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Research Article

On Perturbation of Convoluted C-Regularized Operator Families

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Of concern are two classes of convoluted C-regularized operator families: convoluted C-cosine operator families and convoluted C-semigroups. We obtain new and general multiplicative and additive perturbation theorems for these convoluted C-regularized operator families. Two examples are given to illustrate our abstract results.

1. Introduction

It is well known that the cosine operator families (resp., the C_0 semigroups) and the fractionally integrated C-cosine operator families (resp., integrated C-semigroups) are important tools in studying incomplete second-order (resp., first-order) abstract Cauchy problems (cf., e.g., [1–17]). As an extension of the cosine operator families (resp., the C_0 semigroups) as well as the fractionally integrated C-cosine operator families (resp., integrated C-semigroups), the convoluted C-cosine operator families (resp., convoluted C-semigroups) (cf., e.g., [15, 18, 19]) are also good operator families in dealing with ill-posed incomplete second order (resp. first order) abstract Cauchy problems.

In last two decades, there are many works on the perturbations on the C-regularized operator families (cf., e.g., [16, 20–24]). In the present paper, we will study the multiplicative and additive perturbation for two classes of convoluted C-regularized operator families: convoluted C-cosine operator families and convoluted C-semigroups, and our purpose is to obtain some new and general perturbation theorems for these convoluted C-regularized operator families and to make the results new even for convoluted n-times integrated C-cosine operator families (resp., convoluted n-times integrated C-semigroups) ($n \in \mathbb{N}_0$, where $\mathbb{N}_0$ denotes the nonnegative integers).

Throughout this paper, $\mathbb{N}$, $\mathbb{R}$, $\mathbb{C}$ denote the set of positive integers, the real numbers, and the complex plane, respectively. X denotes a nontrivial complex Banach space, and $L(X)$ denotes the space of bounded linear operators from X into X. In the sequel, we assume that $C \in L(X)$ is an injective operator. $\mathbf{C}([a,b], X)$ denotes the space of all continuous functions from $[a,b]$ to X. For a closed linear operator A on X, its domain, range, resolvent set, and the C-resolvent set are denoted by $D(A)$, $R(A)$, $\rho(A)$, and $\rho_c(A)$, respectively, where $\rho_c(A)$ is defined by

$$\rho_c(A) := \{\lambda \in \mathbb{C} : R(C) \subset R(\lambda - A),\ \lambda - A \text{ is injective}\}. \tag{1}$$

$K \in \mathbf{C}([0,\infty), \mathbb{C})$ is an exponentially bounded function, and for $\beta \in \mathbb{R}$,

$$\mathscr{L}[K(t)](\lambda) \neq 0 \quad (\operatorname{Re}\lambda > \beta), \tag{2}$$

where $\mathscr{L}[K(t)](\lambda)$ is the Laplace transform of $K(t)$ as in the monograph [15]. We define

$$\Theta(t) := \int_0^t K(s)\,ds, \quad t \geq 0. \tag{3}$$

Next, we recall some notations and basic results from [15, 19] about the convoluted C-cosine operator families and convoluted C-semigroups.

The following definition is the convoluted version of [15, Chapter 1, Definition 4.1].

Definition 1. Let $\omega \geq 0$ and $(\omega^2,\infty) \subset \rho_c(A)$. Let $\{C_K(t)\}_{t\geq 0}(C_K(t) \in L(X), t \geq 0)$ be a strongly continuous operator family such that

$$\|C_K(t)\| \leq Me^{\omega t}, \quad t \geq 0, \tag{4}$$

for some $M > 0$, and

$$\lambda\left(\lambda^2 - A\right)^{-1}Cx = \frac{1}{\mathscr{L}[K(t)](\lambda)}\int_0^\infty e^{-\lambda t}C_K(t)\,x\,dt, \qquad \operatorname{Re}\lambda > \max(\omega,\beta),\ x \in X. \tag{5}$$

Then, A is called a subgenerator of the exponentially bounded K-convoluted C-cosine operator family $\{C_K(t)\}_{t\geq 0}$. Moreover, the operator $\overline{A} := C^{-1}AC$ is called the generator of the $\{C_K(t)\}_{t\geq 0}$.

Proposition 2. *Let A be a closed operator and $\{C_K(t)\}_{t\geq 0}$ a strongly continuous, exponentially bounded operator family. Then A is the subgenerator of a K-convoluted C-cosine operator family $\{C_K(t)\}_{t\geq 0}$ if and only if*

(1) $C_K(t)C = CC_K(t)$, $t \geq 0$;

(2) $C_K(t)A \subset AC_K(t)$, $t \geq 0$, *and*

$$A\int_0^t\int_0^s C_K(\sigma)\,x\,d\sigma\,ds = C_K(t)\,x - \Theta(t)\,Cx, \qquad t \geq 0,\ x \in X. \tag{6}$$

Remark 3. If A is the subgenerator of a K-convoluted C-cosine operator family, then $CA \subseteq AC$.

Definition 4. Let $0 \leq \omega < \infty$ and $(\omega,\infty) \subset \rho_c(A)$. Let $\{T_K(t)\}_{t\geq 0}$ be a strongly continuous operator family such that

$$\|T_K(t)\| \leq Me^{\omega t}, \quad t \geq 0, \tag{7}$$

for some $M > 0$, and

$$(\lambda - A)^{-1}Cx = \frac{1}{\mathscr{L}[K(t)](\lambda)}\int_0^\infty e^{-\lambda t}T_K(t)\,x\,dt, \qquad \operatorname{Re}\lambda > \max\{\omega,\beta\},\ x \in X. \tag{8}$$

Then, A is called a subgenerator of an exponentially bounded K-convoluted C-semigroup $\{T_K(t)\}_{t\geq 0}$. Moreover, the operator $\overline{A} := C^{-1}AC$ is called the generator of the $\{T_K(t)\}_{t\geq 0}$.

Proposition 5. *Let A be a closed operator, and $\{T_K(t)\}_{t\geq 0}$ a strongly continuous, exponentially bounded operator family. Then, A is the subgenerator of a K-convoluted C-semigroup $\{T_K(t)\}_{t\geq 0}$ if and only if*

(1) $T_K(t)C = CT_K(t)$, $t \geq 0$;

(2) $T_K(t)A \subset AT_K(t)$, $t \geq 0$, *and*

$$A\int_0^t T_K(s)\,x\,ds = T_K(t)\,x - \Theta(t)\,Cx, \quad t \geq 0,\ x \in X. \tag{9}$$

Remark 6. From [15], we know that the C-cosine operator families (resp., C-semigroups) are exactly the 0-times integrated C-cosine operator families (resp., the 0-times integrated C-semigroups). Let $\Gamma(\cdot)$ be the well-known Gamma function, and

$$K(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}. \tag{10}$$

Then, by Propositions 2 and 5, we get results for the α-times integrated C-cosine operator families (resp., α-times integrated C-semigroups) as well as C-cosine operator families (resp., C-semigroups). For more information on various C operator families, we refer the reader to, for example, [3, 6–8, 14, 15, 17, 22] and references therein.

2. Multiplicative Perturbation Theorems

Lemma 7. *Suppose that A is a subgenerator of an exponentially bounded K-convoluted C-cosine operator family on X. If $\rho(A) \neq \emptyset$, then $C^{-1}AC = A$.*

Proof. For any $\lambda_0 \in \rho(A)$ and $x \in D(C^{-1}AC)$, let

$$y = \lambda_0 x - C^{-1}ACx. \tag{11}$$

Then,

$$\begin{gathered}(\lambda_0 - A)^{-1}C = C(\lambda_0 - A)^{-1},\\ Cx = (\lambda_0 - A)^{-1}Cy = C(\lambda_0 - A)^{-1}y.\end{gathered} \tag{12}$$

Therefore,

$$x = (\lambda_0 - A)^{-1}y \in D(A). \tag{13}$$

This means that $C^{-1}AC \subseteq A$. Thus, by Remark 3, we see that $C^{-1}AC = A$. □

Theorem 8. *Let A be a closed linear operator on X and $\mathscr{R} \in L(X)$. Assume that there exists an injective operator C on X satisfying $CA \subseteq AC, \mathscr{R}C = C\mathscr{R}$. Then, the following statements hold.*

(1) *If $\mathscr{R}A$ subgenerates an exponentially bounded K-convoluted C-cosine operator family on X, then $A\mathscr{R}$ subgenerates an exponentially bounded K-convoluted C-cosine operator family on X.*

(2) *If $A\mathscr{R}$ subgenerates an exponentially bounded K-convoluted C-cosine operator family on X and $\rho(\mathscr{R}A) \neq \emptyset$, then $\mathscr{R}A$ generates an exponentially bounded K-convoluted C-cosine operator family on X.*

Proof. (1) Assume that $\mathscr{R}A$ subgenerates an exponentially bounded K-convoluted C-cosine operator family $\{C_K(t)\}_{t\geq 0}$ on X.

In this case, it is easy to see that for any $t \geq 0$, the operator

$$x \longmapsto A \int_0^t S_K(s)\mathcal{R}x ds \tag{14}$$

is bounded, since

$$\int_0^t S_K(s)\mathcal{R}x ds \in D(\mathcal{R}A), \tag{15}$$

where $S_K(t) = \int_0^t C_K(s)ds$. Now, for each $t \geq 0$, we define a bounded linear operator as follows:

$$\widehat{C}_K(t)x = \Theta(t)Cx + A\int_0^t S_K(s)\mathcal{R}x ds. \tag{16}$$

Clearly, the graph norms of $\mathcal{R}A$ and A are equivalent. Therefore, noting that $\mathcal{R}A$ subgenerates an exponentially bounded K-convoluted C-cosine operator family $\{C_K(t)\}_{t\geq 0}$ on X, we obtain, for every $t_1, t_2 \geq 0$, and $x \in X$, that there exists a constant M_1 such that

$$\begin{aligned}
&\left\|\widehat{C}_K(t_1)x - \widehat{C}_K(t_2)x\right\| \\
&\quad \leq \left\|\Theta(t_1)Cx - \Theta(t_2)Cx\right\| \\
&\qquad + \left\|A\left(\int_0^{t_1} S_K(s)\mathcal{R}x ds - \int_0^{t_2} S_K(s)\mathcal{R}x ds\right)\right\| \\
&\quad \leq \left\|\Theta(t_1)Cx - \Theta(t_2)Cx\right\| \\
&\qquad + M_1\left(\left\|\mathcal{R}A\left(\int_0^{t_1} S_K(s)\mathcal{R}x ds - \int_0^{t_2} S_K(s)\mathcal{R}x ds\right)\right\|\right. \\
&\qquad\qquad \left. + \left\|\int_0^{t_1} S_K(s)\mathcal{R}x ds - \int_0^{t_2} S_K(s)\mathcal{R}x ds\right\|\right) \\
&\quad = (M_1+1)\left\|\Theta(t_1)Cx - \Theta(t_2)Cx\right\| \\
&\qquad + M_1\left\|C_K(t_1)x - C_K(t_2)x\right\| \\
&\qquad + \left\|\int_0^{t_1} S_K(s)\mathcal{R}x ds \right. \\
&\qquad\qquad \left. - \int_0^{t_2} S_K(s)\mathcal{R}x ds\right\| \longrightarrow 0, \quad \text{as } t_1 \longrightarrow t_2.
\end{aligned} \tag{17}$$

Hence, $\widehat{C}_K(\cdot)$ is strongly continuous.

Similarly, we can prove that $\widehat{C}_K(\cdot)$ is exponentially bounded; that is, there exists a constant $\widehat{M} > 0$ such that

$$\left\|\widehat{C}_K(t)\right\| \leq \widehat{M}e^{\omega t}, \quad t \geq 0. \tag{18}$$

As in the monograph [15], we write

$$\begin{gathered}
\mathcal{L}\left[\widehat{C}_K(t)\right](\lambda)x = \int_0^\infty e^{-\lambda t}\widehat{C}_K(t)x dt, \\
\text{for } \operatorname{Re}\lambda > \max(\omega,\beta), \ x \in X.
\end{gathered} \tag{19}$$

Then, by (16), we have

$$\begin{aligned}
\mathcal{L}\left[\widehat{C}_K(t)\right](\lambda)x &= \frac{\mathcal{L}[K(t)](\lambda)}{\lambda}Cx \\
&\quad + A\frac{\mathcal{L}[K(t)](\lambda)}{\lambda}\left(\lambda^2 - \mathcal{R}A\right)^{-1}C\mathcal{R}x.
\end{aligned} \tag{20}$$

Hence,

$$\begin{aligned}
&\mathcal{R}\mathcal{L}\left[\widehat{C}_K(t)\right](\lambda)x \\
&\quad = \frac{\mathcal{L}[K(t)](\lambda)}{\lambda} \\
&\qquad \times C\left[\mathcal{R}x + \mathcal{R}A\left(\lambda^2 - \mathcal{R}A\right)^{-1}C\mathcal{R}x\right] \\
&\quad = \lambda\mathcal{L}[K(t)](\lambda)\left(\lambda^2 - \mathcal{R}A\right)^{-1}C\mathcal{R}x \\
&\quad \in D(A).
\end{aligned} \tag{21}$$

Furthermore,

$$\begin{aligned}
&\left(\lambda^2 - A\mathcal{R}\right)\mathcal{L}\left[\widehat{C}_K(t)\right](\lambda)x \\
&\quad = \lambda^2\mathcal{L}\left[\widehat{C}_K(t)\right](\lambda)x \\
&\qquad - \lambda\mathcal{L}[K(t)](\lambda)A\left(\lambda^2 - \mathcal{R}A\right)^{-1}C\mathcal{R}x \\
&\quad = \lambda\mathcal{L}[K(t)](\lambda)Cx.
\end{aligned} \tag{22}$$

On the other hand, for each $x \in D(A\mathcal{R})$, $\operatorname{Re}\lambda > \max(\omega,\beta)$, we obtain

$$\begin{aligned}
&\frac{\mathcal{L}[K(t)](\lambda)}{\lambda}\left[C + A\left(\lambda^2 - \mathcal{R}A\right)^{-1}C\mathcal{R}\right]\left(\lambda^2 - A\mathcal{R}\right)x \\
&\quad = \lambda\mathcal{L}[K(t)](\lambda)Cx.
\end{aligned} \tag{23}$$

Therefore,

$$\lambda\left(\lambda^2 - A\mathcal{R}\right)^{-1}C = \frac{1}{\lambda}\left[I + A\left(\lambda^2 - \mathcal{R}A\right)^{-1}\mathcal{R}\right]C. \tag{24}$$

It follows from (20) that

$$\mathcal{L}\left[\widehat{C}_K(t)\right](\lambda)x = \lambda\mathcal{L}[K(t)](\lambda)\left(\lambda^2 - A\mathcal{R}\right)^{-1}Cx. \tag{25}$$

Thus, by Definition 1, we know that $A\mathcal{R}$ subgenerates an exponentially bounded K-convoluted C-cosine operator family on X.

(2) Assume that $A\mathcal{R}$ subgenerates an exponentially bounded K-convoluted C-cosine operator family on X and $\rho(\mathcal{R}A) \neq \emptyset$, and let

$$\begin{gathered}
\lambda_0 \in \rho(\mathcal{R}A), \\
E = (\lambda_0 - \mathcal{R}A)\mathcal{R}, \qquad F = A(\lambda_0 - \mathcal{R}A)^{-1}.
\end{gathered} \tag{26}$$

It is not hard to see that E is closed operator on X and

$$F \in L(X), \qquad CE \subseteq EC, \qquad FC = CF. \tag{27}$$

Since $FE = A\mathcal{R}$ subgenerates an exponentially bounded K-convoluted C-cosine operator family on X, we know from (1) that the operator $EF = \mathcal{R}A$ subgenerates an exponentially bounded K-convoluted C-cosine operator family on X.

Noting that $\rho(\mathcal{R}A) \neq \emptyset$ and in view of Lemma 7, we see that $\mathcal{R}A$ generates an exponentially bounded K-convoluted C-cosine operator family on X. □

Theorem 9. *Let A be a subgenerator of an exponentially bounded K-convoluted C-cosine operator family $\{C_K(t)\}_{t\geq 0}$ on X,*

$$S_K(t) = \int_0^t C_K(s)\,ds, \quad t \geq 0, \tag{28}$$

$B \in L(X)$, and $R(B) \subset R(C)$. Suppose that

(H1) *there exists an operator $\mathcal{F} : X \to X$ such that*

$$\mathcal{F}S_K(t)\,x := G_K(t)\,x \in \mathbf{C}([0,\infty), X) \tag{29}$$

is Laplace transformable, and

$$\mathcal{L}(G_K)(\lambda) = \left(\lambda^2 - A\right)^{-1} Cx, \quad x \in X; \tag{30}$$

(H2) *for any $\Phi \in \mathbf{C}([0,\infty), X)$, $\int_0^t G_K(t-s)C^{-1}B\Phi(s)ds \in D(A)$, and*

$$\left\| A \int_0^t G_K(t-s)\,C^{-1}B\Phi(s)\,ds \right\| \leq \widetilde{M} \int_0^t e^{\omega(t-s)}\,\|\Phi(s)\|\,ds, \quad t \geq 0, \tag{31}$$

where $\widetilde{M}$ is a constant;

(H3) *there exists an injective operator $C_1 \in L(X)$ such that $R(C_1) \subset R(C)$ and $C_1A(I+B) \subset A(I+B)C_1$.*

Then,

(1) *$A(I+B)$ subgenerates an exponentially bounded K-convoluted C_1-cosine operator family,*

(2) *if $\rho(A) \neq \emptyset$, then $A(I+B)$ generates an exponentially bounded K-convoluted C_1-cosine operator family;*

(3) *if $\rho((I+B)A) \neq \emptyset$ and $BC_1 = C_1B$, $C_1A \subseteq AC_1$, then $(I+B)A$ generates an exponentially bounded K-convoluted C_1-cosine operator family on X.*

Proof. (1) For each $x \in X, t \geq 0$, define

$$\overline{C}_0(t)\,x = C_K(t)\,x,$$
$$\overline{C}_n(t)\,x = A\int_0^t G_K(t-s)\,C^{-1}B\overline{C}_{n-1}(s)\,x\,ds, \quad n = 1, 2, \ldots. \tag{32}$$

Then, the operator family $\{\overline{C}_n(t)\}_{t\geq 0}$ has the following properties:

(i) for any $x \in X, \overline{C}_n(t)x \in \mathbf{C}([0,\infty), X)$;

(ii) $\|\overline{C}_n(t)\| \leq (M\widetilde{M}^n t^n/n!)e^{\omega t}$, $t \geq 0, \forall n \in \mathbb{N}_0$.

Therefore, the following series

$$\sum_{n=0}^{\infty} \overline{C}_n(t)\,C^{-1}C_1, \quad t \geq 0, \tag{33}$$

is uniformly convergent on every compact interval in t, and we set

$$h(t) = \sum_{n=0}^{\infty} \overline{C}_n(t)\,C^{-1}C_1, \quad t \geq 0. \tag{34}$$

Clearly,

$$\|h(t)\| \leq M_1 e^{(\omega+\widetilde{M})t}, \quad t \geq 0, \tag{35}$$

where $M_1 = M\left\|C^{-1}C_1\right\|$, and

$$t \longrightarrow h(t)\,x \text{ is continuous on } [0,\infty) \text{ for any } x \in X. \tag{36}$$

Moreover,

$$h(t)\,x = C_K(t)\,C^{-1}C_1x + A\int_0^t G_K(t-s)\,C^{-1}Bh(s)\,x\,ds, \quad x \in X,\ t \geq 0. \tag{37}$$

As in the monograph [15], we write, for sufficiently large λ,

$$\mathcal{L}[h(t)](\lambda)\,x = \int_0^{\infty} e^{-\lambda t}h(t)\,x\,dt, \quad x \in X. \tag{38}$$

Thus, by (5), we have

$$\mathcal{L}[h(t)](\lambda)\,x = \lambda\mathcal{L}[K(t)](\lambda)\left(\lambda^2 - A\right)^{-1}C_1x + A\left(\lambda^2 - A\right)^{-1}B\mathcal{L}[h(t)](\lambda)\,x, \quad x \in X. \tag{39}$$

This implies that

$$\begin{aligned} &R((I+B)\,\mathcal{L}[h(t)](\lambda)) \subseteq D(A), \\ &\left(\lambda^2 - A(I+B)\right)\mathcal{L}[h(t)](\lambda)\,x \\ &\quad = \lambda^2\mathcal{L}[h(t)](\lambda)\,x - \lambda\mathcal{L}[K(t)](\lambda)\,A\left(\lambda^2 - A\right)^{-1}C_1x \\ &\quad\quad - \lambda^2 A\left(\lambda^2 - A\right)^{-1}B\mathcal{L}[h(t)](\lambda)\,x \\ &\quad = \lambda\mathcal{L}[K(t)](\lambda)\,C_1x, \quad x \in X. \end{aligned} \tag{40}$$

Let

$$U(t)x = A\int_0^t G_K(t-s)C^{-1}Bxds, \quad x \in X,\ t \geq 0. \tag{41}$$

Then, for large λ, we have

$$\|\lambda\mathscr{L}[U(t)](\lambda)\| = \left\|\lambda\int_0^\infty e^{-\lambda t}U(t)\,dt\right\| \leq \frac{\widetilde{M}}{\lambda-\omega}. \tag{42}$$

So, for sufficiently large λ,

$$\|\lambda\mathscr{L}[K(t)](\lambda)\| = \left\|A(\lambda^2-A)^{-1}B\right\| < 1. \tag{43}$$

This means that the operator $I - A(\lambda^2-A)^{-1}B$ is invertible.

On the other hand, since $\lambda^2 - A$ and $I - A(\lambda^2-A)^{-1}B$ are injective, and

$$(\lambda^2-A)\left(I-A(\lambda^2-A)^{-1}B\right)x = (\lambda^2 - A(I+B))x, \quad x \in D(A(I+B)), \tag{44}$$

we infer that $\lambda^2 - A(I+B)$ is injective. This together with (40) implies that

$$\lambda(\lambda^2 - A(I+B))^{-1}C_1x = \frac{1}{\mathscr{L}[K(t)](\lambda)}\int_0^\infty e^{-\lambda t}h(t)\,xdt. \tag{45}$$

By Definition 1, we know that $A(I+B)$ subgenerates an exponentially bounded K-convoluted C_1-cosine operator family on X.

(2) By the proof of (1), we see that the operator $I - A(\lambda^2 - A)^{-1}B$ is invertible, and $\rho(A) \neq \emptyset$ implies that

$$\rho(A(I+B)) \neq \emptyset. \tag{46}$$

In view of Lemma 7, we get

$$C_1^{-1}A(I+B)C_1 = A(I+B). \tag{47}$$

(3) By virtue of Theorem 8 (2), we have the conclusion. □

Remark 10. (1) It is easy to see that if we take

$$\mathscr{F}S_K(t)x := \left(\mathscr{L}^{-1}\left(\frac{1}{\mathscr{L}[K(t)](\lambda)}\right) * S_K\right)(t)x, \tag{48}$$

then (H1) is satisfied.

(2) In Theorem 9, if we take

$$K(t) = \frac{t^{n-1}}{\Gamma(n)}, \qquad \mathscr{F} := \frac{d^n}{dt^n}, \quad n \in \mathbb{N}, \tag{49}$$

then we obtain the perturbations for n-times integrated C-cosine operator families.

(3) In Theorem 9, if we take

$$K(t) \equiv \frac{1}{t} \quad (t \neq 0) \tag{50}$$

and $\mathscr{F} := I$, then we have the multiplicative perturbations on the exponentially bounded C-cosine operator families.

By Theorem 9, we can immediately deduce the following theorem on K-convoluted C-semigroups.

Theorem 11. *Let A be a subgenerator of an exponentially bounded K-convoluted C-semigroup $\{T_K(t)\}_{t\geq 0}$ on X, $B \in L(X)$ and $R(B) \subset R(C)$. Suppose that*

(H1) *there exists an operator $\mathscr{F} : X \to X$ such that*

$$\mathscr{F}T_K(t)x := H_K(t)x \in \mathbf{C}([0,\infty), X) \tag{51}$$

is Laplace transformable, and

$$\mathscr{L}(H_K)(\lambda) = (\lambda - A)^{-1}Cx, \quad x \in X; \tag{52}$$

(H2) *for any $\Phi \in \mathbf{C}([0,\infty), X)$, $\int_0^t H_K(t-s)C^{-1}B\Phi(s)ds \in D(A)$, and*

$$\left\|A\int_0^t H_K(t-s)C^{-1}B\Phi(s)\,ds\right\| \leq \overline{M}\int_0^t e^{\omega(t-s)}\|\Phi(s)\|\,ds, \quad t \geq 0, \tag{53}$$

where $\overline{M}$ is a constant;

(H3) *there exists an injective operator $C_1 \in L(X)$ such that $R(C_1) \subset R(C)$ and $C_1A(I+B) \subset A(I+B)C_1$.*

Then,

(1) *$A(I+B)$ subgenerates an exponentially bounded K-convoluted C_1-semigroup on X;*

(2) *if $\rho(A) \neq \emptyset$, then $A(I+B)$ generates an exponentially bounded K-convoluted C_1-semigroup on X.*

(3) *if $\rho((I+B)A) \neq \emptyset$, then $(I+B)A$ generates an exponentially bounded K-convoluted C_1-semigroup on X.*

Remark 12. (1) In Theorem 11, if we take

$$K(t) := \frac{t^{n-1}}{\Gamma(n)}, \qquad \mathscr{F} := \frac{d^n}{dt^n}, \quad n \in \mathbb{N}, \tag{54}$$

then we obtain the perturbations for n-times integrated C-semigroups.

(2) In Theorem 11, if we take

$$K(t) := \frac{1}{t} \quad (t \neq 0) \tag{55}$$

and $\mathscr{F} := I$, then we have the multiplicative perturbations on the exponentially bounded C-semigroups.

3. Additive Perturbation Theorem

Theorem 13. *Let $B \in L(X)$, $R(B) \subset R(C)$, and there exists an injective operator $C_1 \in L(X)$ such that $R(C_1) \subset R(C)$ and $C_1(A+B) \subset (A+B)C_1$.*

(i) *Suppose that A is a subgenerator of an exponentially bounded K-convoluted C-cosine operator family $\{C_K(t)\}_{t\geq 0}$ on X. If there exists an operator $\mathscr{F} : X \to X$ such that*

$$\mathscr{F}C_K(t)x := G_K(t)x \in \mathbf{C}([0,\infty), X) \tag{56}$$

is Laplace transformable, and

$$\mathscr{L}(G_K)(\lambda) = (\lambda^2 - A)^{-1} Cx, \quad x \in X, \tag{57}$$

then $A + B$ subgenerates an exponentially bounded K-convoluted C_1-cosine operator family $\{\widehat{C}_K(t)\}_{t\geq 0}$ on X, where

$$\widehat{C}_K(t)x = C_K(t)C^{-1}C_1 x + \int_0^t S_K(t-s)C^{-1}B\widehat{C}_K(t)xds, \quad t \geq 0,\ x \in X,$$
$$S_K(t)x = \int_0^t C_K(s)xds, \quad t \geq 0,\ x \in X. \tag{58}$$

(ii) *Suppose that A is a subgenerator of an exponentially bounded K-convoluted C-semigroup $\{T_K(t)\}_{t\geq 0}$ on X. If there exists an operator $\mathscr{F} : X \to X$ such that*

$$\mathscr{F} T_K(t)x := H_K(t)x \in \mathbf{C}([0,\infty), X) \tag{59}$$

is Laplace transformable, and

$$\mathscr{L}(H_K)(\lambda) = (\lambda - A)^{-1} Cx,\ x \in X, \tag{60}$$

then $A + B$ subgenerates an exponentially bounded K-convoluted C_1-semigroup $\{\widehat{T}_K(t)\}_{t\geq 0}$ on X, where

$$\widehat{T}_K(t)x = T_K(t)C^{-1}C_1 x + \int_0^t T_K(t-s)C^{-1}B\widehat{T}_K(s)xds, \quad t \geq 0,\ x \in X. \tag{61}$$

Proof. Replacing (37) with the following equality:

$$h(t)x = C_K(t)C^{-1}C_1 x + \int_0^t G_K(t-s)C^{-1}Bh(s)xds, \quad x \in X,\ t \geq 0, \tag{62}$$

and by the arguments similar to those in the proof of Theorem 9, we can prove (i).

Point (ii) can also be deduced by a similar way. □

Remark 14. In Theorem 13, if we take

$$K(t) = \frac{t^{n-1}}{\Gamma(n)}, \qquad \mathscr{F} := \frac{d^n}{dt^n}, \quad n \in \mathbb{N}_0, \tag{63}$$

then we obtain an additive perturbation theorem for the exponentially bounded n-times integrated C_1-cosine operator families (resp., n-times integrated C_1-semigroups) as well as C_1-cosine operator families (resp., 0-times integrated C_1-semigroup).

4. Examples

Example 1. Let

$$X := C_0(\mathbb{R}) \oplus C_0(\mathbb{R}) \oplus C_0(\mathbb{R}),$$
$$A(f,g,h)(\cdot) := \left(f', g', \left(\chi_{[0,\infty)} - \chi_{(-\infty,0]}\right)h\right), \tag{64}$$

where

$$(f,g,h) \in D(A) = \left\{(f,g,h) \in X : f' \in C_0(\mathbb{R}), g' \in C_0(\mathbb{R}), h(0) = 0\right\},$$
$$C(f,g,h) := (f, g, \sin(\cdot)h(\cdot)), \quad f,g,h \in C_0(\mathbb{R}). \tag{65}$$

As in [22, Examples 8.1 and 8.2], we can prove that A is the generator of an exponentially bounded once integrated C-semigroup ([15]).

Define

$$B(f,g,h)(t) = \left(e^{-t}\cos t \int_0^t f(s)ds,\ e^{-2t}\cos 2t \times \int_0^t g(s)ds,\ te^{-6t}\sin t \cdot h(t)\right), \tag{66}$$

for every $t \in \mathbb{R}$, and $f, g, h \in C_0(\mathbb{R})$. Then, we can simply verify $B \in L(X)$, $R(B) \subset C(D(A))$, and

$$BC(f,g,h) = CB(f,g,h), \quad (f,g,h) \in X. \tag{67}$$

Therefore, taking

$$K(t) \equiv 1, \qquad \mathscr{F} := \frac{d}{dt} \tag{68}$$

and using Remark 12 (1), we know that $A(I + B)$ subgenerates an exponentially bounded once integrated C-semigroup on X.

Example 2. Let $X_1 = L^\infty(\mathbb{R})$, $X_2 = L^2(\mathbb{R})$,

$$A_1 = \frac{d^2}{d\xi^2}, \qquad D(A_1) = W^{2,\infty}(\mathbb{R}),$$
$$A_2 = \frac{d^2}{d\xi^2}, \qquad D(A_2) = H^2(\mathbb{R}). \tag{69}$$

It follows from [15] that A_1 generates an exponentially bounded C_1-cosine operator family $C_1(\cdot)$ on X_1, where $C_1 = (1 - d^2/d\xi^2)^{-1}$. Moreover, it is well known that A_2 generates a strongly continuous cosine operator family $C_2(\cdot)$ on X_2.

Let

$$b_1(\cdot) \in W^{4,\infty}(\mathbb{R}), \qquad b_2(\cdot) \in H^2(\mathbb{R}), \tag{70}$$

and define $B_1 : X_2 \to X_1$, $B_2 : X_1 \to X_2$ as follows:

$$(B_1\phi)(\xi) = b_1(\xi)\int_0^1 \phi(\sigma)d\sigma,$$
$$(B_2\phi)(\xi) = b_2(\xi)\int_0^1 \phi(\sigma)d\sigma. \tag{71}$$

Set $X = X_1 \times X_2$,

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}, \qquad D(A) := D(A_1) \times D(A_2),$$
$$B = \begin{pmatrix} 0 & B_1 \\ B_2 & 0 \end{pmatrix}, \qquad D(B) := X. \tag{72}$$

Clearly, $\rho(A) \neq \emptyset$ and $D(A_1) = R(C_1)$. Take

$$\lambda_0 \in \rho(A), \qquad C = (\lambda_0 - A)^{-1}. \tag{73}$$

Then, A generates an exponentially bounded C-cosine operator family $C(\cdot)$ on X, where

$$C(t) = \begin{pmatrix} C_1(t) C_1^{-1} (\lambda_0 - A_1)^{-1} & 0 \\ 0 & C_2(t) (\lambda_0 - A_2)^{-1} \end{pmatrix}. \tag{74}$$

Hence,

$$S(t) = \begin{pmatrix} S_1(t) C_1^{-1} (\lambda_0 - A_1)^{-1} & 0 \\ 0 & S_2(t) (\lambda_0 - A_2)^{-1} \end{pmatrix}, \tag{75}$$

where

$$S(t) := \int_0^t C(s)\, ds, \qquad S_1(t) := \int_0^t C_1(s)\, ds, \tag{76}$$

$$S_2(t) := \int_0^t C_2(s)\, ds. \tag{77}$$

Therefore, we have, for each $f = \binom{f_1}{f_2} \in \mathbf{C}([0,\infty), X)$,

$$\int_0^t S(t-s) C^{-1} B f(s)\, ds = \begin{pmatrix} A_1 \int_0^t S_1(t-s) C_1^{-1} B_1 f_2(s)\, ds \\ A_2 \int_0^t S_2(t-s) B_2 f_1(s)\, ds \end{pmatrix}. \tag{78}$$

Since

$$R(B_1) \subset D(A_1 C_1^{-1}), \qquad R(B_2) \subset D(A_2), \tag{79}$$

we see that there exist $M, \omega > 0$ such that

$$\left\| \int_0^t S(t-s) C^{-1} B f(s)\, ds \right\| \leq M \int_0^t e^{\omega(t-s)} \| f(s) \|\, ds, \quad t \geq 0. \tag{80}$$

Consequently, if there exists an injective operator $\widetilde{C} \in L(X)$ such that $R(\widetilde{C}) \subset R(C)$ and $\widetilde{C}(A+B) \subset (A+B)\widetilde{C}$, then taking

$$K(t) \equiv \frac{1}{t}, \quad \mathcal{F} := I \tag{81}$$

and using Remark 14, we know that $A+B$ subgenerates a $\widetilde{C}$-cosine operator family on X.

Moreover, it is not hard to see that there exist $\widehat{M}, \omega > 0$ such that

$$\left\| A \int_0^t S(t-s) C^{-1} B f(s)\, ds \right\| \leq \widehat{M} \int_0^t e^{\omega(t-s)} \| f(s) \|\, ds, \quad t \geq 0. \tag{82}$$

Hence, if there exists an injective operator $\widehat{C} \in L(X)$ such that $R(\widehat{C}) \subset R(C)$ and $\widehat{C}A(I+B) \subset A(I+B)\widehat{C}$, then by Remark 10 (3) ($\rho(A) \neq \emptyset$), we know that $A(I+B)$ generates a $\widehat{C}$-cosine operator family on X.

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