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Research Article

The Fractional Carleson Measures on the Unit Ball of $\mathbb{R}^{n+1}$

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We construct a quantity in terms of L^p integral of the Jacobian of a conformal self-map on the unit ball of $\mathbb{R}^{n+1}$. Then, we characterize the fractional Carleson measures on the unit ball by the quantity.

1. Introduction

Let $\mathbb{D}$ be the unit disc of the complex plane and let $\partial\mathbb{D}$ be the boundary of $\mathbb{D}$. For any arc $I \subseteq \partial\mathbb{D}$, let $|I| = \int_I d\zeta/2\pi$ be the normalized length of I. The Carleson square based on an arc I is the set

$$S(I) = \left\{z \in \mathbb{D} : |z| > 1 - |I|, \frac{z}{|z|} \in I\right\}. \tag{1}$$

We set $S(I) = \mathbb{D}$ when $I = \partial\mathbb{D}$. Then, for $s > 0$, a nonnegative measure ν on $\mathbb{D}$ is called an s-Carleson measure if there exists a constant $C > 0$ such that

$$\nu(S(I)) \le C|I|^s, \quad \forall I \subset \partial\mathbb{D}. \tag{2}$$

ν is said to be a compact s-Carleson measure if ν is an s-Carleson measure and

$$\lim_{|I|\to 0} \frac{\nu(S(I))}{|I|^s} = 0. \tag{3}$$

The 1-Carleson measure is the classical Carleson measure (see [1, 2]).

Carleson measures are related to certain holomorphic function spaces, such as BMO, Morrey spaces, and Q spaces in a natural way (see [2–5]). Besides these, much research has been done about the characterizations of Carleson measures (see [6–10]). A well-known result is that a nonnegative measure ν on $\mathbb{D}$ is a 1-Carleson measure if and only if

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \frac{1-|a|^2}{|1-\bar{a}z|^2}\, d\nu(z) < \infty \tag{4}$$

(see [2]). The s-Carleson measures can be characterized by modifying (4) (see [4, 7, 10–12]). For instance, in [4], for $\tau, s > 0$, it is shown that a nonnegative measure ν on $\mathbb{D}$ is an s-Carleson measure or a compact s-Carleson measure if and only if

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \frac{\left(1-|a|^2\right)^\tau}{|1-\bar{a}z|^{\tau+s}} d\nu(z) < \infty$$
$$\text{or} \quad \lim_{|a|\to 1^-} \int_{\mathbb{D}} \frac{\left(1-|a|^2\right)^\tau}{|1-\bar{a}z|^{\tau+s}} d\nu(z) = 0. \tag{5}$$

Recently, Wu [13] provided a different way for the characterization of s-Carleson measures in terms of the L^p estimates instead of L^1 in (5). Suppose that $\tau, s > 0$, $1/s \le p < \infty$, and ν is a nonnegative measure on $\mathbb{D}$. For $a \in \mathbb{D}$, $\zeta \in \partial\mathbb{D}$, write

$$\mathscr{A}_\nu(a,\tau,s,p) = \left\| \int_{\Gamma(\zeta)} \frac{\left(1-|a|^2\right)^\tau}{|1-\bar{a}z|^{\tau+s}} \frac{d\nu(z)}{1-|z|} \right\|_{L^p(\partial\mathbb{D})}, \tag{6}$$

where $\Gamma(\zeta) = \{z \in \mathbb{D} : |\zeta - z| < 2(1-|z|)\}$ is the cone in $\mathbb{D}$ with the vertex ζ. In [13], it is shown that

(1) $\sup_{a\in\mathbb{D}} \mathscr{A}_\nu(a,\tau,s,p) < \infty$ if and only if ν is an $(s+1-1/p)$-Carleson measure;

(2) $\mathscr{A}_\nu(a,\tau,s,p) \to 0$ as $|a| \to 1^-$ if and only if ν is a compact $(s+1-1/p)$-Carleson measure.

The relations among the Carleson measures, quantities $\mathscr{A}_\nu(a,\tau,s,p)$, and some function spaces defined on $\partial\mathbb{D}$

are also displayed, which are applied to characterizing the boundedness and compactness of Volterra-type operators from Hardy spaces to some holomorphic spaces. One can refer to [13] for more details.

In order to show that the Jacobian of a conformal self-map of the unit ball $\mathbb{B}$ in $\mathbb{R}^{n+1}$ obeys the weak Harnack inequality, Kotilainen et al. introduced an integral form of the fractional Carleson measures on the unit ball $\mathbb{B}$ (see [14]). For $\omega \in \mathbb{B} \setminus \{0\}$, set

$$E(\omega) = \left\{ z \in \mathbb{B} : \left| z - \frac{\omega}{|\omega|} \right| < 1 - |\omega| \right\}, \tag{7}$$

and set $E(0) = \mathbb{B}$. For $t > 0$, a nonnegative measure μ on $\mathbb{B}$ is called a t-Carleson measure or a compact t-Carleson measure if and only if

$$\sup_{\omega\in\mathbb{B}} \frac{\mu(E(\omega))}{(1-|\omega|)^t} < \infty \quad \text{or} \quad \lim_{|\omega|\to 1^-} \frac{\mu(E(\omega))}{(1-|\omega|)^t} = 0 \tag{8}$$

(see [14, 15]). For $z_0 \in \mathbb{B}$, a conformal self-map T_{z_0} of $\mathbb{B}$ is defined by

$$T_{z_0}(\omega) = \frac{\left(1-|z_0|^2\right)(\omega - z_0) - |\omega - z_0|^2 \omega}{1 + |\omega|^2 |z_0|^2 - 2\omega \cdot z_0}. \tag{9}$$

Let $T'_{z_0}(\omega)$ stand for the Jacobian matrix of T_{z_0} at $\omega \in \mathbb{B}$. Then the Jacobian of T_{z_0} is

$$\left| T'_{z_0}(\omega) \right| = \frac{1 - |z_0|^2}{1 + |\omega|^2 |z_0|^2 - 2\omega \cdot z_0}. \tag{10}$$

For more details about this conformal self-map, one can refer to [16–19]. Then, it is shown in [14] that μ is a t-Carleson measure or a compact t-Carleson measure on $\mathbb{B}$ if and only if

$$\begin{aligned} &\sup_{z_0\in\mathbb{B}} \int_{\mathbb{B}} \left| T'_{z_0}(\omega) \right|^t d\mu(\omega) < \infty \\ &\text{or} \quad \lim_{|z_0|\to 1^-} \int_{\mathbb{B}} \left| T'_{z_0}(\omega) \right|^t d\mu(\omega) = 0, \end{aligned} \tag{11}$$

which is the analogue of (4) and (5).

Pursuing the above, in this paper, analogically to (6), we will construct a quantity on the unit ball $\mathbb{B}$ by using the Jacobian of T_{z_0} and establish the connections between the fractional Carleson measures on $\mathbb{B}$ and the quantity. In Section 2, we give some preliminaries, which contain the fractional Carleson measure defined in terms of tents or Carleson boxes. In Section 3, we state our main results and their proof. The results are the extension of the ones in [13], and the real analysis techniques used in this paper should have an application in studying the operators on the function spaces defined on the unit sphere in future.

2. Preliminaries

Throughout this paper, C denotes a positive constant that may change from one step to the next. For fixed $z \in \mathbb{B}$, we call the set

$$\sigma\left(\frac{z}{|z|}, 1-|z|\right) = \left\{ x \in \mathbb{S}^n : \left| x - \frac{z}{|z|} \right| < 2(1-|z|) \right\} \tag{12}$$

a spherical cap centered on $z/|z|$ with radius $1-|z|$. It is easy to see that $\sigma(z/|z|, 1-|z|)$ is the projection of $E(z)$ on the unit sphere. We always write a spherical cap as σ without pointing out its center and radius, if there are no confusing cases. For a spherical cap σ, we also denote the radius of σ by $r(\sigma)$ and the Lebesgue measure of σ by $|\sigma|$. Clearly, there is the estimate

$$|\sigma| \asymp (r(\sigma))^n, \tag{13}$$

where we say that F and G are equivalent, denoted by $F \asymp G$, if there are two positive constants c and C such that $cF \le G \le CF$. The Carleson box based on a spherical cap σ is defined by

$$S(\sigma) = \left\{ z \in \mathbb{B} : \frac{z}{|z|} \in \sigma,\ 1 - r(\sigma) < |z| \le 1 \right\}. \tag{14}$$

The tent based on σ is defined by

$$T(\sigma) = \left\{ z \in \mathbb{B} : \sigma\left(\frac{z}{|z|}, 1-|z|\right) \subset \sigma \right\}. \tag{15}$$

The cone $\Gamma(x)$ in $\mathbb{B}$ with the vertex $x \in \mathbb{S}^n$ is defined by

$$\Gamma(x) = \left\{ z \in \mathbb{B} : x \in \sigma\left(\frac{z}{|z|}, 1-|z|\right) \right\}. \tag{16}$$

For any fixed $z \in \mathbb{B}$, set

$$\sigma(z) = \left\{ x \in \mathbb{S}^n : z \in \Gamma(x) \right\}. \tag{17}$$

Clearly, if $z \ne 0$, $\sigma(z)$ is just the spherical cap in $\mathbb{S}^n$ with center $z/|z|$ and radius $1-|z|$.

For $z \in \mathbb{B}$ and a measurable function f defined on $\mathbb{S}^n$, we denote by

$$\begin{aligned} P(f)(z) &= \int_{\mathbb{S}^n} f(x)\, p(x,z)\, dx \\ &= \int_{\mathbb{S}^n} f(x) \frac{1-|z|^2}{|\mathbb{S}^n|\, |x-z|^{n+1}} dx \end{aligned} \tag{18}$$

the Poisson extension of f onto $\mathbb{B}$. The nontangential maximal function of $P(f)$ is the function

$$P^*(f)(x) = \sup_{z\in\Gamma(x)} \left| P(f)(z) \right| \tag{19}$$

defined on $\mathbb{S}^n$.

For $g \in L^1(\mathbb{S}^n)$ and any $z \in \mathbb{B}$, we write

$$T(g)(z) = \frac{1}{|\sigma(z)|}\int_{\sigma(z)} g(x)\,dx, \tag{20}$$

which is also an extension of g onto $\mathbb{B}$.

For $x \in \mathbb{S}^n$, we call the function

$$M(f)(x) = \sup_{0<r<1}\frac{1}{|\sigma(x,r)|}\int_{\sigma(x,r)} |f(y)|\,dy \tag{21}$$

the centered Hardy-Littlewood maximal function of f defined on $\mathbb{S}^n$.

The following two lemmas give a lower estimate and an upper estimate for the Poisson integral on $\mathbb{B}$. In the case of $n = 1$, one can refer to [20, Theorems 2.4 and 2.5].

Lemma 1. *There exists a constant C, such that*

$$T(g)(z) < CP(g)(z), \quad \forall z \in \mathbb{B} \tag{22}$$

holds for all $g \geq 0$ and $g \in L^1(\mathbb{S}^n)$, where $P(g)(z)$ is the Poisson extension of g.

Proof. It is sufficient to prove that the estimate

$$\frac{1}{|\sigma(z)|}\chi_{\sigma(z)}(x) \leq C\frac{1-|z|^2}{|x-z|^{n+1}}, \quad \forall x \in \mathbb{S}^n \tag{23}$$

holds for any $z \in \mathbb{B}$. Since $1 - |z| \geq C|x - z|$ as $x \in \sigma(z)$, by the estimate in (13), it is easy to obtain the conclusion. □

Lemma 2. *There exists a constant C, such that*

$$P(g)(z) < CT(P^*(g))(z), \quad \forall z \in \mathbb{B} \tag{24}$$

holds for all $g \geq 0$ and $g \in L^1(\mathbb{S}^n)$.

Proof. Clearly, it is the consequence of the definitions of $T(P^*(g))(z)$ and $\Gamma(x)$. Noticing that $P^*(g)(x) \geq P(g)(z)$ as $x \in \sigma(z)$, we have

$$\begin{aligned} T(P^*(g))(z) &= \frac{1}{|\sigma(z)|}\int_{\sigma(z)} P^*(g)(x)\,dx \\ &\geq P(g)(z)\frac{1}{|\sigma(z)|}\int_{\sigma(z)} dx \end{aligned} \tag{25}$$

for any $z \in \mathbb{B}$. Using (13), it implies the required conclusion. □

The following two lemmas are well known. One is the generalized maximal theorem; the other is the fact that the nontangential maximal function can be pointwise controlled by maximal function (see [21]).

Lemma 3. *Let f be a measurable function defined on $\mathbb{S}^n$.*

(a) *If $f \in L^p(\mathbb{S}^n)$ and $1 \leq p \leq \infty$, $M(f)$ is finite almost everywhere.*

(b) *If $f \in L^1(\mathbb{S}^n)$, then for any $\alpha > 0$,*

$$\left|\{x \in \mathbb{S}^n : M(f)(x) > \alpha\}\right| \leq \frac{C}{\alpha}\int_{\mathbb{S}^n} |f(x)|\,dx. \tag{26}$$

(c) *If $f \in L^p(\mathbb{S}^n)$ and $1 < p \leq \infty$, then $M(f) \in L^p(\mathbb{S}^n)$ and*

$$\|M(f)\|_{L^p(\mathbb{S}^n)} \leq A_p\|f\|_{L^p(\mathbb{S}^n)}, \tag{27}$$

where A_p depends only on C and p.

Lemma 4. *If $f \in L^p(\mathbb{S}^n)$, $p \geq 1$, then $P^*(f)(x) \leq CM(f)(x)$ holds for almost every $x \in \mathbb{S}^n$.*

For the convenience, we define Carleson measures on $\mathbb{B}$ in terms of Carleson boxes or tents.

Definition 5. Let $s > 0$ and σ a spherical cap in $\mathbb{S}^n$. A nonnegative measure μ on $\mathbb{B}$ is called an s-Carleson measure if there exits a constant C such that

$$\mu(S(\sigma)) \leq C|\sigma|^s, \quad \forall \sigma \subset \mathbb{S}^n. \tag{28}$$

μ is called a compact s-Carleson measure if μ is an s-Carleson measure and

$$\lim_{|\sigma|\to 0}\frac{\mu(S(\sigma))}{|\sigma|^s} = 0. \tag{29}$$

Remark 6. (1) Comparing Definition 5 with the definition in (8), one can see that the s in Definition 5 is equal to nt in (8).

(2) For any $\sigma \subseteq \mathbb{S}^n$, we can replace $S(\sigma)$ with $T(\sigma)$ in Definition 5.

(3) For $n = 1$, it goes back to the one in [2].

For $0 < p \leq q < \infty$, a well-known result which is due to Carleson in [1] for $p = q$ and Duren in [22] for $p < q$ says that a nonnegative measure μ on the unit disc $\mathbb{D}$ of the complex plane $\mathbb{C}$ is a bounded (q/p)-Carleson measure if and only if

$$\int_{\mathbb{D}} |f(z)|^q d\mu(z) \leq \|f\|_{H^p}^q, \quad \forall f \in H^p(\mathbb{D}). \tag{30}$$

By Definition 5 and using the real analysis techniques, we obtain the following extension of this result on $\mathbb{B}$.

Theorem 7. *Let μ be a nonnegative measure on $\mathbb{B}$.*

(a) *For $0 < p \leq q < \infty$, let $\phi(z)$ be a μ-measurable function defined on $\mathbb{B}$ and let ϕ^* be the nontangential maximal function of $\phi(z)$. If μ is a (q/p)-Carleson measure on $\mathbb{B}$, then*

$$\int_{\mathbb{B}} |\phi(z)|^q d\mu(z) \leq C\|\phi^*\|_{L^p(\mathbb{S}^n)}^q. \tag{31}$$

(b) *For $1 < p \leq q < \infty$, μ is a (q/p)-Carleson measure on $\mathbb{B}$ if and only if*

$$\int_{\mathbb{B}} |P(f)(z)|^q d\mu(z) \leq C\|f\|_{L^p(\mathbb{S}^n)}^q \tag{32}$$

for any $f \in L^p(\mathbb{S}^n)$, and here $P(f)$ is the Poisson integral of f.

Proof. It is sufficient to (31) if

$$\int_{\mathbb{B}} |\varphi(z)| \, d\mu(z) \le C\|\varphi^*\|_{L^{p/q}(\mathbb{S}^n)}. \tag{33}$$

Indeed, write $\varphi(z) = |\phi(z)|^q$, and above if the inequality holds, then

$$\begin{aligned}\int_{\mathbb{B}} |\phi(z)|^q d\mu(z) &\le C\left(\int_{\mathbb{S}^n} |\varphi^*|^{p/q} dx\right)^{q/p} \\ &= C\left(\int_{\mathbb{S}^n} |\phi^*|^p dx\right)^{q/p},\end{aligned} \tag{34}$$

which is what we need.

Suppose that μ is a (q/p)-Carleson measure on $\mathbb{B}$. Write $\Omega = \{x \in \mathbb{S}^n : \varphi^*(x) > \alpha\}$. Recalling the Whitney decomposition (see [21]), we know that there exists a disjoint collection of spherical caps $\{\sigma^k\}$ such that $\bigcup_k \sigma_*^k = \Omega$, and here σ_*^k is the spherical cap with the same center as σ^k but radius C times. Now, we claim that

$$\{z \in \mathbb{B} : \varphi(z) > \alpha\} \subseteq \bigcup_k T\left(\sigma_*^k\right). \tag{35}$$

Indeed, let $z \in \mathbb{B}$ so that $\varphi(z) > \alpha$. By the definition of φ^*, we have that $\varphi^*(x) > \alpha$ for all $x \in \mathbb{S}^n$ satisfying $x \in \sigma(z/|z|, 1 - |z|)$. Thus, $\sigma(z/|z|, 1 - |z|) \subseteq \Omega = \bigcup_k \sigma_*^k$, and clearly $z \in \bigcup_k T(\sigma_*^k)$.

Clearly, we have

$$\begin{aligned}&\mu(\{z \in \mathbb{B} : \varphi(z) > \alpha\}) \\ &\quad\le \sum_k \mu\left(T\left(\sigma_*^k\right)\right) \le C\sum_k \left|\sigma_*^k\right|^{q/p} \\ &\quad\le C\sum_k \left|\sigma^k\right|^{q/p} \le C|\Omega|^{q/p}.\end{aligned} \tag{36}$$

Thus, we have

$$\begin{aligned}\int_{\mathbb{B}} |\varphi(z)| \, d\mu(z) &= \int_0^\infty \mu\{z \in \mathbb{B} : \varphi(z) > \alpha\} \, d\alpha \\ &\le C\int_0^\infty |\{x \in \mathbb{S}^n : \varphi^*(x) > \alpha\}|^{q/p} d\alpha \\ &\le C\int_0^\infty \left(\int_{\mathbb{S}^n} \chi_{\varphi^*(x)>\alpha} dx\right)^{q/p} d\alpha.\end{aligned} \tag{37}$$

By Minkowski's inequality acting on the last inequality, we have

$$\int_{\mathbb{B}} |\varphi(z)| \, d\mu(z) \le C\left(\int_{\mathbb{S}^n} |\varphi^*|^{p/q} dx\right)^{q/p} = C\|\varphi^*\|_{L^{p/q}(\mathbb{S}^n)}. \tag{38}$$

Then, (31) follows.

Now, turn to the proof of the "only if" part of (b). If μ is a (q/p)-Carleson measure, we can obtain

$$\int_{\mathbb{B}} |P(f)(z)|^q d\mu(z) \le C\|P^*(f)\|_{L^p(\mathbb{S}^n)}^q \tag{39}$$

as a direct sequence of (a). Noting Lemmas 3 and 4, we complete the proof of the "only if" part of (b).

The proof of the "if" part of (b) is easy. For any spherical cap σ in $\mathbb{S}^n$, if $z \in T(\sigma)$, we have $\sigma(z) \subseteq \sigma$. Let $f = \chi_\sigma(x)$. Since the simple fact $|x - z| \le C(1 - |z|)$ as $x \in \sigma(z)$ and the estimate in (13), we obtain the useful estimate

$$\begin{aligned}P(f)(z) &= \int_{\mathbb{S}^n} \chi_\sigma(x) \, p(x, z) \, dx \\ &\ge C\int_{\sigma(z)} \frac{1}{(1 - |z|)^n} dx \ge C.\end{aligned} \tag{40}$$

Now, it is easy to see

$$\begin{aligned}\mu(T(\sigma)) &\le C\int_{T(\sigma)} |P(f)(z)|^q d\mu(z) \\ &\le C\left(\int_{\mathbb{S}^n} |\chi_\sigma(x)|^p dx\right)^{q/p} = C|\sigma|^{q/p},\end{aligned} \tag{41}$$

which is to say that μ is a (q/p)-Carleson measure. The proof of (b) is completed. □

Let σ be a spherical cap in $\mathbb{S}^n$ with center $x_0 = z_0/|z_0|$ and radius $r(\sigma) = 1 - |z_0|$. Let j be a nonnegative integer and, N be the greatest integer less than $\log_2(1/r(\sigma))$. Denote the spherical cap with the same center as σ but radius $2^j r(\sigma)$ by σ_j. Then, $\sigma_0 = \sigma$ and $\sigma_{N+1} = \mathbb{S}^n$. Moreover, for any spherical cap $\sigma \subseteq \mathbb{S}^n$, we have

$$\mathbb{B} = S(\sigma) \bigcup \left(\bigcup_{j=0}^{N} \left(S\left(\sigma_{j+1}\right) \setminus S\left(\sigma_j\right)\right)\right). \tag{42}$$

Lemma 8. *For fixed $z_0 \in \mathbb{B}$, let σ be a spherical cap in $\mathbb{S}^n$ with center $x_0 = z_0/|z_0|$ and radius $r(\sigma) = 1 - |z_0|$. Then, one has the following estimates:*

(i) $1 + |\omega|^2|z_0|^2 - 2\omega \cdot z_0 \asymp (r(\sigma))^2$, *if* $\omega \in S(\sigma)$,

(ii) $1+|\omega|^2|z_0|^2-2\omega \cdot z_0 \asymp 2^{2j}(r(\sigma))^2$, *if* $\omega \in S(\sigma_{j+1}) \setminus S(\sigma_j)$, $j \in [0, N]$.

Proof. If $\omega \in S(\sigma)$, then we have the estimates

$$\begin{aligned}&1 + |\omega|^2|z_0|^2 - 2\omega \cdot z_0 \\ &\quad\ge 1 + |\omega|^2|z_0|^2 - 2|\omega||z_0| \\ &\quad= (1 - |\omega||z_0|)^2 \\ &\quad\ge (1 - |z_0|)^2, \\ &1 + |\omega|^2|z_0|^2 - 2\omega \cdot z_0 \\ &\quad= 1 + |\omega|^2|z_0|^2 - \left(|z_0|^2 + |\omega|^2 - |z_0 - \omega|^2\right) \\ &\quad= \left(1 - |\omega|^2\right)\left(1 - |z_0|^2\right) + |z_0 - \omega|^2 \\ &\quad\le C(1 - |z_0|)^2.\end{aligned} \tag{43}$$

Part (i) is yielded by the above.

For part (ii), if $\omega \in S(\sigma_{j+1}) \setminus S(\sigma_j)$, we have

$$\begin{aligned} 1 + |\omega|^2|z_0|^2 - 2\omega\cdot z_0 &= 1 + |\omega|^2|z_0|^2 \\ &\quad - \left(|z_0|^2 + |\omega|^2 - |z_0-\omega|^2\right) \\ &= \left(1-|\omega|^2\right)\left(1-|z_0|^2\right) + |z_0-\omega|^2 \\ &\asymp 2^{2j}(r(\sigma))^2. \end{aligned} \tag{44}$$

The proof is completed. □

Combining together Definition 5, the decomposition of $\mathbb{B}$ in (42), and Lemma 8, we can obtain the following characterization for s-Carleson measures on $\mathbb{B}$, which is similar to [14, Theorems 2.3 and 2.4].

Theorem 9. *Let $s > 0$ and $0 < \tau < \infty$. A nonnegative measure μ on $\mathbb{B}$ is an s-Carleson measure or a compact s-Carleson measure if and only if*

$$\sup_{z_0\in\mathbb{B}} \int_{\mathbb{B}} \left(\frac{\left(1-|z_0|^2\right)^\tau}{\left(1+|\omega|^2|z_0|^2-2\omega\cdot z_0\right)^{(1+\tau)/2}}\right)^{ns} d\mu(\omega) < \infty, \tag{45}$$

or

$$\lim_{|z_0|\to 1^-} \int_{\mathbb{B}} \left(\frac{\left(1-|z_0|^2\right)^\tau}{\left(1+|\omega|^2|z_0|^2-2\omega\cdot z_0\right)^{(1+\tau)/2}}\right)^{ns} d\mu(\omega) = 0. \tag{46}$$

In particular, when $\tau = 1$, μ is an s-Carleson measure or a compact s-Carleson measure on $\mathbb{B}$ if and only if

$$\begin{aligned} &\sup_{z_0\in\mathbb{B}} \int_{\mathbb{B}} \left|T'_{z_0}(\omega)\right|^{ns} d\mu(\omega) < \infty, \\ &\text{or} \quad \lim_{|z_0|\to 1^-} \int_{\mathbb{B}} \left|T'_{z_0}(\omega)\right|^{ns} d\mu(\omega) = 0. \end{aligned} \tag{47}$$

3. The Carleson Measures Characterized by L^p Behaviors

By the estimate in (13) and Theorem 9, we observe that

$$\begin{aligned} &\int_{\mathbb{B}} \left|T'_{z_0}(\omega)\right|^{ns} d\mu(\omega) \\ &\asymp \int_{\mathbb{B}} \int_{S^n} \chi_{\Gamma(x)}(\omega) \left(\frac{1-|z_0|^2}{1+|\omega|^2|z_0|^2-2\omega\cdot z_0}\right)^{ns} \\ &\qquad \times \frac{dx}{(1-|\omega|)^n} d\mu(\omega) \\ &\asymp \int_{\mathbb{S}^n} \int_{\Gamma(x)} \left(\frac{1-|z_0|^2}{1+|\omega|^2|z_0|^2-2\omega\cdot z_0}\right)^{ns} \frac{d\mu(\omega)}{(1-|\omega|)^n} dx \\ &= \left\| \int_{\Gamma(x)} \left|T'_{z_0}(\omega)\right|^{ns} \frac{d\mu(\omega)}{(1-|\omega|)^n} \right\|_{L^1(\mathbb{S}^n)}. \end{aligned} \tag{48}$$

For $z_0 \in \mathbb{B}$, $s \in \mathbb{R}$, and $0 < p \le \infty$, write

$$\mathfrak{R}_\mu(z_0, s, p) = \left\| \int_{\Gamma(x)} \left|T'_{z_0}(\omega)\right|^{ns} \frac{d\mu(\omega)}{(1-|\omega|)^n} \right\|_{L^p(\mathbb{S}^n)}. \tag{49}$$

By Theorem 9, it is clear that μ is an s-Carleson measure on $\mathbb{B}$ if and only if $\sup_{z_0\in B}\mathfrak{R}_\mu(z_0, s, 1) < \infty$.

For a spherical cap $\sigma \subseteq \mathbb{S}^n$ and $0 < p \le \infty$, define

$$\mathfrak{I}_\mu(S(\sigma), p) = \left\| \int_{\Gamma(x)\cap S(\sigma)} \frac{d\mu(\omega)}{(1-|\omega|)^n} \right\|_{L^p(\mathbb{S}^n)}. \tag{50}$$

It is also clear that μ is an s-Carleson measure on $\mathbb{B}$ if and only if $\mathfrak{I}_\mu(S(\sigma), 1) \le C|\sigma|^s$ for all $\sigma \subseteq \mathbb{S}^n$.

Now, we are going to state our main arguments in this section, which are devoted to establishing the connections between s-Carleson measures on $\mathbb{B}$ and

$$\sup_{z_0\in\mathbb{B}} \mathfrak{R}_\mu(z_0, s, p) < \infty \quad \text{or} \quad \mathfrak{I}_\mu(S(\sigma), p) \le C|\sigma|^s. \tag{51}$$

Here, we emphasiz that the following results have been achieved when $n = 1$ (see [13]).

Theorem 10. *Let $s > 0$, $0 < p \le \infty$, and μ a nonnegative measure on $\mathbb{B}$.*

(i) $\sup_{z_0\in\mathbb{B}}\mathfrak{R}_\mu(z_0, s, p) < \infty$ *if and only if* $\mathfrak{I}_\mu(S(\sigma), p) \le C|\sigma|^s$ *for all spherical cap* $\sigma \subset \mathbb{S}^n$.

(ii) $\lim_{|z_0|\to 1^-}\mathfrak{R}_\mu(z_0, s, p) = 0$ *if and only if* $\mathfrak{I}_\mu(S(\sigma), p) = o(|\sigma|^s)$ *for all spherical cap* $\sigma \subset \mathbb{S}^n$.

Remark 11. (1) In Theorem 10, the Carleson box $S(\sigma)$ can be replaced by the tent $T(\sigma)$.

(2) The results in the above theorem hold for $s \in \mathbb{R}$ on $\mathbb{D}$ (see [13, Theorem 1]), but they do only for $s > 0$ here.

Proof. For any spherical cap σ, there must exist $z_0 \in \mathbb{B}$ such that $\sigma = \sigma(z_0/|z_0|, 1-|z_0|)$. By Lemma 8, we have

$$\left(\frac{1-|z_0|^2}{1+|\omega|^2|z_0|^2-2\omega\cdot z_0}\right)^{ns} \asymp |\sigma|^{-s}, \quad \omega \in S(\sigma). \tag{52}$$

Thus,

$$\begin{aligned} &\int_{\Gamma(x)\cap S(\sigma)} \frac{d\mu(\omega)}{(1-|\omega|)^n} \\ &\le C|\sigma|^s \int_{\Gamma(x)} \left(\frac{1-|z_0|^2}{1+|\omega|^2|z_0|^2-2\omega\cdot z_0}\right)^{ns} \frac{d\mu(\omega)}{(1-|\omega|)^n}. \end{aligned} \tag{53}$$

The "only if" parts of (i) and (ii) are concluded from the above inequality.

To prove the "if" part of (i), let $\sigma = \sigma(z_0/|z_0|, 1-|z_0|)$ and $\sigma_{-1} = \phi$. By the decomposition of $\mathbb{B}$ in (42), it is clear that

$$\Gamma(x) = \Gamma(x) \cap \mathbb{B} = \bigcup_{j=0}^{N+1} \left(\Gamma(x) \cap \left(S(\sigma_j) \setminus S(\sigma_{j-1}) \right) \right). \quad (54)$$

By Lemma 8, we have

$$\begin{aligned} &\int_{\Gamma(x)} \left(\frac{1-|z_0|^2}{1+|\omega|^2|z_0|^2 - 2\omega\cdot z_0} \right)^{ns} \frac{d\mu(\omega)}{(1-|\omega|)^n} \\ &= \sum_{j=0}^{N+1} \int_{\Gamma(x)\cap(S(\sigma_j)\setminus S(\sigma_{j-1}))} \left(\frac{1-|z_0|^2}{1+|\omega|^2|z_0|^2 - 2\omega\cdot z_0} \right)^{ns} \\ &\quad \times \frac{d\mu(\omega)}{(1-|\omega|)^n} \\ &\le C \sum_{j=0}^{N+1} \int_{\Gamma(x)\cap S(\sigma_j)} \frac{|\sigma|^s}{2^{2jns}|\sigma|^{2s}} \frac{d\mu(\omega)}{(1-|\omega|)^n}. \end{aligned} \quad (55)$$

Taking L^p norm on both sides of the above inequality, if $1 \le p \le \infty$, we have

$$\begin{aligned} \mathfrak{R}_\mu(z_0, s, p) &\le C \sum_{j=0}^{N+1} \frac{|\sigma|^s}{2^{2jns}|\sigma|^{2s}} \mathfrak{I}_\mu\left(S(\sigma_j), p\right) \\ &\le C \sum_{j=0}^{N+1} \frac{|\sigma|^s}{2^{2jns}|\sigma|^{2s}} \left|\sigma_j\right|^s \\ &\le C \sum_{j=0}^{N+1} 2^{-jns} \\ &\le C. \end{aligned} \quad (56)$$

If $0 < p < 1$, then

$$\begin{aligned} \left(\mathfrak{R}_\mu(z_0, s, p)\right)^p &\le C \sum_{j=0}^{N+1} \left(\frac{|\sigma|^s}{2^{2jns}|\sigma|^{2s}} \mathfrak{I}_\mu\left(S(\sigma_j), p\right) \right)^p \\ &\le C \sum_{j=0}^{N+1} \left(\frac{|\sigma|^s}{2^{2jns}|\sigma|^{2s}} \left|\sigma_j\right|^s \right)^p \\ &\le C \sum_{j=0}^{N+1} 2^{-jnps} \\ &\le C. \end{aligned} \quad (57)$$

The proof of the "if" part of (i) is completed.

With the same technique used in the proof of "if" part of (i), one can obtain the "if" part of (ii). Suppose that $1 \le p \le \infty$. We observe first that for $\varepsilon > 0$, there must exist an integer $m > 0$ such that

$$C \sum_{j=m}^{N+1} \frac{1}{2^{jns}} < \frac{\varepsilon}{2}. \quad (58)$$

Then, we have

$$\begin{aligned} \mathfrak{R}_\mu(z_0, s, p) &\le C \sum_{j=0}^{N+1} \frac{|\sigma|^s}{2^{2jns}|\sigma|^{2s}} \mathfrak{I}_\mu\left(S(\sigma_j), p\right) \\ &\le C \sum_{j=0}^{m-1} \frac{|\sigma|^s}{2^{2jns}|\sigma|^{2s}} \mathfrak{I}_\mu\left(S(\sigma_m), p\right) + \frac{\varepsilon}{2} \\ &\le \frac{C}{1-2^{-2ns}} \frac{2^{mns}}{|\sigma_m|^s} \mathfrak{I}_\mu\left(S(\sigma_m), p\right) + \frac{\varepsilon}{2}. \end{aligned} \quad (59)$$

By the assumption of the "if" part of (ii), there must exist $\delta > 0$ such that if $|\sigma_m| < \delta$

$$\mathfrak{I}_\mu\left(S(\sigma_m), p\right) < \frac{1-2^{-2ns}}{C2^{mns}} \frac{\varepsilon}{2} \left(|\sigma_m|^s\right). \quad (60)$$

If $(1-|z_0|)^n \asymp |\sigma| < 2^{-mn}\delta$, then we have

$$\mathfrak{R}_\mu(z_0, s, p) < \varepsilon, \quad (61)$$

which is the desired result. With the same process, we can obtain the results when $0 < p < 1$. The proof of Theorem 10 is completed. □

Theorem 12. *Suppose that $s > 0$, $1 \le p < \infty$, and μ is a nonnegative measure on $\mathbb{B}$.*

(i) $\sup_{z_0\in\mathbb{B}} \mathfrak{R}_\mu(z_0, s, p) < \infty$ *or* $\mathfrak{I}_\mu(S(\sigma), p) \le C|\sigma|^s$ *for all spherical cap* $\sigma \subset \mathbb{S}^n$ *if and only if* $|P(f)(z)|d\mu(z)$ *is an s-Carleson measure for all* $f \in L^{p/(p-1)}(\mathbb{S}^n)$.

(ii) $\lim_{|z_0|\to 1^-} \mathfrak{R}_\mu(z_0, s, p) = 0$ *or* $\mathfrak{I}_\mu(S(\sigma), p) = o(|\sigma|^s)$ *for all spherical cap* $\sigma \subset \mathbb{S}^n$ *if and only if* $|P(f)(z)|d\mu(z)$ *is a compact s-Carleson measure for all* $f \in L^{p/(p-1)}(\mathbb{S}^n)$.

Proof. In the situation of $p = 1$, the results are deduced by Theorems 9 and 10 and the estimate

$$\begin{aligned} &\mathfrak{R}_\mu(z_0, s, 1) \\ &\quad = \int_{\mathbb{S}^n} \int_{\Gamma(x)} \left(\frac{1-|z_0|^2}{1+|\omega|^2|z_0|^2 - 2\omega\cdot z_0} \right)^{ns} \frac{d\mu(\omega)}{(1-|\omega|)^n} dx \\ &\quad \asymp \int_{\mathbb{B}} \left(\frac{1-|z_0|^2}{1+|\omega|^2|z_0|^2 - 2\omega\cdot z_0} \right)^{ns} d\mu(\omega). \end{aligned} \quad (62)$$

For $1 < p < \infty$, it is well known that $P^*(f) \in L^{p/(p-1)}(\mathbb{S}^n)$ if $f(x) \in L^{p/(p-1)}(\mathbb{S}^n)$. By Lemma 2, we have

$$P(f)(z) \le T(P^*(f))(z), \quad \forall z \in \mathbb{B}. \quad (63)$$

Now, by the estimate in (13) and Hölder's inequality, we have

$$\begin{aligned}\int_{S(\sigma)} P(f)(z)\,d\mu(z) &\le \int_{S(\sigma)} T(P^*(f))(z)\,d\mu(z)\\ &\le C\int_{S(\sigma)} \frac{1}{|\sigma(z)|}\int_{\sigma(z)} P^*(f)(x)\,dx\,d\mu(z)\\ &\le C\int_{\mathbb{S}^n} P^*(f)(x)\int_{S(\sigma)\bigcap\Gamma(x)} \frac{d\mu(z)}{(1-|z|)^n}dx\\ &\le C\|P^*(f)\|_{L^{p/(p-1)}(\mathbb{S}^n)}\mathfrak{J}_\mu(S(\sigma),p).\end{aligned} \tag{64}$$

Combining with Theorem 10, we complete the proof of "only if" parts.

Turn to the proof of "if" parts. For $f\in L^{p/(p-1)}(\mathbb{S}^n)$ and $f\ge 0$, suppose that $|P(f)(z)|d\mu(z)$ is an s-Carleson measure or a compact s-Carleson measure. By Lemma 1, we know that

$$T(f)(z)\le CP(f)(z),\quad \forall z\in\mathbb{B}, \tag{65}$$

which implies that $T(f)(z)d\mu(z)$ is an s-Carleson measure or a compact s-Carleson measure. By the estimate in (13), we have

$$\int_{\mathbb{S}^n} f(x)\int_{\Gamma(x)\bigcap S(\sigma)} \frac{d\mu(z)}{(1-|z|)^n}dx \asymp C\int_{S(\sigma)} T(f)(z)\,d\mu(z). \tag{66}$$

Using the above estimate, the $L^{p/(p-1)}$ duality, and also Theorem 10, we deduce the desired results. □

Now, we are going to consider the case of $p=\infty$.

Theorem 13. *Let $s>0$ and μ a nonnegative measure on $\mathbb{B}$. Then, the following are equivalent.*

(a1) $\sup_{z_0\in\mathbb{B}}\mathfrak{R}_\mu(z_0,s,\infty)<\infty$ *or* $\mathfrak{J}_\mu(S(\sigma),\infty)\le C|\sigma|^s$ *for all spherical cap* $\sigma\subseteq S^n$.

(b1) $|T(f)(z)|d\mu(z)$ *is an s-Carleson measure for all* $f\in L^1(\mathbb{S}^n)$.

Moreover, the following are equivalent.

(a2) $\sup_{z_0\in\mathbb{B}}\mathfrak{R}_\mu(z_0,s,\infty)<\infty$ *or* $\mathfrak{J}_\mu(S(\sigma),\infty)\le C|\sigma|^s$ *for all spherical cap* $\sigma\subseteq\mathbb{S}^n$.

(b2) $|P(f)(z)|d\mu(z)$ *is an s-Carleson measure for all* $f\in H^1(\mathbb{S}^n)$.

(c2) μ *is an $(s+1)$-Carleson measure.*

The equivalence above holds for the compact case also.

Proof. For $f(x)\in L^1(\mathbb{S}^n)$ and $f\ge 0$, we have

$$\int_{\mathbb{S}^n} f(x)\int_{\Gamma(x)\bigcap S(\sigma)} \frac{d\mu(z)}{(1-|z|)^n}dx \asymp C\int_{S(\sigma)} T(f)(z)\,d\mu(z), \tag{67}$$

which implies the equivalence of (a1) and (b1) by duality theorem and Theorem 10.

Invoking Lemma 2, we observe that "(a2)⇒(b2)" is a direct consequence of the equivalence of (a1) and (b1).

To prove "(b2)⇒(c2)," let z_0 be any fixed point in $\mathbb{B}$ and let

$$f(x)=\left(\frac{1-|z_0|^2}{1+|x|^2|z_0|^2-2x\cdot z_0}\right)^n \tag{68}$$

be a function defined on $\mathbb{S}^n$. For $z\in T(\sigma(z_0))$, one should observe the fact $\sigma(z)\subseteq\sigma(z_0)$ and

$$\begin{aligned}P(f)(z)&=\int_{\mathbb{S}^n} f(x)\,p(x,z)\,dx\\ &\ge C\frac{1}{|\sigma(z_0)|}\int_{\sigma(z_0)} p(x,z)\,dx\\ &\ge C\frac{1}{|\sigma(z_0)|}\int_{\sigma(z)}\frac{1-|z|}{|x-z|^{n+1}}dx\\ &\ge C\frac{1}{|\sigma(z_0)|}\end{aligned} \tag{69}$$

by Lemma 8 and the estimate (13). Then, the assumption that $|P(f)(z)|d\mu(z)$ is an s-Carleson measure implies that

$$\begin{aligned}\frac{1}{|\sigma(z_0)|}\int_{T(\sigma(z_0))} d\mu(z) &\le C\int_{T(\sigma(z_0))} |P(f)(z)|\,d\mu(z)\\ &\le C|\sigma(z_0)|^s\end{aligned} \tag{70}$$

holds for any $z_0\in\mathbb{B}$, which is to say that μ is an $(s+1)$-Carleson measure.

To prove "(c2)⇒(a2)," assume that μ is an $(s+1)$-Carleson measure. For any $z_0\in\mathbb{B}$, let σ be the sphere cap $\sigma=\sigma(z_0/|z_0|,1-|z_0|)$. Let i be a nonnegative integer. For some $x\in\mathbb{S}^n$, write $\sigma^*=\sigma(x,2(1-|z_0|))$. Let σ_i^* be the spherical cap with the same center as σ^* but radius $2^{-i}r(\sigma^*)$. We choose a number ε small enough such that

$$\Gamma(x)\bigcap S(\sigma^*)\subset\bigcup_{i=0}^{\infty}\widehat{S(\sigma_i^*)}_\varepsilon. \tag{71}$$

Here $\widehat{S(\sigma_i^*)}_\varepsilon=\{z\in S(\sigma_i^*):1-|z|\ge\varepsilon r(\sigma_i^*)\}$ is the top of $S(\sigma_i^*)$. We observe the fact that

$$\begin{aligned}\Gamma(x)\bigcap S(\sigma)&=\left\{z\in B: x\in\sigma\left(\frac{z}{|z|},1-|z|\right), z\in S(\sigma)\right\}\\ &\subseteq\left\{z\in B:\frac{z}{|z|}\in\sigma(x,1-|z_0|),|z_0|\le|z|\right\}\\ &\subseteq S(\sigma^*)\bigcap\Gamma(x)\end{aligned} \tag{72}$$

holds for any $x \in \mathbb{S}^n$. Now, it is easy to see that for any $x \in \mathbb{S}^n$

$$\begin{aligned}\int_{\Gamma(x)\bigcap S(\sigma)} \frac{d\mu(z)}{(1-|z|)^n} &\leq \int_{S(\sigma^*)\bigcap \Gamma(x)} \frac{d\mu(z)}{(1-|z|)^n} \\ &\leq \sum_{i=0}^{\infty} \int_{\widehat{S(\sigma_i^*)}_\varepsilon} \frac{d\mu(z)}{(1-|z|)^n} \\ &\leq C\sum_{i=0}^{\infty} \int_{\widehat{S(\sigma_i^*)}_\varepsilon} |\sigma_i^*|^{-1} d\mu(z) \\ &\leq C\sum_{i=0}^{\infty} |\sigma_i^*|^s \\ &\leq C|\sigma^*|^s \sum_{i=0}^{\infty} 2^{-isn},\end{aligned} \tag{73}$$

which is to say that $\mathfrak{I}_\mu(S(\sigma),\infty) \leq C|\sigma|^s$. The proof is completed. □

Theorem 14. *Let $s > 0$, $\max(1/s,1) \leq p < \infty$, and μ a nonnegative measure on $\mathbb{B}$.*

(i) $\sup_{z_0\in\mathbb{B}} \mathfrak{R}_\mu(z_0,s,p) < \infty$ *or* $\mathfrak{I}_\mu(S(\sigma),p) \leq C|\sigma|^s$ *for all spherical cap $\sigma \subseteq \mathbb{S}^n$ if and only if μ is an $(s+1-1/p)$-Carleson measure.*

(ii) $\lim_{|z_0|\to 1^-} \mathfrak{R}_\mu(z_0,s,p) = 0$ *or* $\mathfrak{I}_\mu(S(\sigma),p) = o(|\sigma|^s)$ *for all spherical cap $\sigma \subseteq \mathbb{S}^n$ if and only if μ is a compact $(s+1-1/p)$-Carleson measure.*

Proof. To prove the "if" parts, suppose that μ is an $(s+1-1/p)$-Carleson measure or a compact $(s+1-1/p)$-Carleson measure. By Theorem 10, it is sufficient to prove that

$$\mathfrak{I}_\mu(S(\sigma),p) \leq C(\mu(S(\sigma)))^{s/(s+1-1/p)} \tag{74}$$

holds for all $\sigma \subseteq \mathbb{S}^n$.

For $p=1$, the results are trivial because of the fact of $\mathfrak{I}_\mu(S(\sigma),1) \asymp \mu(S(\sigma))$.

For $p>1$, let $g \in L^{p'}(\mathbb{S}^n)$ and $g \geq 0$ (here p' is the conjugate of p, that is, $1/p+1/p' = 1$). By Lemma 1, we have

$$T(g)(z) \leq CP(g)(z), \quad \forall z \in \mathbb{B}. \tag{75}$$

Let $q = p'(s+1-1/p)$. It is easy to see that $q > p' > 1$ and $q' = (s+1-1/p)/s$ (here q' is the conjugate of q). Now, by the previous assumption, we have that μ is a (q/p')-Carleson measure. Using Hölder's inequality and Theorem 7, we have

$$\begin{aligned}&\int_{\mathbb{S}^n} g(x) \int_{\Gamma(x)\bigcap S(\sigma)} \frac{d\mu(z)}{(1-|z|)^n} dx \\ &\leq C\int_{S(\sigma)} T(g)(z)\, d\mu(z) \\ &\leq C\left(\int_{S(\sigma)} |P(g)(z)|^q d\mu(z)\right)^{1/q} \left(\int_{S(\sigma)} d\mu(z)\right)^{1/q'} \\ &\leq C\|g\|_{L^{p'}(\mathbb{S}^n)} \mu(S(\sigma))^{s/(s+1-1/p)}.\end{aligned} \tag{76}$$

By the duality theorem, we conclude that

$$\mathfrak{I}_\mu(S(\sigma),p) \leq C\mu(S(\sigma))^{s/(s+1-1/p)}, \tag{77}$$

which completes the proof of "if" parts.

We now turn to the "only if" parts. Assume that $\mathfrak{I}_\mu(S(\sigma),p) \leq C|\sigma|^s$ or $\mathfrak{I}_\mu(S(\sigma),p) = o(|\sigma|^s)$ for all spherical cap $\sigma \subseteq \mathbb{S}^n$. For a spherical cap $\sigma \subseteq \mathbb{S}^n$, we have

$$\sigma\left(\frac{z}{|z|}, 1-|z|\right) \subset \sigma, \quad \forall z \in T(\sigma). \tag{78}$$

Then, by the estimate in (13) and Fubini's theorem, we obtain that

$$\begin{aligned}\frac{1}{|\sigma|}\int_{T(\sigma)} d\mu(z) &\leq C\frac{1}{|\sigma|}\int_{T(\sigma)}\int_{\sigma\cap\sigma(z)} \frac{dx}{(1-|z|)^n} d\mu(z) \\ &\leq C\int_\sigma \int_{T(\sigma)\bigcap\Gamma(x)} \frac{d\mu(z)}{(1-|z|)^n}\frac{dx}{|\sigma|}.\end{aligned} \tag{79}$$

Since $p \geq 1$, Jensen's inequality and the above inequality imply that

$$\begin{aligned}&\frac{1}{|\sigma|}\int_{T(\sigma)} d\mu(z) \\ &\leq C|\sigma|^{-1/p}\left(\int_{\mathbb{S}^n}\left(\int_{T(\sigma)\bigcap\Gamma(x)} \frac{d\mu(z)}{(1-|z|)^n}\right)^p dx\right)^{1/p} \\ &\leq C|\sigma|^{-1/p}\mathfrak{I}_\mu(T(\sigma),p).\end{aligned} \tag{80}$$

Moreover, by the assumption and the remarks below Definition 5 and Theorem 10, we have

$$\mathfrak{I}_\mu(T(\sigma),p) \leq C|\sigma|^s \quad \text{or} \quad \mathfrak{I}_\mu(T(\sigma),p) = o(|\sigma|^s). \tag{81}$$

Then, we conclude that μ is an $(s+1-1/p)$-Carleson measure or a compact $(s+1-1/p)$-Carleson measure. The proof of Theorem 14 is completed. □

Theorems 10 and 14 imply the following characterizations for s-Carleson measures on $\mathbb{B}$ as $s \geq 1$.

Corollary 15. *Let $s \geq 1$, $1 \leq p < \infty$ and μ a nonnegative measure on $\mathbb{B}$.*

(i) *μ is an s-Carleson measure on $\mathbb{B}$ if and only if*

$$\begin{aligned}&\sup_{z_0\in\mathbb{B}} \mathfrak{R}_\mu\left(z_0, s-1+\frac{1}{p}, p\right) < \infty \\ &\textit{or} \quad \mathfrak{I}_\mu(S(\sigma),p) \leq C|\sigma|^{s-1+\frac{1}{p}} \quad \forall\sigma \subseteq \mathbb{S}^n.\end{aligned} \tag{82}$$

(ii) *μ is a compact s-Carleson measure on $\mathbb{B}$ if and only if*

$$\begin{aligned}&\sup_{|z_0|\to 1^-} \mathfrak{R}_\mu\left(z_0, s-1+\frac{1}{p}, p\right) = 0 \\ &\textit{or} \quad \mathfrak{I}_\mu(S(\sigma),p) = o\left(|\sigma|^{s-1+1/p}\right) \quad \forall\sigma \subseteq \mathbb{S}^n.\end{aligned} \tag{83}$$

Remark 16. By Theorem 13, the above corollary holds if $s > 1$ and $p = \infty$.

For $0 < s < 1$, we have the following results.

Theorem 17. *Let $0 < s < 1$ and μ a nonnegative measure on $\mathbb{B}$.*

(a) *If $0 < p \leq 1$ and μ is an s-Carleson measure on $\mathbb{B}$, then*

$$\sup_{z_0 \in \mathbb{B}} \mathfrak{R}_\mu \left(z_0, s - 1 + \frac{1}{p}, p \right) < \infty,$$
$$\mathfrak{I}_\mu (S(\sigma), p) \leq C|\sigma|^{s-1+1/p} \quad \forall \sigma \subseteq \mathbb{S}^n. \tag{84}$$

(b) *If $1 \leq p < 1/(1-s)$ and $\sup_{z_0 \in \mathbb{B}} \mathfrak{R}_\mu(z_0, s-1+1/p, p) < \infty$ or $\mathfrak{I}_\mu(S(\sigma), p) \leq C|\sigma|^{s-1+1/p}$ for all $\sigma \subseteq \mathbb{S}^n$, then μ is an s-Carleson measure.*

(c) *If $1/(1-s) \leq p < \infty$, then $\mathfrak{I}_\mu(S(\sigma), p) \leq C|\sigma|^{s-1+1/p}$ for all $\sigma \subseteq \mathbb{S}^n$ if and only if*

$$\int_{\mathbb{B}} |P(f)(z)|^{p-1} d\mu(z) \leq C\|f\|_{L^p(\mathbb{S}^n)}^{p-1}, \quad \forall f \in L^p(\mathbb{S}^n). \tag{85}$$

Proof. To prove (a), it is sufficient to prove

$$\mathfrak{I}_\mu (T(\sigma), p) \leq C|\sigma|^{1/p-1} \mu(T(\sigma)) \tag{86}$$

for any spherical cap $\sigma \subseteq \mathbb{S}^n$. For $0 < p \leq 1$ and any spherical cap σ, by Hölder's inequality and Fubini's theorem, it is clear that

$$\begin{aligned}
&\left(\mathfrak{I}_\mu (T(\sigma), p) \right)^p \\
&\quad = \int_{\mathbb{S}^n} \left(\int_{\Gamma(x) \bigcap T(\sigma)} \frac{d\mu(z)}{(1-|z|)^n} \right)^p dx \\
&\quad = \int_{\sigma} \left(\int_{\Gamma(x) \bigcap T(\sigma)} \frac{d\mu(z)}{(1-|z|)^n} \right)^p dx \\
&\quad \leq C \left(\int_\sigma dx \right)^{1-p} \left(\int_\sigma \int_{\Gamma(x) \bigcap T(\sigma)} \frac{d\mu(z)}{(1-|z|)^n} dx \right)^p \\
&\quad \leq C|\sigma|^{1-p} \left(\int_{T(\sigma)} d\mu(z) \right)^p \\
&\quad \leq C|\sigma|^{1-p} (\mu(T(\sigma)))^p.
\end{aligned} \tag{87}$$

Then, by the assumption that μ is an s-Carleson measure, it is easy to see that $\mathfrak{I}_\mu(T(\sigma), p) \leq C|\sigma|^{s-1+1/p}$.

For (b), let $f(x) = \chi_\sigma(x)$, and then $\|f\|_{L^{p'}(\mathbb{S}^n)} = |\sigma|^{1/p'}$ (here p' is also the conjugate of p). Noticing the fact that $\sigma(z/|z|, 1-|z|) \subset \sigma$ for any $z \in T(\sigma)$, we obtain

$$\begin{aligned}
\mu(T(\sigma)) &= \int_{T(\sigma)} d\mu(z) \\
&\leq C \int_{T(\sigma)} \frac{1}{|\sigma(z)|} \int_{\sigma(z)} f(x)\, dx\, d\mu(z) \\
&\leq C \int_{\mathbb{S}^n} f(x) \int_{\Gamma(x) \bigcap T(\sigma)} \frac{d\mu(x)}{(1-|z|)^n} dx \\
&\leq C\|f\|_{L^{p'}(\mathbb{S}^n)} \mathfrak{I}_\mu (T(\sigma), p) \\
&\leq C|\sigma|^s.
\end{aligned} \tag{88}$$

By Theorem 10, we complete the proof of (b).

For (c), since condition $\mathfrak{I}_\mu(S(\sigma), p) \leq C|\sigma|^{s-1+1/p}$ for all $\sigma \subseteq \mathbb{S}^n$ is equivalent to $\mathfrak{I}_\mu(\mathbb{B}, p) \leq C$, it is the direct consequence of the estimate for $g \geq 0$

$$\int_{\mathbb{B}} T(g)(z)\, d\mu(z) \asymp \int_{\mathbb{S}^n} g(x) \int_{\Gamma(x)} \frac{d\mu(z)}{(1-|z|)^n} dx \tag{89}$$

and the duality theorem with appropriate choices of g. □

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