be obtained.

Theorem 10. *Let $1 < p, q < \infty$, $0 < p_1 \leq p_2 \leq \infty$, $\omega \in A_1$ and $\Omega \in L^s(S^{n-1})$ with homogeneous of degree zero for some $1 < s \leq \infty$. Suppose that $b \in C\dot{M}O^{p\max\{q,q'\}}(\omega)$ and $\lambda \geq 0$, then*

(a) *H^b_Ω is bounded from $M\dot{K}^{\alpha,\lambda}_{p_1,q}(\omega,\omega)$ to $M\dot{K}^{\alpha,\lambda}_{p_2,q}(\omega,\omega^{1-q})$ if $\alpha < n\delta/q' + \lambda$;*

(b) *$H^{*,b}_\Omega$ is bounded from $M\dot{K}^{\alpha,\lambda}_{p_1,q}(\omega,\omega)$ to $M\dot{K}^{\alpha,\lambda}_{p_2,q}(\omega,\omega^{1-q})$ if $\alpha > -n\delta/q + \lambda$.*

Theorem 11. *Let $1 < p, q < \infty$, $1/q = (1/p) - (\beta/n)$, $\omega \in A_1$, and $\Omega \in L^s(S^{n-1})$ with homogeneous of degree zero for some $1 < s \leq \infty$. Suppose that $b \in Lip_{\beta,\omega}$ for $0 < \beta < 1$; then H^b_Ω and $H^{*,b}_\Omega$ are bounded from $L^p(\omega)$ to $L^q(\omega^{1-q})$.*

Theorem 12. *Let $0 < p_1 \leq p_2 < \infty$, $1 < q_1, q_2 < \infty$, $1/q_2 = (1/q_1) - (\beta/n)$, $\omega \in A_1$, and $\Omega \in L^s(S^{n-1})$ with homogeneous of degree zero for some $1 < s \leq \infty$. Suppose that $b \in Lip_{\beta,\omega}$ for $0 < \beta < 1$; then*

(a) *H^b_Ω is bounded from $\dot{K}^{\alpha,p_1}_{q_1}(\omega,\omega)$ to $\dot{K}^{\alpha,p_2}_{q_2}(\omega,\omega^{1-q_2})$ if $\alpha < n\delta/q_1'$;*

(b) *$H^{*,b}_\Omega$ is bounded from $\dot{K}^{\alpha,p_1}_{q_1}(\omega,\omega)$ to $\dot{K}^{\alpha,p_2}_{q_2}(\omega,\omega^{1-q_2})$ if $\alpha > -n\delta/q_2$.*

Remark 13. Let $\alpha = 0$, $p_1 = q_1 = p$, and $p_2 = q_2 = q$ in Theorem 12; then Theorem 11 can be obtained.

Theorem 14. *Let $0 < p_1 \leq p_2 < \infty$, $1 < q_1, q_2 < \infty$, $1/q_2 = (1/q_1) - (\beta/n)$, $\omega \in A_1$, and $\Omega \in L^s(S^{n-1})$ with homogeneous of degree zero for some $1 < s \leq \infty$. Suppose that $b \in Lip_{\beta,\omega}$ for $0 < \beta < 1$ and $\lambda \geq 0$; then*

(a) *H^b_Ω is bounded from $M\dot{K}^{\alpha,\lambda}_{p_1,q_1}(\omega,\omega)$ to $M\dot{K}^{\alpha,\lambda}_{p_2,q_2}(\omega,\omega^{1-q_2})$ if $\alpha < n\delta/q_1' + \lambda$;*

(b) *$H^{*,b}_\Omega$ is bounded from $M\dot{K}^{\alpha,\lambda}_{p_1,q_1}(\omega,\omega)$ to $M\dot{K}^{\alpha,\lambda}_{p_2,q_2}(\omega,\omega^{1-q_2})$ if $\alpha > -n\delta/q_2 + \lambda$.*

Remark 15. As $\Omega = 1$, our results are consistent with the main results in [5].

3. Proofs of the Main Results

In this section, we shall give the proofs of Theorems 7, 10, 12, and 14. In order to do this, we shall need the following lemmas.

Lemma 16 (see [10]). *Let $\omega \in A_1$. Then there exist constants C_1, C_2, and $0 < \delta < 1$ depending only on A_1-constant of ω, such that for any measurable subset E of a ball B,*

$$C_1\frac{|E|}{|B|} \leq \frac{\omega(E)}{\omega(B)} \leq C_2\left(\frac{|E|}{|B|}\right)^\delta. \tag{18}$$

Remark 17. If $\omega \in A_1$, it is easy to see from Lemma 16 that there exists a constant C and δ $(0 < \delta < 1)$ such that $\omega(B_k)/\omega(B_j) \leq C2^{(k-j)n}$ as $k > j$ and $\omega(B_k)/\omega(B_j) \leq C2^{(k-j)n\delta}$ as $k \leq j$.

Lemma 18 (see [5]). *Let $\omega \in A_1$ and $b \in C\dot{M}O^p(\omega)$. Then there exists a constant C such that, for any $j > k$,*

$$\left|b_{B_j} - b_{B_k}\right| \le C(j-k)\|b\|_{C\dot{M}O^p(\omega)} \frac{\omega(B_k)}{|B_k|}. \quad (19)$$

Lemma 19 (see [11]). *Let $\omega \in A_1$ and $b \in Lip_{\beta,\omega}$. Then there exists a constant C such that, for any $j > k$,*

$$\left|b_{B_j} - b_{B_k}\right| \le C(j-k)\|b\|_{Lip_{\beta,\omega}} \omega\left(B_j\right)^{\beta/n} \frac{\omega(B_k)}{|B_k|}. \quad (20)$$

Lemma 20 (see [11]). *Let $\omega \in A_1$. Then, for any $1 \le p < \infty$,*

$$\int_B \omega(x)^{1-p'} dx \le C|B|^{p'} \omega(B)^{1-p'}, \quad (21)$$

where $1/p + 1/p' = 1$.

We are now in a position to give the proof of Theorem 7.

Proof of Theorem 7. (a) From the definition in (9), we readily see that

$$\begin{aligned}
&\left\|\left(H_\Omega^b f\right)\chi_k\right\|_{L^q(\omega^{1-q})}^q \\
&\quad\le C2^{-kqn}\int_{\Delta k}\left(\sum_{i=-\infty}^{k}\int_{\Delta i}\left|f(y)\Omega(x-y)\right.\right. \\
&\qquad\qquad \left.\left.\times\left(b(x)-b_{B_k}\right)\right|dy\right)^q \\
&\qquad\times \omega(x)^{1-q}dx \\
&\quad + C2^{-kqn}\int_{\Delta k}\left(\sum_{i=-\infty}^{k}\int_{\Delta i}\left|f(y)\Omega(x-y)\right.\right. \\
&\qquad\qquad \left.\left.\times\left(b(y)-b_{B_k}\right)\right|dy\right)^q \\
&\qquad\times \omega(x)^{1-q}dx \\
&\quad = I + II.
\end{aligned} \quad (22)$$

For I, noticing that $\omega \in A_1 \subset A_q$ and then by Lemma 20 and Hölder's inequality it follows that

$$\begin{aligned}
&\int_{\Delta i}|f(y)|dy \\
&\quad\le \left(\int_{\Delta i}|f(y)|^q\omega(y)dy\right)^{1/q}\left(\int_{\Delta i}\omega(y)^{1-q'}dy\right)^{1/q'} \\
&\quad\le \left\|f\chi_i\right\|_{L^q(\omega)}\left|B_i\right|\omega(B_i)^{-1/q}.
\end{aligned} \quad (23)$$

When $\Omega \in L^\infty(S^{n-1})$, by inequality (23) and Remark 17, we have

$$\begin{aligned}
I &\le C2^{-kqn}\int_{\Delta k}\left|b(x)-b_{B_k}\right|^q\omega(x)^{1-q} \\
&\qquad\times\left(\sum_{i=-\infty}^{k}\int_{\Delta i}\left|f(y)\Omega(x-y)\right|dy\right)^q dx \\
&\le C2^{-kqn}\|b\|_{C\dot{M}O^q(\omega)}^q\omega(B_k) \\
&\qquad\times\left(\sum_{i=-\infty}^{k}\|\Omega\|_{L^\infty(S^{n-1})}\int_{\Delta i}|f(y)|dy\right)^q \\
&\le C\|\Omega\|_{L^\infty(S^{n-1})}^q 2^{-kqn}\|b\|_{C\dot{M}O^q(\omega)}^q\omega(B_k) \\
&\qquad\times\left(\sum_{i=-\infty}^{k}\left\|f\chi_i\right\|_{L^q(\omega)}\left|B_i\right|\omega(B_i)^{-1/q}\right)^q \\
&= C\|\Omega\|_{L^\infty(S^{n-1})}^q\|b\|_{C\dot{M}O^q(\omega)}^q \\
&\qquad\times\left(\sum_{i=-\infty}^{k}2^{(i-k)n}\left(\frac{\omega(B_k)}{\omega(B_i)}\right)^{1/q}\left\|f\chi_i\right\|_{L^q(\omega)}\right)^q \\
&\le C\|\Omega\|_{L^\infty(S^{n-1})}^q\|b\|_{C\dot{M}O^q(\omega)}^q \\
&\qquad\times\left(\sum_{i=-\infty}^{k}2^{(i-k)n/q'}\left\|f\chi_i\right\|_{L^q(\omega)}\right)^q.
\end{aligned} \quad (24)$$

When $\Omega \in L^s(S^{n-1})$, $1 < s < \infty$, also by inequality (23) and Remark 17 we can obtain that

$$\begin{aligned}
I &\le C2^{-kqn}\int_{\Delta k}\left|b(x)-b_{B_k}\right|^q\omega(x)^{1-q} \\
&\qquad\times\left(\sum_{i=-\infty}^{k}\int_{\Delta i}\left|f(y)\Omega(x-y)\right|dy\right)^q dx \\
&\le C2^{-kqn}\|b\|_{C\dot{M}O^q(\omega)}^q\omega(B_k) \\
&\qquad\times\left(\sum_{i=-\infty}^{k}\left(\int_{\Delta i}|f(y)|^{s'}dy\right)^{1/s'}\right. \\
&\qquad\qquad\left.\times\left(\int_{\Delta i}|\Omega(x-y)|^s dy\right)^{1/s}\right)^q \\
&\le C2^{-kqn}\|b\|_{C\dot{M}O^q(\omega)}^q\omega(B_k) \\
&\qquad\times\left(\sum_{i=-\infty}^{k}\|\Omega\|_{L^s(S^{n-1})}\int_{\Delta i}|f(y)|dy\right)^q \\
&\le C\|\Omega\|_{L^s(S^{n-1})}^q\|b\|_{C\dot{M}O^q(\omega)}^q
\end{aligned}$$

$$\times\left(\sum_{i=-\infty}^{k}2^{-kn}\omega(B_k)^{1/q}\|f\chi_i\|_{L^q(\omega)}|B_i|\,\omega(B_i)^{-1/q}\right)^q$$
$$=C\|\Omega\|^q_{L^s(S^{n-1})}\|b\|^q_{C\dot{M}O^q(\omega)}$$
$$\times\left(\sum_{i=-\infty}^{k}2^{(i-k)n}\left(\frac{\omega(B_k)}{\omega(B_i)}\right)^{1/q}\|f\chi_i\|_{L^q(\omega)}\right)^q$$
$$\le C\|\Omega\|^q_{L^s(S^{n-1})}\|b\|^q_{C\dot{M}O^q(\omega)}$$
$$\times\left(\sum_{i=-\infty}^{k}2^{(i-k)n/q'}\|f\chi_i\|_{L^q(\omega)}\right)^q. \tag{25}$$

Hence, for $1<s\le\infty$, applying inequalities (24) and (25), we have

$$I\le C\|b\|^q_{C\dot{M}O^q(\omega)}\left(\sum_{i=-\infty}^{k}2^{(i-k)n/q'}\|f\chi_i\|_{L^q(\omega)}\right)^q. \tag{26}$$

Now we turn our attention to the estimate of II. According to the conditions, we can further decompose II as

$$II\le C2^{-kqn}$$
$$\times\int_{\Delta k}\left(\sum_{i=-\infty}^{k}\int_{\Delta i}\left|f(y)\,\Omega(x-y)\left(b(y)-b_{B_i}\right)\right|dy\right)^q$$
$$\times\,\omega(x)^{1-q}dx$$
$$+C2^{-kqn}\int_{\Delta k}\left(\sum_{i=-\infty}^{k}\int_{\Delta i}\left|f(y)\,\Omega(x-y)\right.\right.$$
$$\left.\left.\times\left(b_{B_k}-b_{B_i}\right)\right|dy\right)^q$$
$$\times\,\omega(x)^{1-q}dx$$
$$=II_1+II_2. \tag{27}$$

Below we shall give the estimates of II_1. By Hölder's inequality, we can deduce

$$\int_{\Delta i}\left|f(y)\left(b(y)-b_{B_i}\right)\right|dy$$
$$\le\left(\int_{\Delta i}|f(y)|^q\omega(y)\,dy\right)^{1/q}$$
$$\times\left(\int_{\Delta i}\left|b(y)-b_{B_i}\right|^{q'}\omega(y)^{-q'/q}dy\right)^{1/q'} \tag{28}$$
$$\le\|f\chi_i\|_{L^q(\omega)}\|b\|_{C\dot{M}O^{q'}(\omega)}\omega(B_i)^{1/q'}.$$

When $\Omega\in L^\infty(S^{n-1})$, using Lemma 20, we can obtain the following result by inequality (28) and Remark 17 that

$$II_1\le C2^{-kqn}\|\Omega\|^q_{L^\infty(S^{n-1})}$$
$$\times\int_{\Delta k}\left(\sum_{i=-\infty}^{k}\int_{\Delta i}\left|f(y)\left(b(y)-b_{B_i}\right)\right|dy\right)^q\omega(x)^{1-q}dx$$
$$\le C\|\Omega\|^q_{L^\infty(S^{n-1})}2^{-kqn}|B_k|^q\omega(B_k)^{1-q}$$
$$\times\left(\sum_{i=-\infty}^{k}\|f\chi_i\|_{L^q(\omega)}\|b\|_{C\dot{M}O^{q'}(\omega)}\omega(B_i)^{1/q'}\right)^q$$
$$=C\|\Omega\|^q_{L^\infty(S^{n-1})}\|b\|^q_{C\dot{M}O^{q'}(\omega)}$$
$$\times\left(\sum_{i=-\infty}^{k}\left(\frac{\omega(B_i)}{\omega(B_k)}\right)^{1-1/q}\|f\chi_i\|_{L^q(\omega)}\right)^q$$
$$\le C\|\Omega\|^q_{L^\infty(S^{n-1})}\|b\|^q_{C\dot{M}O^{q'}(\omega)}$$
$$\times\left(\sum_{i=-\infty}^{k}2^{(i-k)n\delta/q'}\|f\chi_i\|_{L^q(\omega)}\right)^q. \tag{29}$$

When $\Omega\in L^s(S^{n-1})$, $1<s<\infty$, noticing inequality (28) and then by using Hölder's inequality, Lemma 20, and Remark 17, we can obtain that

$$II_1\le C2^{-kqn}$$
$$\times\int_{\Delta k}\left(\sum_{i=-\infty}^{k}\left(\int_{\Delta i}\left|f(y)\left(b(y)-b_{B_i}\right)\right|^{s'}dy\right)^{1/s'}\right.$$
$$\left.\times\left(\int_{\Delta i}|\Omega(x-y)|^s dy\right)^{1/s}\right)^q\omega(x)^{1-q}dx$$
$$\le C2^{-kqn}\int_{\Delta k}\omega(x)^{1-q}dx$$
$$\times\left(\sum_{i=-\infty}^{k}\|\Omega\|_{L^s(S^{n-1})}\right.$$
$$\left.\times\int_{\Delta i}\left|f(y)(b(y)-b_{B_i})\right|dy\right)^q$$
$$\le C2^{-kqn}|B_k|^q\omega(B_k)^{1-q}$$
$$\times\left(\sum_{i=-\infty}^{k}\|\Omega\|_{L^s(S^{n-1})}\int_{\Delta i}\left|f(y)\left(b(y)-b_{B_i}\right)\right|dy\right)^q$$
$$\le C\|\Omega\|^q_{L^s(S^{n-1})}2^{-kqn}|B_k|^q\omega(B_k)^{1-q}$$
$$\times\left(\sum_{i=-\infty}^{k}\|f\chi_i\|_{L^q(\omega)}\|b\|_{C\dot{M}O^{q'}(\omega)}\omega(B_i)^{1/q'}\right)^q$$

$$
\begin{aligned}
&\le C\|\Omega\|^q_{L^s(S^{n-1})}\|b\|^q_{C\dot{M}O^{q'}(\omega)} \\
&\quad\times\left(\sum_{i=-\infty}^{k}\left(\frac{\omega(B_i)}{\omega(B_k)}\right)^{1-1/q}\|f\chi_i\|_{L^q(\omega)}\right)^q \\
&\le C\|\Omega\|^q_{L^s(S^{n-1})}\|b\|^q_{C\dot{M}O^{q'}(\omega)} \\
&\quad\times\left(\sum_{i=-\infty}^{k}2^{(i-k)n\delta/q'}\|f\chi_i\|_{L^q(\omega)}\right)^q .
\end{aligned}
\tag{30}
$$

Hence, for $1 < s \le \infty$, by estimates (29) and (30), we get that

$$
II_1 \le C\|b\|^q_{C\dot{M}O^{q'}(\omega)}\left(\sum_{i=-\infty}^{k}2^{(i-k)n\delta/q'}\|f\chi_i\|_{L^q(\omega)}\right)^q . \tag{31}
$$

Now, let us deal with the last term II_2. When $\Omega \in L^\infty(S^{n-1})$, noting inequality (23), by using Lemma 18, Lemma 20, and Remark 17, we have that

$$
\begin{aligned}
II_2 &\le C2^{-kqn} \\
&\quad\times\int_{\Delta k}\left(\sum_{i=-\infty}^{k}\int_{\Delta i}\left|f(y)\,\Omega(x-y)(k-i)\right.\right. \\
&\qquad\left.\left.\times\ \|b\|_{C\dot{M}O^p(\omega)}\frac{\omega(B_i)}{|B_i|}\right|dy\right)^q \\
&\quad\times\omega(x)^{1-q}dx \\
&\le C2^{-kqn}\|b\|^q_{C\dot{M}O^p(\omega)}|B_k|^q\omega(B_k)^{1-q} \\
&\quad\times\left(\sum_{i=-\infty}^{k}(k-i)\|\Omega\|_{L^\infty(S^{n-1})}\frac{\omega(B_i)}{|B_i|}\int_{\Delta i}|f(y)|\,dy\right)^q \\
&\le C\|\Omega\|^q_{L^\infty(S^{n-1})}\|b\|^q_{C\dot{M}O^p(\omega)}\omega(B_k)^{1-q} \\
&\quad\times\left(\sum_{i=-\infty}^{k}(k-i)\frac{\omega(B_i)}{|B_i|}\|f\chi_i\|_{L^q(\omega)}|B_i|\,\omega(B_i)^{-1/q}\right)^q \\
&= C\|\Omega\|^q_{L^\infty(S^{n-1})}\|b\|^q_{C\dot{M}O^p(\omega)} \\
&\quad\times\left(\sum_{i=-\infty}^{k}(k-i)\left(\frac{\omega(B_i)}{\omega(B_k)}\right)^{1-1/q}\|f\chi_i\|_{L^q(\omega)}\right)^q \\
&\le C\|\Omega\|^q_{L^\infty(S^{n-1})}\|b\|^q_{C\dot{M}O^p(\omega)} \\
&\quad\times\left(\sum_{i=-\infty}^{k}(k-i)\,2^{(i-k)n\delta/q'}\|f\chi_i\|_{L^q(\omega)}\right)^q .
\end{aligned}
\tag{32}
$$

When $\Omega \in L^s(S^{n-1}), 1 < s < \infty$, by using Lemma 18, inequalities (21) and (23), Remark 17, and an application of Hölder's inequality give us that

$$
\begin{aligned}
II_2 &\le C2^{-kqn} \\
&\quad\times\int_{\Delta k}\left(\sum_{i=-\infty}^{k}\int_{\Delta i}\left|f(y)\,\Omega(x-y)\right.\right. \\
&\qquad\left.\left.\times\ (k-i)\,\|b\|_{C\dot{M}O^p(\omega)}\frac{\omega(B_i)}{|B_i|}\right|dy\right)^q \\
&\quad\times\omega(x)^{1-q}dx \\
&\le C2^{-kqn}\|b\|^q_{C\dot{M}O^p(\omega)} \\
&\quad\times\int_{\Delta k}\left(\sum_{i=-\infty}^{k}(k-i)\,\frac{\omega(B_i)}{|B_i|}\left(\int_{\Delta i}|f(y)|^{s'}dy\right)^{1/s'}\right. \\
&\qquad\left.\times\left(\int_{\Delta i}|\Omega(x-y)|^s dy\right)^{1/s}\right)^q \\
&\quad\times\omega(x)^{1-q}dx \\
&\le C2^{-kqn}\|b\|^q_{C\dot{M}O^p(\omega)}|B_k|^q\omega(B_k)^{1-q} \\
&\quad\times\left(\sum_{i=-\infty}^{k}(k-i)\|\Omega\|_{L^s(S^{n-1})}\frac{\omega(B_i)}{|B_i|}\int_{\Delta i}|f(y)|\,dy\right)^q \\
&\le C\|\Omega\|^q_{L^q(S^{n-1})}\|b\|^q_{C\dot{M}O^p(\omega)}\omega(B_k)^{1-q} \\
&\quad\times\left(\sum_{i=-\infty}^{k}(k-i)\,\frac{\omega(B_i)}{|B_i|}\|f\chi_i\|_{L^q(\omega)}|B_i|\,\omega(B_i)^{-1/q}\right)^q \\
&= C\,\|\Omega\|^q_{L^s(S^{n-1})}\|b\|^q_{C\dot{M}O^p(\omega)} \\
&\quad\times\left(\sum_{i=-\infty}^{k}(k-i)\left(\frac{\omega(B_i)}{\omega(B_k)}\right)^{1-1/q}\|f\chi_i\|_{L^q(\omega)}\right)^q \\
&\le C\|\Omega\|^q_{L^s(S^{n-1})}\|b\|^q_{C\dot{M}O^p(\omega)} \\
&\quad\times\left(\sum_{i=-\infty}^{k}(k-i)2^{(i-k)n\delta/q'}\|f\chi_i\|_{L^q(\omega)}\right)^q .
\end{aligned}
\tag{33}
$$

As before, for $1 < s \le \infty$, using inequalities (32) and (33), we can get

$$
II_2 \le C\|b\|^q_{C\dot{M}O^p(\omega)}\left(\sum_{i=-\infty}^{k}(k-i)\,2^{(i-k)n\delta/q'}\|f\chi_i\|_{L^q(\omega)}\right)^q . \tag{34}
$$

Summing up the above estimates for I, II_1, and II_2, it is true that

$$\begin{aligned}
&\left\|H_\Omega^b f\right\|_{\dot K_q^{\alpha,p_2}(\omega,\omega^{1-q})}\\
&\leq\left(\sum_{k=-\infty}^{\infty}\omega(B_k)^{\alpha p_1/n}\left\|\left(H_\Omega^b f\right)\chi_k\right\|_{L^q(\omega^{1-q})}^{p_1}\right)^{1/p_1}\\
&\leq C\|b\|_{C\dot M O^q(\omega)}\\
&\quad\times\left(\sum_{k=-\infty}^{\infty}\omega(B_k)^{\alpha p_1/n}\right.\\
&\qquad\left.\times\left(\sum_{i=-\infty}^{k}2^{(i-k)n/q'}\|f\chi_i\|_{L^q(\omega)}\right)^{p_1}\right)^{1/p_1}\\
&\quad+C\|b\|_{C\dot M O^{q'}(\omega)}\\
&\quad\times\left(\sum_{k=-\infty}^{\infty}\omega(B_k)^{\alpha p_1/n}\right.\\
&\qquad\left.\times\left(\sum_{i=-\infty}^{k}2^{(i-k)n\delta/q'}\|f\chi_i\|_{L^q(\omega)}\right)^{p_1}\right)^{1/p_1}\\
&\quad+C\|b\|_{C\dot M O^p(\omega)}\\
&\quad\times\left(\sum_{k=-\infty}^{\infty}\omega(B_k)^{\alpha p_1/n}\right.\\
&\qquad\left.\times\left(\sum_{i=-\infty}^{k}(k-i)\,2^{(i-k)n\delta/q'}\|f\chi_i\|_{L^q(\omega)}\right)^{p_1}\right)^{1/p_1}\\
&:=S.
\end{aligned}\tag{35}$$

Consequently,

$$\begin{aligned}
S^{p_1}&\leq C\|b\|_{C\dot M O^{p\max\{q,q'\}}(\omega)}^{p_1}\sum_{k=-\infty}^{\infty}\omega(B_k)^{\alpha p_1/n}\\
&\quad\times\left(\sum_{i=-\infty}^{k}(k-i)\,2^{(i-k)n\delta/q'}\|f\chi_i\|_{L^q(\omega)}\right)^{p_1}\\
&\leq C\|b\|_{C\dot M O^{p\max\{q,q'\}}(\omega)}^{p_1}\\
&\quad\times\sum_{k=-\infty}^{\infty}\left(\sum^{k}(k-i)\,\omega(B_i)^{\alpha/n}2^{(i-k)(n\delta/q'-\alpha)}\right.\\
&\qquad\left.\times\|f\chi_i\|_{L^q(\omega)}\right)^{p_1}.
\end{aligned}\tag{36}$$

We shall consider two cases. As $0<p_1\leq 1$, noticing that $\alpha<n\delta/q'$, we can deduce

$$\begin{aligned}
S^{p_1}&\leq C\|b\|_{C\dot M O^{p\max\{q,q'\}}(\omega)}^{p_1}\\
&\quad\times\sum_{k=-\infty}^{\infty}\sum_{i=-\infty}^{k}(k-i)^{p_1}\omega(B_i)^{\alpha p_1/n}2^{(i-k)(n\delta/q'-\alpha)p_1}\\
&\qquad\times\|f\chi_i\|_{L^q(\omega)}^{p_1}\\
&=C\|b\|_{C\dot M O^{p\max\{q,q'\}}(\omega)}^{p_1}\\
&\quad\times\sum_{i=-\infty}^{\infty}\omega(B_i)^{\alpha p_1/n}\|f\chi_i\|_{L^q(\omega)}^{p_1}\\
&\quad\times\sum_{k=i}^{\infty}(k-i)^{p_1}2^{(i-k)(n\delta/q'-\alpha)p_1}\\
&=C\|b\|_{C\dot M O^{p\max\{q,q'\}}(\omega)}^{p_1}\|f\|_{\dot K_q^{\alpha,p_1}(\omega,\omega)}^{p_1}.
\end{aligned}\tag{37}$$

For the case of $1<p_1<\infty$, it follows from Hölder's inequality and $\alpha<n\delta/q'$ that

$$\begin{aligned}
S^{p_1}&\leq C\|b\|_{C\dot M O^{p\max\{q,q'\}}(\omega)}^{p_1}\\
&\quad\times\sum_{k=-\infty}^{\infty}\sum_{i=-\infty}^{k}\omega(B_i)^{\alpha p_1/n}2^{(i-k)(n\delta/q'-\alpha)p_1/2}\\
&\qquad\times\|f\chi_i\|_{L^q(\omega)}^{p_1}\\
&\qquad\times\left(\sum_{i=-\infty}^{k}(k-i)^{p_1'}2^{(i-k)(n\delta/q'-\alpha)p_1'/2}\right)^{p_1/p_1'}\\
&=C\|b\|_{C\dot M O^{p\max\{q,q'\}}(\omega)}^{p_1}\\
&\quad\times\sum_{i=-\infty}^{\infty}\omega(B_i)^{\alpha p_1/n}\|f\chi_i\|_{L^q(\omega)}^{p_1}\sum_{k=i}^{\infty}2^{(i-k)(n\delta/q'-\alpha)p_1/2}\\
&=C\,\|b\|_{C\dot M O^{p\max\{q,q'\}}(\omega)}^{p_1}\|f\|_{\dot K_q^{\alpha,p_1}(\omega,\omega)}^{p_1}.
\end{aligned}\tag{38}$$

Therefore, we conclude the proof of Theorem 7(a).

(b) Following the same procedure as that of Theorem 7(a), we can also show the conclusion of (b); the details are omitted here. □

Proof of Theorem 10. We only give the proof of (a) as $\lambda>0$. The proof of (b) is similar to that for (a). By using (15) and the above estimates for I, II_1, and II_2 as those of the proof of Theorem 7, we can obtain that

$$\begin{aligned}
&\left\|H^b_\Omega f\right\|_{M\dot{K}^{\alpha,\lambda}_{p_2,q}(\omega,\omega^{1-q})}\\
&\quad\le C\|b\|_{C\dot{M}O^{q}(\omega)}\sup_{k_0\in Z}\omega\left(B_{k_0}\right)^{-\lambda/n}\\
&\quad\quad\times\left(\sum_{k=-\infty}^{k_0}\omega\left(B_k\right)^{\alpha p_2/n}\right.\\
&\quad\quad\quad\left.\times\left(\sum_{i=-\infty}^{k}2^{(i-k)n/q'}\|f\chi_i\|_{L^q(\omega)}\right)^{p_2}\right)^{1/p_2}\\
&\quad\quad+C\|b\|_{C\dot{M}O^{q'}(\omega)}\sup_{k_0\in Z}\omega\left(B_{k_0}\right)^{-\lambda/n}\\
&\quad\quad\times\left(\sum_{k=-\infty}^{k_0}\omega\left(B_k\right)^{\alpha p_2/n}\right.\\
&\quad\quad\quad\left.\times\left(\sum_{i=-\infty}^{k}2^{(i-k)n\delta/q'}\|f\chi_i\|_{L^q(\omega)}\right)^{p_2}\right)^{1/p_2}\\
&\quad\quad+C\|b\|_{C\dot{M}O^{p}(\omega))}\sup_{k_0\in Z}\omega\left(B_{k_0}\right)^{-\lambda/n}\\
&\quad\quad\times\left(\sum_{k=-\infty}^{k_0}\omega\left(B_k\right)^{\alpha p_2/n}\right.\\
&\quad\quad\quad\left.\times\left(\sum_{i=-\infty}^{k}(k-i)\,2^{(i-k)n\delta/q'}\|f\chi_i\|_{L^q(\omega)}\right)^{p_2}\right)^{1/p_2}\\
&\quad\le C\|b\|_{C\dot{M}O^{p\max\{q,q'\}}(\omega)}\sup_{k_0\in Z}\omega\left(B_{k_0}\right)^{-\lambda/n}\\
&\quad\quad\times\left(\sum_{k=-\infty}^{k_0}\omega\left(B_k\right)^{\alpha p_2/n}\right.\\
&\quad\quad\quad\left.\times\left(\sum_{i=-\infty}^{k}(k-i)2^{(i-k)n\delta/q'}\|f\chi_i\|_{L^q(\omega)}\right)^{p_2}\right)^{1/p_2}\\
&\quad\le C\|b\|_{C\dot{M}O^{p\max\{q,q'\}}(\omega)}\sup_{k_0\in Z}\omega\left(B_{k_0}\right)^{-\lambda/n}\\
&\quad\quad\times\left(\sum_{k=-\infty}^{k_0}\omega\left(B_k\right)^{\lambda p_2/n}\right.\\
&\quad\quad\quad\times\left(\sum_{i=-\infty}^{k}(k-i)\,2^{(i-k)n\delta/q'}\right.\\
&\quad\quad\quad\times\left(\frac{\omega(B_k)}{\omega(B_i)}\right)^{(\alpha-\lambda)/n}\omega(B_i)^{-\lambda/n}\\
&\quad\quad\quad\left.\left.\times\left(\sum_{l=-\infty}^{i}\omega(B_l)^{\alpha p_1/n}\|f\chi_l\|^{p_1}_{L^q(\omega)}\right)^{1/p_1}\right)^{p_2}\right)^{1/p_2}\\
&\quad\le C\|b\|_{C\dot{M}O^{p\max\{q,q'\}}(\omega)}\sup_{k_0\in Z}\omega\left(B_{k_0}\right)^{-\lambda/n}\\
&\quad\quad\times\left(\sum_{k=-\infty}^{k_0}\omega\left(B_k\right)^{\lambda p_2/n}\right.\\
&\quad\quad\quad\times\left(\sum_{i=-\infty}^{k}(k-i)\,2^{(i-k)(n\delta/q'-\alpha+\lambda)}\right.\\
&\quad\quad\quad\left.\left.\times\|f\|_{M\dot{K}^{\alpha,\lambda}_{p_1,q}(\omega,\omega)}\right)^{p_2}\right)^{1/p_2}\\
&\quad= C\|b\|_{C\dot{M}O^{p\max\{q,q'\}}(\omega)}\sup_{k_0\in Z}\omega\left(B_{k_0}\right)^{-\lambda/n}\\
&\quad\quad\times\left(\sum_{k=-\infty}^{k_0}\omega\left(B_k\right)^{\lambda p_2/n}\right)^{1/p_2}\|f\|_{M\dot{K}^{\alpha,\lambda}_{p_1,q}(\omega,\omega)}\\
&\quad\le C\|b\|_{C\dot{M}O^{p\max\{q,q'\}}(\omega)}\|f\|_{M\dot{K}^{\alpha,\lambda}_{p_1,q}(\omega,\omega)}.
\end{aligned}\tag{39}$$

This completes the proof of Theorem 10. □

Proof of Theorem 12. (a) Let $\alpha < n\delta/q_1'$. We follow the strategy of the proof of Theorem 7; we can also write

$$\begin{aligned}
&\left\|\left(H^b_\Omega f\right)\chi_k\right\|^{q_2}_{L^{q_2}(\omega^{1-q_2})}\\
&\quad\le C2^{-knq_2}\int_{\Delta_k}\left(\sum_{i=-\infty}^{k}\int_{\Delta_i}\left(b(x)-b_{B_k}\right)\right.\\
&\quad\quad\left.\times\Omega(x-y)f(y)\mid dy\right)^{q_2}\\
&\quad\quad\times\omega(x)^{1-q_2}dx\\
&\quad\quad+C2^{-knq_2}\int_{\Delta_k}\left(\sum_{i=-\infty}^{k}\int_{\Delta_i}\left(b(y)-b_{B_k}\right)\right.\\
&\quad\quad\left.\times\Omega(x-y)f(y)\mid dy\right)^{q_2}\\
&\quad\quad\times\omega(x)^{1-q_2}dx\\
&\quad:=J+JJ.
\end{aligned}\tag{40}$$

For J, when $\Omega \in L^{\infty}(S^{n-1})$, noticing that $1/q_1 = 1/q_2 + \beta/n$ and $\omega \in A_1 \subset A_{q_1}$, by inequality (23) and Remark 17, we obtain that

$$\begin{aligned}
J &\le C2^{-knq_2} \int_{\Delta_k} \left|b(x) - b_{B_k}\right|^{q_2} \omega(x)^{1-q_2} \\
&\quad \times \left(\sum_{i=-\infty}^{k} \|\Omega\|_{L^{\infty}(S^{n-1})} \int_{\Delta_i} |f(y)|\, dy \right)^{q_2} dx \\
&\le C\|\Omega\|_{L^{\infty}(S^{n-1})}^{q_2} 2^{-knq_2} \|b\|_{Lip_{\beta,\omega}}^{q_2} \\
&\quad \times \omega(B_k)^{1+\beta q_2/n} \left(\sum_{i=-\infty}^{k} \int_{\Delta_i} |f(y)|\, dy \right)^{q_2} \\
&\le C\|\Omega\|_{L^{\infty}(S^{n-1})}^{q_2} 2^{-knq_2} \|b\|_{Lip_{\beta,\omega}}^{q_2} \omega(B_k)^{1+\beta q_2/n} \\
&\quad \times \left(\sum_{i=-\infty}^{k} \|f\chi_i\|_{L^{q_1}(\omega)} |B_i| \omega(B_i)^{1/q_1'-1} \right)^{q_2} \\
&= C\|\Omega\|_{L^{\infty}(S^{n-1})}^{q_2} \|b\|_{Lip_{\beta,\omega}}^{q_2} \\
&\quad \times \left(\sum_{i=-\infty}^{k} 2^{(i-k)n} \left(\frac{\omega(B_k)}{\omega(B_i)} \right)^{1/q_1} \|f\chi_i\|_{L^{q_1}(\omega)} \right)^{q_2} \\
&\le C\|\Omega\|_{L^{\infty}(S^{n-1})}^{q_2} \|b\|_{Lip_{\beta,\omega}}^{q_2} \\
&\quad \times \left(\sum_{i=-\infty}^{k} 2^{(i-k)n/q_1'} \|f\chi_i\|_{L^{q_1}(\omega)} \right)^{q_2}.
\end{aligned} \tag{41}$$

As $\Omega \in L^s(S^{n-1})$, $1 < s < \infty$, observing that $1/q_1 = 1/q_2 + \beta/n$ and $\omega \in A_1 \subset A_{q_1}$, by the inequality (23) and Remark 17, again using Hölder's inequality, we can obtain that

$$\begin{aligned}
J &\le C2^{-knq_2} \int_{\Delta_k} \left|b(x) - b_{B_k}\right|^{q_2} \omega(x)^{1-q_2} \\
&\quad \times \left(\sum_{i=-\infty}^{k} \left(\int_{\Delta_i} |\Omega(x-y)|^s dy \right)^{1/s} \right. \\
&\quad \times \left. \left(\int_{\Delta_i} |f(y)|^{s'} dy \right)^{1/s'} \right)^{q_2} dx \\
&\le C2^{-knq_2} \|b\|_{Lip_{\beta,\omega}}^{q_2} \omega(B_k)^{1+\beta q_2/n} \\
&\quad \times \left(\sum_{i=-\infty}^{k} \|\Omega\|_{L^s(S^{n-1})} \int_{\Delta_i} |f(y)|\, dy \right)^{q_2} \\
&\le C\|\Omega\|_{L^s(S^{n-1})}^{q_2} 2^{-knq_2} \|b\|_{Lip_{\beta,\omega}}^{q_2} \omega(B_k)^{1+\beta q_2/n} \\
&\quad \times \left(\sum_{i=-\infty}^{k} \|f\chi_i\|_{L^{q_1}(\omega)} |B_i| \omega(B_i)^{-1/q_1} \right)^{q_2} \\
&= C\|\Omega\|_{L^s(S^{n-1})}^{q_2} \|b\|_{Lip_{\beta,\omega}}^{q_2} \\
&\quad \times \left(\sum_{i=-\infty}^{k} 2^{(i-k)n} \left(\frac{\omega(B_k)}{\omega(B_i)} \right)^{1/q_1} \|f\chi_i\|_{L^{q_1}(\omega)} \right)^{q_2} \\
&\le C\|\Omega\|_{L^s(S^{n-1})}^{q_2} \|b\|_{Lip_{\beta,\omega}}^{q_2} \\
&\quad \times \left(\sum_{i=-\infty}^{k} 2^{(i-k)n/q_1'} \|f\chi_i\|_{L^{q_1}(\omega)} \right)^{q_2}.
\end{aligned} \tag{42}$$

Therefore, for $1 < s \le \infty$, by estimates (41) and (42), we obtain

$$J \le C\|b\|_{Lip_{\beta,\omega}}^{q_2} \left(\sum_{i=-\infty}^{k} 2^{(i-k)n/q_1'} \|f\chi_i\|_{L^{q_1}(\omega)} \right)^{q_2}. \tag{43}$$

Now we come to estimate the other term JJ. Again, we shall further decompose JJ as

$$\begin{aligned}
JJ &\le C2^{-knq_2} \\
&\quad \times \int_{\Delta_k} \left(\sum_{i=-\infty}^{k} \int_{\Delta_i} \left|\left(b(y) - b_{B_i}\right) \Omega(x-y) f(y)\right| dy \right)^{q_2} \\
&\quad \times \omega(x)^{1-q_2} dx + C2^{-knq_2} \\
&\quad \times \int_{\Delta_k} \left(\sum_{i=-\infty}^{k} \int_{\Delta_i} \left|\left(b_{B_k} - b_{B_i}\right) \Omega(x-y) f(y)\right| dy \right)^{q_2} \\
&\quad \times \omega(x)^{1-q_2} dx \\
&:= JJ_1 + JJ_2.
\end{aligned} \tag{44}$$

We first turn to deal with the term JJ_1. By Hölder's inequality, we can obtain that

$$\begin{aligned}
&\int_{\Delta i} \left|f(y)\left(b(y) - b_{B_i}\right)\right| dy \\
&\quad \le \left(\int_{\Delta i} |f(y)|^{q_1} \omega(y)\, dy \right)^{1/q_1} \\
&\quad \quad \times \left(\int_{\Delta i} \left|b(y) - b_{B_i}\right|^{q_1'} \omega(y)^{-q_1'/q_1} dy \right)^{1/q_1'} \\
&\quad \le \|b\|_{Lip_{\beta,\omega}} \omega(B_i)^{1/q_1'+\beta/n} \|f\chi_i\|_{L^{q_1}(\omega)}.
\end{aligned} \tag{45}$$

As $\Omega \in L^{\infty}(S^{n-1})$, noticing that $1/q_1 = 1/q_2 + \beta/n$ and $\omega \in A_1 \subset A_{q_2'}$, we can deduce from inequality (45), Lemma 20, and Remark 17 that

$$\begin{aligned}
JJ_1 &\le C2^{-knq_2} \\
&\quad \times \int_{\Delta_k} \left(\sum_{i=-\infty}^{k} \|\Omega\|_{L^{\infty}(S^{n-1})} \int_{\Delta_i} \left|\left(b(y) - b_{B_i}\right) f(y)\right| dy \right)^{q_2} \\
&\quad \times \omega(x)^{1-q_2} dx \\
&\le C2^{-knq_2} |B_k|^{q_2} \omega(B_k)^{1-q_2} \\
&\quad \times \left(\sum_{i=-\infty}^{k} \|\Omega\|_{L^{\infty}(S^{n-1})} \int_{\Delta_i} \left|\left(b(y) - b_{B_i}\right) f(y)\right| dy \right)^{q_2} \\
&\le C\|\Omega\|^{q_2}_{L^{\infty}(S^{n-1})} 2^{-knq_2} |B_k|^{q_2} \omega(B_k)^{1-q_2} \\
&\quad \times \left(\sum_{i=-\infty}^{k} \|b\|_{Lip_{\beta,\omega}} \omega(B_i)^{1/q_1'+\beta/n} \|f\chi_i\|_{L^{q_1}(\omega)} \right)^{q_2} \\
&= C\|\Omega\|^{q_2}_{L^{\infty}(S^{n-1})} \|b\|^{q_2}_{Lip_{\beta,\omega}} \\
&\quad \times \left(\sum_{i=-\infty}^{k} \left(\frac{\omega(B_i)}{\omega(B_k)} \right)^{1-1/q_2} \|f\chi_i\|_{L^{q_1}(\omega)} \right)^{q_2} \\
&\le C\|\Omega\|^{q_2}_{L^{\infty}(S^{n-1})} \|b\|^{q_2}_{Lip_{\beta,\omega}} \\
&\quad \times \left(\sum_{i=-\infty}^{k} 2^{(i-k)n\delta/q_2'} \|f\chi_i\|_{L^{q_1}(\omega)} \right)^{q_2}.
\end{aligned} \tag{46}$$

As $\Omega \in L^{s}(S^{n-1})$, observing that $1/q_1 = 1/q_2 + \beta/n$ and $\omega \in A_1 \subset A_{q_2'}$ and then by using Hölder's inequality, inequality (45), Lemma 20, and Remark 17, we can obtain that

$$\begin{aligned}
JJ_1 &\le C2^{-knq_2} \\
&\quad \times \int_{\Delta_k} \left(\sum_{i=-\infty}^{k} \left(\int_{\Delta_i} |\Omega(x-y)|^{s} dy \right)^{1/s} \right. \\
&\qquad \left. \times \left(\int_{\Delta_i} \left|\left(b(y) - b_{B_i}\right) f(y)\right|^{s'} dy \right)^{1/s'} \right)^{q_2} \\
&\quad \times \omega(x)^{1-q_2} dx \\
&\le C2^{-knq_2} \int_{\Delta_k} \omega(x)^{1-q_2} dx \\
&\quad \times \left(\sum_{i=-\infty}^{k} \|\Omega\|_{L^{s}(S^{n-1})} \int_{\Delta_i} \left|\left(b(y) - b_{B_i}\right) f(y)\right| dy \right)^{q_2} \\
&\le C2^{-knq_2} \int_{\Delta_k} \omega(x)^{1-q_2} dx \\
&\quad \times \left(\sum_{i=-\infty}^{k} \|\Omega\|_{L^{s}(S^{n-1})} \|b\|_{Lip_{\beta,\omega}} \omega(B_i)^{1/q_1'+\beta/n} \|f\chi_i\|_{L^{q_1}(\omega)} \right)^{q_2} \\
&\le C\|\Omega\|^{q_2}_{L^{s}(S^{n-1})} 2^{-knq_2} |B_k|^{q_2} \omega(B_k)^{1-q_2} \\
&\quad \times \left(\sum_{i=-\infty}^{k} \|b\|_{Lip_{\beta,\omega}} \omega(B_i)^{1/q_1'+\beta/n} \|f\chi_i\|_{L^{q_1}(\omega)} \right)^{q_2} \\
&= C\|\Omega\|^{q_2}_{L^{s}(S^{n-1})} \|b\|^{q_2}_{Lip_{\beta,\omega}} \\
&\quad \times \left(\sum_{i=-\infty}^{k} \left(\frac{\omega(B_i)}{\omega(B_k)} \right)^{1-1/q_2} \|f\chi_i\|_{L^{q_1}(\omega)} \right)^{q_2} \\
&\le C\|\Omega\|^{q_2}_{L^{s}(S^{n-1})} \|b\|^{q_2}_{Lip_{\beta,\omega}} \\
&\quad \times \left(\sum_{i=-\infty}^{k} 2^{(i-k)n\delta/q_2'} \|f\chi_i\|_{L^{q_1}(\omega)} \right)^{q_2}.
\end{aligned} \tag{47}$$

Hence, for $1 < s \le \infty$, by combining inequalities (46) and (47), we have

$$JJ_1 \le C\|b\|^{q_2}_{Lip_{\beta,\omega}} \left(\sum_{i=-\infty}^{k} 2^{(i-k)n\delta/q_2'} \|f\chi_i\|_{L^{q_1}(\omega)} \right)^{q_2}. \tag{48}$$

Finally, let us deal with the term JJ_2. When $\Omega \in L^{\infty}(S^{n-1})$, noticing that $\omega \in A_1 \subset A_{q_2'}$, by using Lemma 19, inequality (23), and Remark 17, we can get that

$$\begin{aligned}
JJ_2 &\le C2^{-knq_2} \\
&\quad \times \int_{\Delta_k} \left(\sum_{i=-\infty}^{k} \int_{\Delta_i} (k-i) \|b\|_{Lip_{\beta,\omega}} \omega(B_k)^{\beta/n} \frac{\omega(B_i)}{|B_i|} \right. \\
&\qquad \left. \times |\Omega(x-y) f(y)| dy \right)^{q_2} \\
&\quad \times \omega(x)^{1-q_2} dx \\
&\le C\|b\|^{q_2}_{Lip_{\beta,\omega}} 2^{-knq_2} \\
&\quad \times \int_{\Delta_k} \left(\sum_{i=-\infty}^{k} (k-i) \omega(B_k)^{\beta/n} \frac{\omega(B_i)}{|B_i|} \|\Omega\|_{L^{\infty}(S^{n-1})} \right. \\
&\qquad \left. \times \int_{\Delta_i} |f(y)| dy \right)^{q_2} \omega(x)^{1-q_2} dx
\end{aligned}$$

$$
\begin{aligned}
&\le C\|\Omega\|_{L^\infty(S^{n-1})}^{q_2}\|b\|_{Lip_{\beta,\omega}}^{q_2}\omega(B_k)^{1-q_2}\\
&\quad\times\left(\sum_{i=-\infty}^{k}(k-i)\,\omega(B_k)^{\beta/n}\right.\\
&\qquad\left.\times\frac{\omega(B_i)}{|B_i|}|B_i|\omega(B_i)^{-1/q_1}\|f\chi_i\|_{L^{q_1}(\omega)}\right)^{q_2}\\
&= C\|\Omega\|_{L^\infty(S^{n-1})}^{q_2}\|b\|_{Lip_{\beta,\omega}}^{q_2}\\
&\quad\times\left(\sum_{i=-\infty}^{k}(k-i)\left(\frac{\omega(B_i)}{\omega(B_k)}\right)^{1-1/q_1}\|f\chi_i\|_{L^{q_1}(\omega)}\right)^{q_2}\\
&\le C\|\Omega\|_{L^\infty(S^{n-1})}^{q_2}\|b\|_{Lip_{\beta,\omega}}^{q_2}\\
&\quad\times\left(\sum_{i=-\infty}^{k}(k-i)\,2^{(i-k)n\delta/q_1'}\|f\chi_i\|_{L^{q_1}(\omega)}\right)^{q_2}.
\end{aligned}
\tag{49}
$$

When $\Omega\in L^s(S^{n-1})$, observing that $\omega\in A_1\subset A_{q_2'}$, it follows from Hölder's inequality, Lemma 19, inequality (23), and Remark 17 that

$$
\begin{aligned}
JJ_2 &\le C2^{-knq_2}\\
&\quad\times\int_{\Delta_k}\left(\sum_{i=-\infty}^{k}\int_{\Delta_i}(k-i)\,\|b\|_{Lip_{\beta,\omega}}\omega(B_k)^{\beta/n}\right.\\
&\qquad\left.\times\frac{\omega(B_i)}{|B_i|}\left|\Omega(x-y)f(y)\right|dy\right)^{q_2}\\
&\quad\times\omega(x)^{1-q_2}dx\\
&\le C\|b\|_{Lip_{\beta,\omega}}^{q_2}2^{-knq_2}\\
&\quad\times\int_{\Delta_k}\left(\sum_{i=-\infty}^{k}(k-i)\,\omega(B_k)^{\beta/n}\frac{\omega(B_i)}{|B_i|}\right.\\
&\qquad\times\left(\int_{\Delta i}|f(y)|^{s'}dy\right)^{1/s'}\\
&\qquad\left.\times\left(\int_{\Delta i}|\Omega(x-y)|^{s}dy\right)^{1/s}\right)^{q_2}\\
&\quad\times\omega(x)^{1-q_2}dx\\
&\le C\|b\|_{Lip_{\beta,\omega}}^{q_2}\omega(B_k)^{1-q_2}\\
&\quad\times\left(\sum_{i=-\infty}^{k}(k-i)\,\omega(B_k)^{\beta/n}\frac{\omega(B_i)}{|B_i|}\right.\\
&\qquad\left.\times\|\Omega\|_{L^s(S^{n-1})}\int_{\Delta_i}|f(y)|\,dy\right)^{q_2}
\end{aligned}
$$

$$
\begin{aligned}
&\le C\|\Omega\|_{L^s(S^{n-1})}^{q_2}\|b\|_{Lip_{\beta,\omega}}^{q_2}(B_k)^{1-q_2}\\
&\quad\times\left(\sum_{i=-\infty}^{k}(k-i)\,\omega(B_k)^{\beta/n}\right.\\
&\qquad\left.\times\frac{\omega(B_i)}{|B_i|}|B_i|\,\omega(B_i)^{-1/q_1}\|f\chi_i\|_{L^{q_1}(\omega)}\right)^{q_2}\\
&= C\|\Omega\|_{L^s(S^{n-1})}^{q_2}\|b\|_{Lip_{\beta,\omega}}^{q_2}\\
&\quad\times\left(\sum_{i=-\infty}^{k}(k-i)\left(\frac{\omega(B_i)}{\omega(B_k)}\right)^{1-1/q_1}\|f\chi_i\|_{L^{q_1}(\omega)}\right)^{q_2}\\
&\le C\|\Omega\|_{L^s(S^{n-1})}^{q_2}\|b\|_{Lip_{\beta,\omega}}^{q_2}\\
&\quad\times\left(\sum_{i=-\infty}^{k}(k-i)\,2^{(i-k)n\delta/q_1'}\|f\chi_i\|_{L^{q_1}(\omega)}\right)^{q_2}.
\end{aligned}
\tag{50}
$$

As before, for $1<s\le\infty$, from inequalities (49), and (50), we obtain that

$$
\begin{aligned}
JJ_2 &\le C\|b\|_{Lip_{\beta,\omega}}^{q_2}\\
&\quad\times\left(\sum_{i=-\infty}^{k}(k-i)\,2^{(i-k)n\delta/q_1'}\|f\chi_i\|_{L^{q_1}(\omega)}\right)^{q_2}.
\end{aligned}
\tag{51}
$$

The remaining proof is similar to the proof of Theorem 7 in case (a); we omit the details here. Thus we complete the proof of Theorem 12 in case (a).

(b) When $\alpha>-n\delta/q_2$, by a direct calculation, we can see

$$
\begin{aligned}
&\left\|\left(H_\Omega^{*,b}f\right)\chi_k\right\|_{L^{q_2}(\omega^{1-q_2})}^{q_2}\\
&\quad\le C\int_{\Delta_k}\left(\sum_{i=k}^{\infty}2^{-in}\right.\\
&\qquad\left.\times\int_{\Delta_i}\left(b(x)-b_{B_k}\right)\Omega(x-y)\,f(y)\mid dy\right)^{q_2}\\
&\qquad\times\omega(x)^{1-q_2}dx\\
&\quad+C\int_{\Delta_k}\left(\sum_{i=k}^{\infty}2^{-in}\right.\\
&\qquad\left.\times\int_{\Delta_i}\left(b(y)-b_{B_k}\right)\Omega(x-y)\,f(y)\mid dy\right)^{q_2}\\
&\qquad\times\omega(x)^{1-q_2}dx\\
&\quad:=J'+JJ'.
\end{aligned}
\tag{52}
$$

When $\Omega \in L^{\infty}(S^{n-1})$, note that $1/q_1 = 1/q_2 + \beta/n$ and $\omega \in A_1 \subset A_{q_1}$. Applying Hölder's inequality, inequality (23), and Remark 17, we can deduce

$$\begin{aligned} J' &\le C \int_{\Delta_k} \left|b(x) - b_{B_k}\right|^{q_2} \omega(x)^{1-q_2} \\ &\quad \times \left(\sum_{i=k}^{\infty} 2^{-in} \|\Omega\|_{L^{\infty}(S^{n-1})} \int_{\Delta_i} |f(y)|\, dy \right)^{q_2} dx \\ &\le C \|\Omega\|_{L^{\infty}(S^{n-1})}^{q_2} \|b\|_{Lip_{\beta,\omega}}^{q_2} \omega(B_k)^{1+\beta q_2/n} \\ &\quad \times \left(\sum_{i=k}^{\infty} 2^{-in} \int_{\Delta_i} |f(y)|\, dy \right)^{q_2} \\ &\le C \|\Omega\|_{L^{\infty}(S^{n-1})}^{q_2} \|b\|_{Lip_{\beta,\omega}}^{q_2} \omega(B_k)^{1+\beta q_2/n} \\ &\quad \times \left(\sum_{i=k}^{\infty} 2^{-in} \|f\chi_i\|_{L^{q_1}(\omega)} |B_i| \, \omega(B_i)^{1/q_1'-1} \right)^{q_2} \\ &= C \|\Omega\|_{L^{\infty}(S^{n-1})}^{q_2} \|b\|_{Lip_{\beta,\omega}}^{q_2} \\ &\quad \times \left(\sum_{i=k}^{\infty} \left(\frac{\omega(B_k)}{\omega(B_i)} \right)^{1/q_1} \|f\chi_i\|_{L^{q_1}(\omega)} \right)^{q_2} \\ &\le C \|\Omega\|_{L^{\infty}(S^{n-1})}^{q_2} \|b\|_{Lip_{\beta,\omega}}^{q_2} \\ &\quad \times \left(\sum_{i=k}^{\infty} 2^{(k-i)n\delta/q_1} \|f\chi_i\|_{L^{q_1}(\omega)} \right)^{q_2}. \end{aligned} \tag{53}$$

When $\Omega \in L^{s}(S^{n-1})$, $1 < s < \infty$, observe that $1/q_1 = 1/q_2 + \beta/n$ and $\omega \in A_1 \subset A_{q_1}$; then by using Hölder's inequality, inequality (23), and Remark 17, we can obtain

$$\begin{aligned} J' &\le C \int_{\Delta_k} \left|b(x) - b_{B_k}\right|^{q_2} \omega(x)^{1-q_2} \\ &\quad \times \left(\sum_{i=k}^{\infty} 2^{-in} \left(\int_{\Delta_i} |\Omega(x-y)|^{s} dy \right)^{1/s} \right. \\ &\quad \times \left. \left(\int_{\Delta_i} |f(y)|^{s'} dy \right)^{1/s'} \right)^{q_2} dx \\ &\le C \|b\|_{Lip_{\beta,\omega}}^{q_2} \omega(B_k)^{1+\beta q_2/n} \\ &\quad \times \left(\sum_{i=k}^{\infty} 2^{-in} \|\Omega\|_{L^{s}(S^{n-1})} \int_{\Delta_i} |f(y)|\, dy \right)^{q_2} \\ &\le C \|\Omega\|_{L^{s}(S^{n-1})}^{q_2} \|b\|_{Lip_{\beta,\omega}}^{q_2} \omega(B_k)^{1+\beta q_2/n} \\ &\quad \times \left(\sum_{i=k}^{\infty} 2^{-in} \|f\chi_i\|_{L^{q_1}(\omega)} |B_i| \, \omega(B_i)^{-1/q_1} \right)^{q_2} \\ &= C \|\Omega\|_{L^{s}(S^{n-1})}^{q_2} \|b\|_{Lip_{\beta,\omega}}^{q_2} \\ &\quad \times \left(\sum_{i=k}^{\infty} \left(\frac{\omega(B_k)}{\omega(B_i)} \right)^{1/q_1} \|f\chi_i\|_{L^{q_1}(\omega)} \right)^{q_2} \\ &\le C \|\Omega\|_{L^{s}(S^{n-1})}^{q_2} \|b\|_{Lip_{\beta,\omega}}^{q_2} \\ &\quad \times \left(\sum_{i=k}^{\infty} 2^{(k-i)n\delta/q_1} \|f\chi_i\|_{L^{q_1}(\omega)} \right)^{q_2}. \end{aligned} \tag{54}$$

Hence, for $1 < s \le \infty$, by estimates (53) and (54), we get

$$J' \le C \|b\|_{Lip_{\beta,\omega}}^{q_2} \left(\sum_{i=k}^{\infty} 2^{(k-i)n\delta/q_1} \|f\chi_i\|_{L^{q_1}(\omega)} \right)^{q_2}. \tag{55}$$

For JJ', using the same arguments as those of JJ, we can see

$$\begin{aligned} JJ' &\le C \int_{\Delta_k} \left(\sum_{i=k}^{\infty} 2^{-in} \right. \\ &\quad \times \left. \int_{\Delta_i} \left| \left(b(y) - b_{B_i}\right) \Omega(x-y) f(y) \right| dy \right)^{q_2} \\ &\quad \times \omega(x)^{1-q_2} dx \\ &\quad + C \int_{\Delta_k} \left(\sum_{i=k}^{\infty} 2^{-in} \right. \\ &\quad \times \left. \int_{\Delta_i} \left| \left(b_{B_k} - b_{B_i}\right) \Omega(x-y) f(y) \right| dy \right)^{q_2} \\ &\quad \times \omega(x)^{1-q_2} dx \\ &:= JJ_1' + JJ_2'. \end{aligned} \tag{56}$$

Similar to the estimate of JJ_1, JJ_2 of (a), we have

$$\begin{aligned} JJ_1' &\le C \|b\|_{Lip_{\beta,\omega}}^{q_2} \left(\sum_{i=k}^{\infty} 2^{(k-i)n/q_2} \|f\chi_i\|_{L^{q_1}(\omega)} \right)^{q_2} \\ JJ_2' &\le C \|b\|_{Lip_{\beta,\omega}}^{q_2} \left(\sum_{i=k}^{\infty} 2^{(k-i)n\delta/q_2} \|f\chi_i\|_{L^{q_1}(\omega)} \right)^{q_2}. \end{aligned} \tag{57}$$

The remaining proof is similar to the proof of (a) of Theorem 7, so that the proof of (b) can be obtained easily. So we conclude the proof of Theorem 12. □

Proof of Theorem 14. (a) Let $\alpha < n\delta/q_1' + \lambda$. By the definition of Morry-Herz spaces and combining inequalities (43), (48), and (51) in the above, it is not difficult to see that

$$\begin{aligned}
&\left\| H_\Omega^b f \right\|_{M\dot{K}_{p_2,q_2}^{\alpha,\lambda}(\omega,\omega^{1-q_2})} \\
&\quad \le C\|b\|_{Lip_{\beta,\omega}} \sup_{k_0\in Z} \omega\left(B_{k_0}\right)^{-\lambda/n} \\
&\quad\quad \times \left(\sum_{k=-\infty}^{k_0} \omega\left(B_k\right)^{\alpha p_2/n} \right. \\
&\quad\quad\quad \left. \times \left(\sum_{i=-\infty}^{k} 2^{(i-k)n/q_1'} \|f\chi_i\|_{L^{q_1}(\omega)} \right)^{p_2} \right)^{1/p_2} \\
&\quad\quad + C\|b\|_{Lip_{\beta,\omega}} \sup_{k_0\in Z} \omega\left(B_{k_0}\right)^{-\lambda/n} \\
&\quad\quad \times \left(\sum_{k=-\infty}^{k_0} \omega\left(B_k\right)^{\alpha p_2/n} \right. \\
&\quad\quad\quad \left. \times \left(\sum_{i=-\infty}^{k} 2^{(i-k)n\delta/q_2'} \|f\chi_i\|_{L^{q_1}(\omega)} \right)^{p_2} \right)^{1/p_2} \\
&\quad\quad + C\|b\|_{Lip_{\beta,\omega}} \sup_{k_0\in Z} \omega\left(B_{k_0}\right)^{-\lambda/n} \\
&\quad\quad \times \left(\sum_{k=-\infty}^{k_0} \omega\left(B_k\right)^{\alpha p_2/n} \right. \\
&\quad\quad\quad \left. \times \left(\sum_{i=-\infty}^{k} (k-i)\, 2^{(i-k)n\delta/q_1'} \|f\chi_i\|_{L^{q_1}(\omega)} \right)^{p_2} \right)^{1/p_2} \\
&\quad := E_1 + E_2 + E_3.
\end{aligned} \tag{58}$$

Therefore, by applying the similar argument as that in the proof of Theorem 10, we can obtain that

$$\begin{aligned}
E_1 &\le C\|b\|_{Lip_{\beta,\omega}} \left\| f \right\|_{M\dot{K}_{p_1,q_1}^{\alpha,\lambda}(\omega,\omega)}, \quad \text{as } \alpha < \frac{n}{q_1'} + \lambda. \\
E_2 &\le C\|b\|_{Lip_{\beta,\omega}} \left\| f \right\|_{M\dot{K}_{p_1,q_1}^{\alpha,\lambda}(\omega,\omega)}, \quad \text{as } \alpha < \frac{n\delta}{q_2'} + \lambda. \\
E_3 &\le C\|b\|_{Lip_{\beta,\omega}} \left\| f \right\|_{M\dot{K}_{p_1,q_1}^{\alpha,\lambda}(\omega,\omega)}, \quad \text{as } \alpha < \frac{n\delta}{q_1'} + \lambda.
\end{aligned} \tag{59}$$

Thus, when $\alpha < n\delta/q_1' + \lambda$, we have

$$\left\| H_\Omega^b f \right\|_{M\dot{K}_{p_2,q_2}^{\alpha,\lambda}(\omega,\omega^{1-q_2})} \le C\|b\|_{Lip_{\beta,\omega}} \left\| f \right\|_{M\dot{K}_{p_1,q_1}^{\alpha,\lambda}(\omega,\omega)}. \tag{60}$$

The proof of Theorem 10 (a) is completed.

(b) The proof of case (b) can be obtained similarly, so we omit the details here. □

Acknowledgments

The authors would like to express their deep thanks to the referee for his/her very careful reading and many valuable comments and suggestions. Shuangping Tao is supported by National Natural Foundation of China (Grant no. 11161042 and 11071250).

References

[1] G. H. Hardy, "Note on a theorem of Hilbert," *Mathematische Zeitschrift*, vol. 6, no. 3-4, pp. 314–317, 1920.

[2] M. Christ and L. Grafakos, "Best constants for two nonconvolution inequalities," *Proceedings of the American Mathematical Society*, vol. 123, no. 6, pp. 1687–1693, 1995.

[3] Z. W. Fu, S. Z. Lu, and F. Y. Zhao, "Commutators of n-dimensional rough Hardy operators," *Science China Mathematics*, vol. 54, no. 1, pp. 95–104, 2011.

[4] G. L. Gao, "Boundedness for commutators of n-dimensional rough Hardy operators on Morry-Herz spaces," *Computers and Mathematics with Applications*, vol. 64, no. 4, pp. 544–549, 2012.

[5] G. L. Gao and M. Wang, "Weighted estimates for commutators of higher dimensional Hardy operators," *Applied Mathematics—A Journal of Chinese Universities*, vol. 27, no. 1, pp. 104–111, 2012.

[6] B. Muckenhoupt, "Weighted norm inequalities for the Hardy maximal function," *Transactions of the American Mathematical Society*, vol. 165, pp. 207–226, 1972.

[7] L. Shanzhen and Y. Dachun, "The decomposition of weighted Herz space on $\mathbb{R}^n$ and its applications," *Science in China A*, vol. 38, no. 2, pp. 147–158, 1995.

[8] J. K. Kuang, "Generalized Hausdorff operators on weighted Morry-Herz spaces," *Acta Mathematica Sinica, Chinese Series*, vol. 55, no. 5, pp. 895–902, 2012.

[9] J. García-Cuerva, "Weighted H^p spaces," *Dissertationes Mathematicae*, vol. 162, pp. 1–63, 1979.

[10] J. L. Journé, "Calderón-Zygmund operators, differential operators and the cauchy integral of Calderón," *Lecture Notes in Mathematics*, vol. 994, pp. 1–127, 1983.

[11] Y. Lin, Z. G. Liu, and M. M. Song, "Lipschitz estimates for commutators of singular integral operators on weighted Herz spaces," *The Jordanian Journal of Mathematics and Statistics*, vol. 3, pp. 53–64, 2010.