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Research Article

Martingale Morrey-Hardy and Campanato-Hardy Spaces

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We introduce generalized Morrey-Campanato spaces of martingales, which generalize both martingale Lipschitz spaces introduced by Weisz (1990) and martingale Morrey-Campanato spaces introduced in 2012. We also introduce generalized Morrey-Hardy and Campanato-Hardy spaces of martingales and study Burkholder-type equivalence. We give some results on the boundedness of fractional integrals of martingales on these spaces.

1. Introduction

Lebesgue spaces L_p and Hardy spaces H_p play an important role in martingale theory and in harmonic analysis as well. Morrey-Campanato spaces are very useful to know more precise properties of functions and martingales. It is known that Morrey-Campanato spaces contain L_p, BMO, and Lip_α as special cases; see, for example, [1, 2].

In martingale theory, Weisz [3] introduced martingale Lipschitz spaces for general filtrations $\{\mathcal{F}_n\}_{n\geq 0}$ and proved the duality between martingale Hardy spaces and martingale Lipschitz spaces. This result was extended to generalized martingale Campanato spaces and martingale Orlicz-Hardy spaces in [4]. Recently, martingale Morrey-Campanato spaces were introduced in [5], where each sub-σ-algebra $\mathcal{F}_n$ is generated by countable atoms.

In this paper, we introduce martingale Morrey-Hardy and Campanato-Hardy spaces based on square functions and unify Hardy, Lipschitz, and Morrey-Campanato spaces in [3–5]. To do this, we first introduce generalized martingale Morrey-Campanato spaces by using subfamilies $\{\mathcal{B}_n\}_{n\geq 0}$ of the filtration $\{\mathcal{F}_n\}_{n\geq 0}$ with $\mathcal{B}_n \subset \mathcal{F}_n$ for each $n \geq 0$. We establish Burkholder-type equivalence and discuss equivalence between martingale Morrey spaces and martingale Campanato spaces in a suitable condition. We also establish a John-Nirenberg-type theorem for generalized martingale Campanato-Hardy spaces; see Theorem 15.

On these martingale spaces, we introduce generalized fractional integrals as martingale transforms and prove their boundedness. Our result extends several results in [5–7] to these spaces. The fractional integrals are very useful tools to analyse function spaces in harmonic analysis. Actually, on the Euclidean space, Hardy and Littlewood [8, 9] and Sobolev [10] investigated the fractional integrals to establish the theory of Lebesgue spaces and Lipschitz spaces. Stein and Weiss [11], Taibleson and Weiss [12], and Krantz [13] also investigated the fractional integrals to establish the theory of Hardy spaces; see also [14]. The L_p-L_q boundedness of the fractional integrals is well known as the Hardy-Littlewood-Sobolev theorem derived from [8–10]. This boundedness has been extended to Morrey-Campanato spaces by Peetre [1] and Adams [15]; see also Chiarenza and Frasca [16]. In martingale theory, based on the result on the Walsh multiplier by Watari [7, Theorem 1.1], Chao and Ombe [6] proved the boundedness of the fractional integrals for H_p, L_p, BMO, and Lipschitz spaces of the dyadic martingale. The boundedness of the fractional integrals for martingale Morrey-Campanato spaces was established in [5]. For other types of operators for martingales, see the recent work by Tanaka and Terasawa [17].

At the end of this section, we make some conventions. Throughout this paper, we always use C to denote a positive constant that is independent of the main parameters involved but whose value may differ from line to line. Constants with subscripts, such as C_p, are dependent on the subscripts. If $f \le Cg$, we then write $f \lesssim g$ or $g \gtrsim f$; and if $f \lesssim g \lesssim f$, we then write $f \sim g$.

2. Definitions and Notation

Let (Ω, Σ, P) be a probability space and $\mathcal{F} = \{\mathcal{F}_n\}_{n\ge 0}$ a nondecreasing sequence of sub-σ-algebras of Σ such that $\Sigma = \sigma(\bigcup_n \mathcal{F}_n)$. For the sake of simplicity, let $\mathcal{F}_{-1} = \mathcal{F}_0$. The set $B \in \mathcal{F}_n$ is called atom, more precisely $(\mathcal{F}_n, P)$-atom, if any $A \subset B$, $A \in \mathcal{F}_n$, satisfying $P(A) = P(B)$ or $P(A) = 0$. Denote by $A(\mathcal{F}_n)$ the set of all atoms in $\mathcal{F}_n$.

The expectation operator and the conditional expectation operators relative to $\mathcal{F}_n$ are denoted by E and E_n, respectively. It is known from the Doob theorem that if $p \in (1,\infty)$, then any L_p-bounded martingale converges in L_p. Moreover, if $p \in [1,\infty)$, then, for any $f \in L_p$, its corresponding martingale $(f_n)_{n\ge 0}$ with $f_n = E_n f$ is an L_p-bounded martingale and converges to f in L_p (see, e.g., [18]). For this reason a function $f \in L_1$ and the corresponding martingale $(f_n)_{n\ge0}$ will be denoted by the same symbol f.

Let $\mathcal{M}$ be the set of all martingales such that $f_0 = 0$. For $p \in [1,\infty]$, let L_p^0 be the set of all $f \in L_p$ such that $E_0 f = 0$. For any $f \in L_p^0$, its corresponding martingale $(f_n)_{n\ge0}$ with $f_n = E_n f$ is an L_p-bounded martingale in $\mathcal{M}$. For this reason, we regard L_p^0 as a subset of $\mathcal{M}$.

Let $\mathcal{B} = \{\mathcal{B}_n\}_{n\ge0}$ be subfamilies of $\mathcal{F} = \{\mathcal{F}_n\}_{n\ge0}$ with $\mathcal{B}_n \subset \mathcal{F}_n$ for each $n \ge 0$. We denote by $\mathcal{B} \subset \mathcal{F}$ this relation of $\mathcal{B}$ and $\mathcal{F}$.

In this paper, we always postulate the following condition on $\mathcal{B}$

$$\text{There exists a countable subset } \mathcal{B}_0' \subset \mathcal{B}_0 \text{ such that } P\left(\bigcup_{B\in\mathcal{B}_0'} B\right) = 1. \tag{1}$$

We first define generalized martingale Morrey-Campanato spaces with respect to $\mathcal{B}$ as follows.

Definition 1. Let $\mathcal{B} \subset \mathcal{F}$, $p \in [1,\infty)$, and $\phi : (0,1] \to (0,\infty)$. For $f \in L_1$, let

$$\begin{aligned}
\|f\|_{L_{p,\phi}} &= \|f\|_{L_{p,\phi}(\mathcal{B})} \\
&= \sup_{n\ge0}\sup_{B\in\mathcal{B}_n} \frac{1}{\phi(P(B))}\left(\frac{1}{P(B)}\int_B |f|^p dP\right)^{1/p}, \\
\|f\|_{\mathcal{L}_{p,\phi}} &= \|f\|_{\mathcal{L}_{p,\phi}(\mathcal{B})} \\
&= \sup_{n\ge0}\sup_{B\in\mathcal{B}_n} \frac{1}{\phi(P(B))}\left(\frac{1}{P(B)}\int_B |f-E_nf|^p dP\right)^{1/p}, \\
\|f\|_{\mathcal{L}^-_{p,\phi}} &= \|f\|_{\mathcal{L}^-_{p,\phi}(\mathcal{B})} \\
&= \sup_{n\ge0}\sup_{B\in\mathcal{B}_n} \frac{1}{\phi(P(B))}\left(\frac{1}{P(B)}\int_B |f-E_{n-1}f|^p dP\right)^{1/p},
\end{aligned} \tag{2}$$

and define

$$\begin{aligned}
L_{p,\phi} &= L_{p,\phi}(\mathcal{B}) = \left\{f \in L_p^0 : \|f\|_{L_{p,\phi}} < \infty\right\}, \\
\mathcal{L}_{p,\phi} &= \mathcal{L}_{p,\phi}(\mathcal{B}) = \left\{f \in L_p^0 : \|f\|_{\mathcal{L}_{p,\phi}} < \infty\right\}, \\
\mathcal{L}^-_{p,\phi} &= \mathcal{L}^-_{p,\phi}(\mathcal{B}) = \left\{f \in L_p^0 : \|f\|_{\mathcal{L}^-_{p,\phi}} < \infty\right\}.
\end{aligned} \tag{3}$$

Remark 2. By the condition (1), the functionals $\|f\|_{L_{p,\phi}}$, $\|f\|_{\mathcal{L}_{p,\phi}}$, and $\|f\|_{\mathcal{L}^-_{p,\phi}}$ are norms on L_p^0.

Remark 3. Let $f \in L_1^0$. Then, $f \in \mathcal{L}_{p,\phi}$ if and only if its corresponding martingale $(f_n)_{n\ge0}$ is $\mathcal{L}_{p,\phi}$-bounded; that is, $\sup_{n\ge0}\|f_n\|_{\mathcal{L}_{p,\phi}} < \infty$. The same conclusion holds for $\mathcal{L}^-_{p,\phi}$. Furthermore, if each sub-σ-algebra $\mathcal{F}_n$ is generated by countable atoms, $\mathcal{B} = \{A(\mathcal{F}_n)\}_{n\ge0}$ and ϕ is almost decreasing, then the same conclusion holds for $L_{p,\phi}$. More precisely, see Proposition 8.

Remark 4. In general, $\|f\|_{\mathcal{L}_{p,\phi}} \le 2\|f\|_{L_{p,\phi}}$ and hence $L_{p,\phi} \subset \mathcal{L}_{p,\phi}$. Actually, for any $B \in \mathcal{B}_n$,

$$\begin{aligned}
\left(\int_B |f-E_nf|^p dP\right)^{1/p} &\le \left(\int_B |f|^p dP\right)^{1/p} \\
&\quad + \left(\int_B |E_nf|^p dP\right)^{1/p} \\
&\le 2\left(\int_B |f|^p dP\right)^{1/p}.
\end{aligned} \tag{4}$$

Similarly, $\|f\|_{\mathcal{L}_{p,\phi}} \le 2\|f\|_{\mathcal{L}^-_{p,\phi}}$ and $\mathcal{L}^-_{p,\phi} \subset \mathcal{L}_{p,\phi}$.

Definition 5. For $\phi \equiv 1$, denote $\mathcal{L}_{p,\phi}$ and $\mathcal{L}^-_{p,\phi}$ by BMO_p and BMO_p^-, respectively. For $\phi(r) = r^\alpha$, $\alpha > 0$, denote $\mathcal{L}_{p,\phi}$ and $\mathcal{L}^-_{p,\phi}$ by $\mathrm{Lip}_{p,\alpha}$ and $\mathrm{Lip}^-_{p,\alpha}$, respectively.

If $\phi(r) = r^\lambda$, $\lambda \in (-\infty,\infty)$, then we simply denote $L_{p,\phi}$, $\mathcal{L}_{p,\phi}$, and $\mathcal{L}^-_{p,\phi}$ by $L_{p,\lambda}$, $\mathcal{L}_{p,\lambda}$, and $\mathcal{L}^-_{p,\lambda}$, respectively.

A function $\phi : (0,1] \to (0,\infty)$ is said to be almost increasing (resp., almost decreasing) if there exists a positive constant C such that

$$\phi(s) \le C\phi(t) \quad (\text{resp., } \phi(t) \le C\phi(s)) \quad \text{for } 0 < s \le t \le 1. \tag{5}$$

For the case $\mathcal{B} = \mathcal{F}$, the spaces $\mathrm{BMO}_p(\mathcal{F})$ and $\mathrm{Lip}_{p,\alpha}(\mathcal{F})$ with $\alpha \ge 0$, were introduced by Weisz [3].

Recall that $A(\mathcal{F}_n)$ is the set of all atoms in $\mathcal{F}_n$ and let $\mathcal{A} = \{A(\mathcal{F}_n)\}_{n\ge0}$. Suppose that each sub-$\sigma$-algebra $\mathcal{F}_n$ is generated by countable atoms for the time being. Then, $\mathrm{BMO}_p(\mathcal{F}) = \mathrm{BMO}_p(\mathcal{A})$ and $\mathrm{Lip}_{p,\alpha}(\mathcal{F}) = \mathrm{Lip}_{p,\alpha}(\mathcal{A})$; see [5]. In general, if ϕ is almost increasing, then

$$\begin{aligned}
L_{p,\phi}(\mathcal{F}) &= L_{p,\phi}(\mathcal{A}), \\
\mathcal{L}_{p,\phi}(\mathcal{F}) &= \mathcal{L}_{p,\phi}(\mathcal{A}), \\
\mathcal{L}^-_{p,\phi}(\mathcal{F}) &= \mathcal{L}^-_{p,\phi}(\mathcal{A}),
\end{aligned} \tag{6}$$

with equivalent norms, respectively. However, if ϕ is not almost increasing, then these equalities fail in general; see [5].

In this paper, we do not always assume that each sub-σ-algebra $\mathcal{F}_n$ is generated by countable atoms. Let

$$A(\mathcal{F}_n)^{\perp} = \{B \in \mathcal{F}_n : P(B \cap A) = 0 \ \forall A \in A(\mathcal{F}_n)\}, \quad (7)$$

and let

$$\mathcal{B}_n = A(\mathcal{F}_n) \cup A(\mathcal{F}_n)^{\perp} \quad (n \geq 0). \quad (8)$$

In this case, if ϕ is almost increasing, then we will show that

$$\begin{aligned} L_{p,\phi}(\mathcal{F}) &= L_{p,\phi}(\mathcal{B}), \\ \mathcal{L}_{p,\phi}(\mathcal{F}) &= \mathcal{L}_{p,\phi}(\mathcal{B}), \\ \mathcal{L}^{-}_{p,\phi}(\mathcal{F}) &= \mathcal{L}^{-}_{p,\phi}(\mathcal{B}), \end{aligned} \quad (9)$$

with equivalent norms, respectively (see Proposition 9). Moreover, if $\mathcal{F}_0$ is nonatomic, then $\mathcal{B}_n = \mathcal{F}_n$ for all $n \geq 0$. If each sub-σ-algebra $\mathcal{F}_n$ is generated by countable atoms, then $\mathcal{B}_n = A(\mathcal{F}_n)$ for all $n \geq 0$. Therefore, our definition generalizes those in [3–5].

Next we define Morrey-Hardy and Campanato-Hardy spaces, based on square functions, with respect to $\mathcal{B}$ as follows. For $f \in \mathcal{M}$, we denote by $S_n(f)$ and $S(f)$ the square function of f:

$$S_n(f) = \left(\sum_{k=0}^{n} |d_k f|^2\right)^{1/2}, \qquad S(f) = \left(\sum_{k=0}^{\infty} |d_k f|^2\right)^{1/2}, \quad (10)$$

where $d_k f = f_k - f_{k-1}$ ($n \geq 0$, with convention $d_0 f = 0$ and $S_{-1}(f) = 0$). We further define

$$S^{(n)}(f) = \left(S(f)^2 - S_n(f)^2\right)^{1/2} = \left(\sum_{k=n+1}^{\infty} |d_k f|^2\right)^{1/2}. \quad (11)$$

Definition 6. Let $\mathcal{B} \subset \mathcal{F}$, $p \in (0, \infty)$, and $\phi : (0, 1] \to (0, \infty)$. For $f = (f_n)_{n \geq 0} \in \mathcal{M}$, let

$$\begin{aligned} \|f\|_{H^S_{p,\phi}} &= \|f\|_{H^S_{p,\phi}(\mathcal{B})} \\ &= \sup_{n \geq 0} \sup_{B \in \mathcal{B}_n} \frac{1}{\phi(P(B))} \left(\frac{1}{P(B)} \int_B S(f)^p \, dP\right)^{1/p}, \\ \|f\|_{\mathcal{H}^S_{p,\phi}} &= \|f\|_{\mathcal{H}^S_{p,\phi}(\mathcal{B})} \\ &= \sup_{n \geq 0} \sup_{B \in \mathcal{B}_n} \frac{1}{\phi(P(B))} \left(\frac{1}{P(B)} \int_B S^{(n)}(f)^p \, dP\right)^{1/p}, \\ \|f\|_{\mathcal{H}^{S-}_{p,\phi}} &= \|f\|_{\mathcal{H}^{S-}_{p,\phi}(\mathcal{B})} \\ &= \sup_{n \geq 0} \sup_{B \in \mathcal{B}_n} \frac{1}{\phi(P(B))} \left(\frac{1}{P(B)} \int_B S^{(n-1)}(f)^p \, dP\right)^{1/p}, \end{aligned} \quad (12)$$

and define

$$\begin{aligned} H^S_{p,\phi} &= H^S_{p,\phi}(\mathcal{B}) = \left\{f \in \mathcal{M} : \|f\|_{H^S_{p,\phi}} < \infty\right\}, \\ \mathcal{H}^S_{p,\phi} &= \mathcal{H}^S_{p,\phi}(\mathcal{B}) = \left\{f \in \mathcal{M} : \|f\|_{\mathcal{H}^S_{p,\phi}} < \infty\right\}, \\ \mathcal{H}^{S-}_{p,\phi} &= \mathcal{H}^{S-}_{p,\phi}(\mathcal{B}) = \left\{f \in \mathcal{M} : \|f\|_{\mathcal{H}^{S-}_{p,\phi}} < \infty\right\}. \end{aligned} \quad (13)$$

By (1), the functionals $\|f\|_{H^S_{p,\phi}}$, $\|f\|_{\mathcal{H}^S_{p,\phi}}$, and $\|f\|_{\mathcal{H}^{S-}_{p,\phi}}$ are quasinorms on $\mathcal{M}$.

Remark 7. If we take $\phi \equiv 1$ and $\mathcal{B} = \mathcal{F}$, then the norm $\|f\|_{\mathcal{H}^{S-}_{p,\phi}}$ coincides with the norm $\|f\|_{\mathrm{BMO}^S_p}$ in [19, Definition 2.45]. In this point, our notation is different from the one in [19].

In the end of this section, we present the definition of regularity on $\mathcal{F}$ and the doubling condition on ϕ. The filtration $\mathcal{F} = \{\mathcal{F}_n\}_{n \geq 0}$ is said to be regular, if there exists a constant $R \geq 2$ such that

$$f_n \leq R f_{n-1} \quad (14)$$

holds for all nonnegative martingales $(f_n)_{n \geq 0}$. We say the smallest constant R satisfying (14) the regularity constant of $\mathcal{F}$. A function $\phi : (0, 1] \to (0, \infty)$ is said to satisfy the doubling condition if there exists a positive constant C_ϕ such that

$$C_\phi^{-1} \leq \frac{\phi(s)}{\phi(t)} \leq C_\phi \quad \forall s, t \in (0, 1] \text{ with } \frac{1}{2} \leq \frac{s}{t} \leq 2. \quad (15)$$

The smallest constant C_ϕ satisfying (15) is called the doubling constant of ϕ.

3. Properties of Morrey-Hardy and Campanato-Hardy Spaces

In this section, we investigate the properties of Morrey-Hardy and Campanato-Hardy spaces. The proofs of the results in this section will be given in Section 6.

First we state basic properties of the norms.

Proposition 8. *Let $\mathcal{B} \subset \mathcal{F}$, $p \in [1, \infty)$ and $\phi : (0, 1] \to (0, \infty)$. Let $f \in L_1$ and let $(f_n)_{n \geq 0}$ be its corresponding martingale; $f_n = E_n f$. Then*

$$\begin{aligned} \|f\|_{L_{p,\phi}} &\leq \sup_{n \geq 0} \|f_n\|_{L_{p,\phi}}, \\ \|f\|_{\mathcal{L}_{p,\phi}} &= \sup_{n \geq 0} \|f_n\|_{\mathcal{L}_{p,\phi}}, \\ \|f\|_{\mathcal{L}^{-}_{p,\phi}} &= \sup_{n \geq 0} \|f_n\|_{\mathcal{L}^{-}_{p,\phi}}. \end{aligned} \quad (16)$$

Moreover, if each sub-σ-algebra $\mathcal{F}_n$ is generated by countable atoms, $\mathcal{B} = \mathcal{A}$, and ϕ is almost decreasing; that is, there exists a positive constant C_0, such that $\phi(t) \leq C_0 \phi(s)$ for $0 < s \leq t \leq 1$, then

$$\|f\|_{L_{p,\phi}} \leq \sup_{n \geq 0} \|f_n\|_{L_{p,\phi}} \leq C_0 \|f\|_{L_{p,\phi}}. \quad (17)$$

Proposition 9. *Let $A(\mathscr{F}_n) \cup A(\mathscr{F}_n)^{\perp} \subset \mathscr{B}_n \subset \mathscr{F}_n (n \geq 0)$. If ϕ is almost increasing; that is, there exists a positive constant C_0, such that $\phi(s) \leq C_0\phi(t)$ for $0 < s \leq t \leq 1$, then*

$$\|f\|_{L_{p,\phi}(\mathscr{B})} \leq \|f\|_{L_{p,\phi}(\mathscr{F})} \leq C_0\|f\|_{L_{p,\phi}(\mathscr{B})}, \tag{18}$$

and the same conclusions hold for $\|\cdot\|_{\mathscr{L}_{p,\phi}}$, $\|\cdot\|_{\mathscr{L}^-_{p,\phi}}$, $\|\cdot\|_{H^S_{p,\phi}}$, $\|\cdot\|_{\mathscr{H}^S_{p,\phi}}$, and $\|\cdot\|_{\mathscr{H}^{S-}_{p,\phi}}$. Consequently,

$$\begin{aligned} L_{p,\phi}(\mathscr{F}) &= L_{p,\phi}(\mathscr{B}), & \mathscr{L}_{p,\phi}(\mathscr{F}) &= \mathscr{L}_{p,\phi}(\mathscr{B}), \\ \mathscr{L}^-_{p,\phi}(\mathscr{F}) &= \mathscr{L}^-_{p,\phi}(\mathscr{B}), & H_{p,\phi}(\mathscr{F}) &= H_{p,\phi}(\mathscr{B}), \\ \mathscr{H}^S_{p,\phi}(\mathscr{F}) &= \mathscr{H}^S_{p,\phi}(\mathscr{B}), & \mathscr{H}^{S-}_{p,\phi}(\mathscr{F}) &= \mathscr{H}^{S-}_{p,\phi}(\mathscr{B}), \end{aligned} \tag{19}$$

with equivalent norms, respectively.

For $p \in (0,\infty)$, let H^S_p be the set of all $f \in \mathscr{M}$ such that $\|S(f)\|_{L_p} < \infty$. Let $\|f\|_{H^S_p} = \|S(f)\|_{L_p}$. Note that if $\phi(r) = r^{-1/p}$ and $\Omega \in \mathscr{B}_0$, then $H^S_{p,\phi} = H^S_p$ and $\|f\|_{H^S_{p,\phi}} = \|f\|_{H^S_p}$.

The following is well known as Burkholder's inequality.

Theorem 10 (Burkholder [20]). *If $p \in (1,\infty)$, then there exist positive constants c_p and C_p, that depend only on p, such that*

$$c_p\|f\|_{L_p} \leq \|f\|_{H^S_p} \leq C_p\|f\|_{L_p} \tag{20}$$

for all $f \in L^0_1 \subset \mathscr{M}$.

For expressions of the constants c_p and C_p, see, for example, [21–23]. See also [24] for Burkholder's inequality on Banach functions spaces.

Our first result is the following, which is an extension of Burkholder's inequality to martingale Campanato spaces.

Theorem 11. *Let $\mathscr{B} \subset \mathscr{F}$, $p \in (1,\infty)$ and $\phi : (0,1] \to (0,\infty)$. Then*

$$c_p\|f\|_{\mathscr{L}_{p,\phi}} \leq \|f\|_{\mathscr{H}^S_{p,\phi}} \leq C_p\|f\|_{\mathscr{L}_{p,\phi}}, \tag{21}$$

$$\frac{c_p}{1+c_p}\|f\|_{\mathscr{L}^-_{p,\phi}} \leq \|f\|_{\mathscr{H}^{S-}_{p,\phi}} \leq \left(2C_p+1\right)\|f\|_{\mathscr{L}^-_{p,\phi}} \tag{22}$$

for all $f \in L^0_1 \subset \mathscr{M}$, where c_p and C_p are the constants in Theorem 10.

Next we give the relations between $\mathscr{L}_{p,\phi}$ and $\mathscr{L}^-_{p,\phi}$ and between $\mathscr{H}^S_{p,\phi}$ and $\mathscr{H}^{S-}_{p,\phi}$. We consider the following condition on $\mathscr{B}$:

$$\{\omega \in \Omega : E_{n-1}[\chi_B](\omega) > 0\} \in \mathscr{B}_{n-1} \quad \forall B \in \mathscr{B}_n \ (n \geq 1). \tag{23}$$

Theorem 12. *Let $\mathscr{B} \subset \mathscr{F}$ and $\phi : (0,1] \to (0,\infty)$. Then*

$$\mathscr{L}_{p,\phi} \supset \mathscr{L}^-_{p,\phi} \quad \text{with } \|f\|_{\mathscr{L}_{p,\phi}} \leq 2\|f\|_{\mathscr{L}^-_{p,\phi}} \text{ for } p \in [1,\infty), \tag{24}$$

$$\mathscr{H}^S_{p,\phi} \supset \mathscr{H}^{S-}_{p,\phi} \quad \text{with } \|f\|_{\mathscr{H}^S_{p,\phi}} \leq \|f\|_{\mathscr{H}^{S-}_{p,\phi}} \text{ for } p \in (0,\infty). \tag{25}$$

Conversely, if $\mathscr{F}$ is regular, $\mathscr{B}$ satisfies (23), and ϕ satisfies the doubling condition, then there exists a positive constant C, dependent only on p, the regularity constant of $\mathscr{F}$, and the doubling constant of ϕ, such that

$$\mathscr{L}_{p,\phi} \subset \mathscr{L}^-_{p,\phi} \quad \text{with } C\|f\|_{\mathscr{L}_{p,\phi}} \geq \|f\|_{\mathscr{L}^-_{p,\phi}} \text{ for } p \in [1,\infty), \tag{26}$$

$$\mathscr{H}^S_{p,\phi} \subset \mathscr{H}^{S-}_{p,\phi} \quad \text{with } C\|f\|_{\mathscr{H}^S_{p,\phi}} \geq \|f\|_{\mathscr{H}^{S-}_{p,\phi}} \text{ for } p \in (0,\infty). \tag{27}$$

We give a relation between martingale Morrey spaces and martingale Campanato spaces in the following form.

Theorem 13. *Suppose that every σ-algebra $\mathscr{F}_n$ is generated by countable atoms. Let $\mathscr{B} = \mathscr{A}$, $p \in [1,\infty)$, and $\phi : (0,1] \to (0,\infty)$. Assume that ϕ satisfies the doubling condition and there exists a positive constant C'_ϕ such that*

$$\int_r^1 \frac{\phi(t)}{t}dt \leq C'_\phi\phi(r) \quad (0 < r < 1). \tag{28}$$

Then, there exists a positive constant C such that

$$\begin{aligned} \frac{1}{2}\|f\|_{\mathscr{L}_{p,\phi}} &\leq \|f\|_{L_{p,\phi}} \leq C\|f\|_{\mathscr{L}^-_{p,\phi}} \quad \forall f \in L^0_1, \\ \|f\|_{\mathscr{H}^{S-}_{p,\phi}} &\leq \|f\|_{H^S_{p,\phi}} \leq C\|f\|_{\mathscr{H}^{S-}_{p,\phi}} \quad \forall f \in \mathscr{M}. \end{aligned} \tag{29}$$

Moreover, if $\mathscr{F}$ is regular, then $\|f\|_{L_{p,\phi}}$, $\|f\|_{\mathscr{L}_{p,\phi}}$, and $\|f\|_{\mathscr{L}^-_{p,\phi}}$ are equivalent to each other, and $\|f\|_{H^S_{p,\phi}}$, $\|f\|_{\mathscr{H}^S_{p,\phi}}$ and $\|f\|_{\mathscr{H}^{S-}_{p,\phi}}$ are equivalent to each other.

Using Theorems 11–13, we have Burkholder-type equivalence for generalized martingale Morrey spaces.

Corollary 14. *Suppose that every σ-algebra $\mathscr{F}_n$ is generated by countable atoms. Let $\mathscr{B} = \mathscr{A}$, $p \in (1,\infty)$, and $\phi : (0,1] \to (0,\infty)$. Assume that ϕ satisfies the doubling condition and there exists a positive constant C'_ϕ such that*

$$\int_r^1 \frac{\phi(t)}{t}dt \leq C'_\phi\phi(r) \quad (0 < r < 1). \tag{30}$$

If $\mathscr{F}$ is regular, then there exist positive constants c and C such that, for all $f \in L^0_1$,

$$c\|f\|_{L_{p,\phi}} \leq \|f\|_{H^S_{p,\phi}} \leq C\|f\|_{L_{p,\phi}}. \tag{31}$$

For the martingale BMO spaces based on square functions, the John-Nirenberg-type equivalence was established by Weisz [25] and [19, Theorem 2.50]. We extend this theorem to the spaces $\mathscr{H}^S_{p,\phi}$ and $\mathscr{H}^{S-}_{p,\phi}$.

Theorem 15. *Let $A(\mathscr{F}_n) \cup A(\mathscr{F}_n)^{\perp} \subset \mathscr{B}_n \subset \mathscr{F}_n$ $(n \geq 0)$, $p \in (0,\infty)$ and $\phi : (0,1] \to (0,\infty)$. Assume that ϕ is almost increasing and satisfies the doubling condition. Then, $\|f\|_{\mathscr{H}^S_{p,\phi}} \leq C_{p,q,\phi}\|f\|_{\mathscr{H}^{S-}_{q,\phi}}$ for all $q \in (0,\infty)$. If we further assume that $\mathscr{F}$ is regular, then $\|f\|_{\mathscr{H}^S_{p,\phi}}$ and $\|f\|_{\mathscr{H}^{S-}_{p,\phi}}$ are equivalent to $\|f\|_{\mathscr{H}^S_{2,\phi}}$.*

4. Fractional Integrals

In this section, we state the results on the boundedness of fractional integrals as martingale transforms. The proofs of the results in this section will be given in Section 7.

Let $(\gamma_n)_{n\geq 0}$ be a sequence of nonnegative bounded functions adapted to $\mathcal{F} = \{\mathcal{F}_n\}_{n\geq 0}$; that is, γ_n is $\mathcal{F}_n$-measurable for every $n \geq 0$. Let I_γ be the martingale transform associate to $(\gamma_n)_{n\geq 0}$; that is,

$$\left(I_\gamma f\right)_n = \sum_{k=0}^{n} \gamma_{k-1} d_k f, \tag{32}$$

with convention $\gamma_{-1} d_0 f = 0$. Note that if $f = (f_n)_{n\geq 0} \in \mathcal{M}$, then $I_\gamma f = ((I_\gamma f)_n)_{n\geq 0} \in \mathcal{M}$.

We now define a generalized fractional integral I_ρ for martingales as a special case of I_γ under the assumption that every σ-algebra $\mathcal{F}_n$ is generated by countable atoms. Our definition generalizes the fractional integral for dyadic martingales introduced in [6, 7]. The idea of I_ρ comes from [26].

Suppose that every σ-algebra $\mathcal{F}_n$ is generated by countable atoms. Let b_n be an $\mathcal{F}_n$-measurable function such that

$$b_n(\omega) = P(B) \quad \text{for a.s. } \omega \in B \text{ with } B \in A(\mathcal{F}_n); \tag{33}$$

that is,

$$b_n = \sum_{B \in A(\mathcal{F}_n)} P(B)\, \chi_B \quad \text{a.s.} \tag{34}$$

For a bounded function $\rho : (0,1] \to (0,\infty)$, we define a generalized fractional integral $I_\rho f = ((I_\rho f)_n)_{n\geq 0}$ of $f = (f_n)_{n\geq 0} \in \mathcal{M}$ by

$$\left(I_\rho f\right)_n = \sum_{k=0}^{n} \rho\left(b_{k-1}\right) d_k f. \tag{35}$$

The generalized fractional integral I_ρ is obtained by taking $\gamma_n = \rho(b_n)$ in (32). If $\rho(r) = r^\alpha$, $\alpha > 0$, then we simply denote I_ρ by I_α.

For quasinormed spaces M_1 and M_2 of martingales, we denote by $B(M_1, M_2)$ the set of all bounded martingale transforms from M_1 to M_2; that is, $T \in B(M_1, M_2)$ means that there exists a positive constant C such that

$$\|Tf\|_{M_2} \leq C \|f\|_{M_1} \tag{36}$$

for all martingales $f = (f_n)_{n\geq 0} \in M_1$.

We first study the boundedness on the spaces $\mathcal{H}^S_{p,\phi}$. On martingale Campanato-Hardy spaces, we consider the fractional integral as a martingale transform associated with monotone multipliers. We say a sequence of nonnegative measurable functions $\gamma = (\gamma_n)_{n\geq 0}$ is almost decreasing if there exists a positive constant C such that

$$\gamma_k(\omega) \leq C \gamma_\ell(\omega) \quad \text{a.s. } \forall k \geq \ell. \tag{37}$$

For an almost decreasing sequence $\gamma = (\gamma_n)_{n\geq 0}$, we define A_γ by

$$A_\gamma = \inf\{C > 0 : C \text{ satisfies } (37)\}. \tag{38}$$

In Theorem 16 below, we do not need any assumption on $\{\mathcal{F}_n\}_{n\geq 0}$.

Theorem 16. *Let $\mathcal{B} \subset \mathcal{F}$, $p \in (0,\infty)$, and $\phi, \psi : (0,1] \to (0,\infty)$. Let $(\gamma_n)_{n\geq 0}$ be a sequence of nonnegative bounded almost decreasing adapted functions, and let I_γ be the martingale transform defined by (32). Assume that*

$$C_{\gamma,\phi,\psi} = \sup_{n\geq 0} \sup_{B \in \mathcal{B}_n} \frac{\phi(P(B))}{\psi(P(B))} \|\gamma_n \chi_B\|_{L_\infty} < \infty. \tag{39}$$

Then

$$I_\gamma \in B\left(\mathcal{H}^S_{p,\phi}, \mathcal{H}^S_{p,\psi}\right) \tag{40}$$

with

$$\left\|I_\gamma f\right\|_{\mathcal{H}^S_{p,\psi}} \leq A_\gamma C_{\gamma,\phi,\psi} \|f\|_{\mathcal{H}^S_{p,\phi}}. \tag{41}$$

If every σ-algebra $\mathcal{F}_n$ is generated by countable atoms, then we can apply Theorem 16 to the generalized fractional integral I_ρ. The following corollary extends [5, Theorem 5.8] to the spaces $\mathcal{H}^S_{p,\phi}$.

Corollary 17. *Assume that every σ-algebra $\mathcal{F}_n$ is generated by countable atoms and $\mathcal{B} = \mathcal{A}$. Let $p \in (0,\infty)$ and $\rho, \phi, \psi : (0,1] \to (0,\infty)$. Suppose that ρ is almost increasing and that*

$$\sup_{0<t\leq 1} \frac{\rho(t)\phi(t)}{\psi(t)} < \infty. \tag{42}$$

Then

$$I_\rho \in B\left(\mathcal{H}^S_{p,\phi}, \mathcal{H}^S_{p,\psi}\right). \tag{43}$$

If one further assumes that $\{\mathcal{F}_n\}_{n\geq 0}$ is regular and that ψ is almost increasing and satisfies the doubling condition, then

$$I_\rho \in B\left(\mathcal{H}^S_{p,\phi}, \mathcal{H}^S_{q,\psi}\right), \quad (p, q \in (0,\infty)). \tag{44}$$

We next study the boundedness on martingale Morrey-Hardy spaces $H^S_{p,\phi}$ and martingale Hardy spaces H^S_p.

Recall that $A(\mathcal{F}_n)^\perp = \mathcal{F}_n$ for all $n \geq 0$ if $\mathcal{F}_0$ is nonatomic.

Proposition 18. *Let $\mathcal{B} \subset \mathcal{F}$, $0 < p < q < \infty$, and $\phi : (0,1] \to (0,\infty)$. Let $(\gamma_n)_{n\geq 0}$ be a sequence of adapted functions. Suppose that $\mathcal{F}_0$ is nonatomic and that $\mathcal{B} = \mathcal{F}$. Assume in addition that ϕ is almost decreasing, that $t^{1/p}\phi(t)$ is almost increasing, and that $\lim_{t\to 0}\phi(t) = \infty$. Then, $I_\gamma \notin B(H^S_{p,\phi}, H^S_{q,\phi^{p/q}}) \setminus \{0\}$.*

According to Proposition 18, to consider the boundedness on $H^S_{p,\phi}$ and H^S_p, we suppose that every σ-algebra $\mathcal{F}_n$ is generated by countable atoms and that $\mathcal{B} = \mathcal{A}$.

In this case, if $\phi(r) = r^{-1/p}$ and $\mathcal{F}_0 = \{\Omega, \emptyset\}$, then $H^S_{p,\phi}$ coincides with H^S_p and $\|f\|_{H^S_{p,\phi}} = \|f\|_{H^S_p}$. However, if $\mathcal{F}_0 \neq \{\Omega, \emptyset\}$, then $H^S_{p,\phi}$ does not coincide with H^S_p in general. We do not always assume that $\mathcal{F}_0 = \{\Omega, \emptyset\}$.

Theorem 19. *Suppose that every σ-algebra $\mathcal{F}_n$ is generated by countable atoms, that $\mathcal{B} = \mathcal{A}$, and that $\{\mathcal{F}_n\}_{n\geq 0}$ is regular. Let*

$0 < p < q < \infty$ *and* $\phi : (0,1] \to (0,\infty)$, *and let* $(\gamma_n)_{n\geq 0}$ *be a sequence of nonnegative bounded adapted functions. Assume that* ϕ *satisfies the doubling condition and that there exists a positive constant* C *such that*

$$\sum_{k=0}^{n} \gamma_{k-1}\phi(b_{k-1})\chi_{\{b_k \neq b_{k-1}\}} + \phi(b_n) \times \sum_{k=n+1}^{\infty} \gamma_{k-1}\chi_{\{b_k \neq b_{k-1}\}} \leq C\phi(b_n)^{p/q} \quad a.s. \tag{45}$$

for all $n \geq 0$, *where* b_k *is the measurable function defined by* (33). *Then*

$$I_\gamma \in B\left(H^S_{p,\phi}, H^S_{q,\phi^{p/q}}\right). \tag{46}$$

Furthermore, if $\phi(t) = t^{-1/p}$, *then*

$$I_\gamma \in B\left(H^S_p, H^S_q\right). \tag{47}$$

As a consequence of Theorem 19, we have the following corollary, which gives an extension of [5, Corollary 5.7] to the spaces $H^S_{p,\phi}$ and gives a martingale Morrey-Hardy version of Gunawan [27, Theorem B]:

Corollary 20. *Suppose that every* σ*-algebra* $\mathcal{F}_n$ *is generated by countable atoms, that* $\mathcal{B} = \mathcal{A}$, *and that* $\{\mathcal{F}_n\}_{n\geq 0}$ *is regular. Let* $0 < p < q < \infty$ *and* $\rho, \phi : (0,1] \to (0,\infty)$. *Assume that* ρ *is bounded, that both* ρ *and* ϕ *satisfy the doubling condition, and that there exists a positive constant* C *such that*

$$\phi(r)\int_0^r \frac{\rho(t)}{t}dt + \int_r^1 \frac{\phi(t)\rho(t)}{t}dt \leq C\phi(r)^{p/q} \quad (0 < r < 1). \tag{48}$$

Then

$$I_\rho \in B\left(H^S_{p,\phi}, H^S_{q,\phi^{p/q}}\right). \tag{49}$$

The following extends the results for dyadic martingales in [6, 7] and the result for $0 < p \leq 1$ in [28].

Corollary 21. *Suppose that every* σ*-algebra* $\mathcal{F}_n$ *is generated by countable atoms, that* $\mathcal{B} = \mathcal{A}$, *and that* $\{\mathcal{F}_n\}_{n\geq 0}$ *is regular. Let* $0 < p < q < \infty$ *and* $-1/p + \alpha = -1/q$. *Then*

$$I_\alpha \in B\left(H^S_p, H^S_q\right). \tag{50}$$

5. Lemmas

We prepare some lemmas to prove the results in Sections 3 and 4.

Lemma 1. *Let* $\mathcal{B}_n$ *satisfy* $A(\mathcal{F}_n) \cup A(\mathcal{F}_n)^\perp \subset \mathcal{B}_n \subset \mathcal{F}_n (n \geq 0)$. *Suppose that* $\phi : (0,1] \to (0,\infty)$ *is almost increasing; that is,* $\phi(r) \leq C_0\phi(s)$ *for all* $0 < r \leq s \leq 1$. *Then, for all nonnegative functions* F,

$$\begin{aligned}\sup_{B\in\mathcal{F}_n} &\frac{1}{\phi(P(B))}\left(\frac{1}{P(B)}\int_B F\,dP\right) \\ &\leq C_0 \sup_{B\in\mathcal{B}_n} \frac{1}{\phi(P(B))}\left(\frac{1}{P(B)}\int_B F\,dP\right).\end{aligned} \tag{51}$$

Proof. Let

$$\mathcal{N} = \sup_{B\in\mathcal{B}_n} \frac{1}{\phi(P(B))}\left(\frac{1}{P(B)}\int_B F\,dP\right). \tag{52}$$

For any $B \in \mathcal{F}_n$, we can choose the sets B_j, $j = 0,1,2,\ldots$ (finite or infinite) such that

$$\begin{aligned} B = \cup_j B_j, \quad B_0 \in A(\mathcal{F}_n)^\perp, \quad B_j \in A(\mathcal{F}_n), \\ j = 1,2,\ldots, \\ P(B) = \sum_j P(B_j). \end{aligned} \tag{53}$$

In this case, $B_j \in \mathcal{B}_n$, $j = 0,1,2,\ldots$, since $A(\mathcal{F}_n) \cup A(\mathcal{F}_n)^\perp \subset \mathcal{B}_n$. Then

$$\begin{aligned} &\frac{1}{\phi(P(B))}\left(\frac{1}{P(B)}\int_B F\,dP\right) \\ &= \frac{1}{\phi(P(B))}\frac{1}{P(B)}\sum_j\left(\int_{B_j} F\,dP\right) \\ &= \sum_j \frac{P(B_j)}{P(B)}\left(\frac{1}{\phi(P(B))P(B_j)}\int_{B_j} F\,dP\right) \\ &\leq \sum_j \frac{P(B_j)}{P(B)}\left(\frac{C_0}{\phi(P(B_j))P(B_j)}\int_{B_j} F\,dP\right) \\ &\leq C_0\sum_j \frac{P(B_j)}{P(B)}\mathcal{N} \\ &= C_0\mathcal{N}. \end{aligned} \tag{54}$$

This shows the conclusion. □

Lemma 2 (see [5, Lemma 3.3]). *Suppose that every* σ*-algebra* $\mathcal{F}_n$ *is generated by countable atoms and that* $\{\mathcal{F}_n\}_{n\geq 0}$ *is regular. Then, every sequence*

$$B_0 \supset B_1 \supset \cdots \supset B_n \supset \cdots, \quad B_n \in A(\mathcal{F}_n), \tag{55}$$

has the following property: for each $n \geq 1$,

$$B_n = B_{n-1} \quad or \quad \left(1 + \frac{1}{R}\right)P(B_n) \leq P(B_{n-1}) \leq RP(B_n), \tag{56}$$

where R *is the constant in* (14).

Lemma 3. *Suppose that every σ-algebra $\mathcal{F}_n$ is generated by countable atoms and that $\{\mathcal{F}_n\}_{n\geq 0}$ is regular. For $B \in A(\mathcal{F}_m)$, let $B_j \in A(\mathcal{F}_j)$ be*

$$B = B_m \subset B_{m-1} \subset \cdots \subset B_0. \tag{57}$$

Let $\phi : (0,1] \to (0,\infty)$. Suppose that ϕ satisfies the doubling condition. Then, there exists a positive constant C, that depends only on ϕ and the regularity constant R, such that

$$\sum_{j=\ell+1}^{m} \phi(b_j)\, \chi_{\{b_j \neq b_{j-1}\}} \leq C \int_{b_m}^{b_\ell} \frac{\phi(t)}{t} dt \quad \text{on } B, \tag{58}$$

where b_j is the function defined by (33).

Proof. Let $J = \{j : b_j \neq b_{j-1}\}$. Then, by Lemma 2, we have

$$\begin{aligned}
&\sum_{j=\ell+1}^{m} \phi(b_j)\, \chi_{\{b_j \neq b_{j-1}\}} \\
&\quad = \sum_{j\in J, \ell<j\leq m} \phi(b_j) \\
&\quad = \sum_{j\in J, \ell<j\leq m} \frac{1}{\log(b_{j-1}/b_j)} \int_{b_j}^{b_{j-1}} \frac{\phi(b_j)}{t} dt \\
&\quad \lesssim \sum_{j\in J, \ell<j\leq m} \int_{b_j}^{b_{j-1}} \frac{\phi(t)}{t} dt \\
&\quad = \int_{b_m}^{b_\ell} \frac{\phi(t)}{t} dt.
\end{aligned} \tag{59}$$

□

In Theorem 13, we do not assume that $\{\mathcal{F}_n\}_{n\geq 0}$ is regular. Hence, we need the following lemma.

Lemma 4. *Let $\phi : (0,1] \to (0,\infty)$. Suppose that every σ-algebra $\mathcal{F}_n$ is generated by countable atoms. For $B \in A(\mathcal{F}_n)$, let $B_j \in A(\mathcal{F}_j)$ be*

$$B = B_n \subset B_{n-1} \subset \cdots \subset B_0. \tag{60}$$

For the sequence $\{B_k\}_{k=0}^n$ above, one defines a decreasing sequence of integers $n_j = n_j(\{B_k\}_{k=0}^n)$ inductively by

$$\begin{aligned}
n_1 &= \sup\{k \in [0,n] \cap \mathbb{Z} : P(B_k) \geq 2P(B)\}, \\
n_j &= \sup\left\{k \in [0, n_{j-1}] \cap \mathbb{Z} : P(B_k) \geq 2P\left(B_{n_{j-1}}\right)\right\} \\
&\qquad (j \geq 2),
\end{aligned} \tag{61}$$

where one uses the convention $\sup\emptyset = -1$. *One further defines*

$$J = \{j : n_j \geq 0\}, \qquad n_j^+ = 1 + n_j. \tag{62}$$

Suppose that ϕ satisfies the doubling condition. Then, there exists a positive constant C, that depends only on ϕ, such that

$$\sum_{j\in J} \phi\left(b_{n_j^+}\right) \leq C \int_{b_n}^{1} \frac{\phi(t)}{t} dt \quad \text{on } B, \tag{63}$$

where b_j is the function defined by (33).

Note that this lemma is the counterpart to the technique in [29, page 1104, line 5].

Proof. By the definition of n_j, if $j \in J$, then

$$b_{n_{j-1}} \leq b_{n_j^+} < 2b_{n_{j-1}} \leq b_{n_j} \quad \text{on } B, \tag{64}$$

where we use the convention $n_0 = n$.

Using the doubling condition on ϕ, we have

$$\sum_{j\in J} \phi\left(b_{n_j^+}\right) \lesssim \sum_{j\in J} \int_{b_{n_{j-1}}}^{2b_{n_{j-1}}} \frac{\phi(t)}{t} dt \leq \int_{b_n}^{1} \frac{\phi(t)}{t} dt \tag{65}$$

because the intervals $(b_{n_{j-1}}, 2b_{n_{j-1}})$ are disjointed by (64). □

In the proof of Theorem 19, we need the following estimates for the square function of $I_\gamma f$.

Lemma 5. *Suppose that every σ-algebra $\mathcal{F}_n$ is generated by countable atoms and that $\{\mathcal{F}_n\}_{n\geq 0}$ is regular. Let $p, q \in (0,\infty)$ with $p < q$. Let $(\gamma_n)_{n\geq 0}$ be a sequence of nonnegative bounded adapted functions. Suppose that $\phi : (0,1] \to (0,\infty)$ satisfies the doubling condition. Assume that there exists a positive constant C such that*

$$\begin{aligned}
&\sum_{k=0}^{n} \gamma_{k-1} \phi(b_{k-1})\, \chi_{\{b_k \neq b_{k-1}\}} + \phi(b_n) \\
&\quad \times \sum_{k=n+1}^{\infty} \gamma_{k-1} \chi_{\{b_k \neq b_{k-1}\}} \leq C\phi(b_n)^{p/q} \quad a.s.
\end{aligned} \tag{66}$$

for all $n \geq 0$, where b_k is the measurable function defined by (33). *Then, for $f \in \mathcal{M}$ with $\|f\|_{H^S_{p,\phi}} = 1$,*

$$\begin{aligned}
S_n(I_\gamma f) &\leq C\phi(b_{n-1})^{p/q}, \\
S^{(n)}(I_\gamma f) &\leq C\phi(b_n)^{p/q-1} S(f)
\end{aligned} \tag{67}$$

for all $n \geq 0$, where C is a positive constant independent of f.

Proof. Let $f \in \mathcal{M}$ such that $\|f\|_{H^S_{p,\phi}} = 1$. We first show that

$$|d_k f| \leq C\phi(b_{k-1}), \tag{68}$$

where C is a positive constant that depends only on ϕ and the regularity constant R. Let $B \in A(\mathcal{F}_k)$. Then, on the set B, keeping in mind that

$$|d_k f| = \left(\frac{1}{P(B)} \int_B |d_k f|^p dP\right)^{1/p}, \tag{69}$$

we have

$$\begin{aligned}
|d_k f| &\leq \left(\frac{1}{P(B)} \int_B S(f)^p dP\right)^{1/p} \\
&\leq \phi(P(B))\, \|f\|_{H^S_{p,\phi}} \lesssim \phi(b_{k-1}).
\end{aligned} \tag{70}$$

We have obtained (68). We now show (67). Using (68) and the assumption (66), we have

$$\begin{aligned} S_n(I_\gamma f)^2 &= \sum_{k=0}^{n} \gamma_{k-1}^2 |d_k f|^2 \\ &\lesssim \sum_{k=0}^{n} \gamma_{k-1}^2 \phi(b_{k-1})^2 \chi_{\{b_k \neq b_{k-1}\}} \\ &\le \left(\sum_{k=0}^{n} \gamma_{k-1} \phi(b_{k-1}) \chi_{\{b_k \neq b_{k-1}\}} \right)^2 \\ &\lesssim \phi(b_n)^{2p/q} \lesssim \phi(b_{n-1})^{2p/q}, \\ S^{(n)}(I_\gamma f)^2 &= \sum_{k=n+1}^{\infty} \gamma_{k-1}^2 |d_k f|^2 \\ &\le S^{(n)}(f)^2 \sum_{k=n+1}^{\infty} \gamma_{k-1}^2 \chi_{\{b_k \neq b_{k-1}\}} \\ &\le S^{(n)}(f)^2 \left(\sum_{k=n+1}^{\infty} \gamma_{k-1} \chi_{\{b_k \neq b_{k-1}\}} \right)^2 \\ &\lesssim \phi(b_n)^{2p/q-2} S^{(n)}(f)^2. \end{aligned} \tag{71}$$

□

Remark 22. In the course of the proof, the embedding $\ell^1 \hookrightarrow \ell^2$ is used. If one does not use the embedding, then

$$\begin{aligned} &\sum_{k=0}^{n} \gamma_{k-1} |d_k f| \lesssim \phi(b_{n-1})^{p/q}, \\ &\sum_{k=n+1}^{\infty} \gamma_{k-1} |d_k f| \lesssim \phi(b_n)^{p/q-1} S^{(n)}(f). \end{aligned} \tag{72}$$

6. Proofs of the Results in Section 3

In this section, we prove the results in Section 3.

Proposition 8 can be proved in the same way as [5, Proposition 2.2], so we omit the proof. Proposition 9 is a direct consequence of Lemma 1. Then, we will prove Theorems 11, 12, and 13.

Recall that $S^{(n)}(f)$ is defined by (11).

6.1. Proof of Theorem 11.

We first show Theorem 11, Burkholder's inequality on generalized martingale Campanato spaces.

Proof of Theorem 11. Let $f \in L_1^0$ and $B \in \mathscr{B}_n$. Then, $f\chi_B - E_n[f\chi_B] \in L_1^0 \subset \mathscr{M}$ and

$$d_k(f\chi_B - E_n[f\chi_B]) = \begin{cases} 0 & \text{if } k \le n, \\ (d_k f)\chi_B & \text{if } k > n. \end{cases} \tag{73}$$

Therefore, we have $S(f\chi_B - E_n[f\chi_B]) = S^{(n)}(f)\chi_B$. Hence, using Theorem 10, we have

$$\begin{aligned} c_p \left(\int_B |f - E_n f|^p dP \right)^{1/p} &= c_p \| f\chi_B - E_n[f\chi_B] \|_{L_p} \\ &\le \| S(f\chi_B - E_n[f\chi_B]) \|_{L_p} \\ &= \left(\int_B S^{(n)}(f)^p dP \right)^{1/p} \\ &\le C_p \| f\chi_B - E_n[f\chi_B] \|_{L_p} \\ &= C_p \left(\int_B |f - E_n f|^p dP \right)^{1/p}. \end{aligned} \tag{74}$$

We have obtained (21).

We next show (22). Using (74), we have

$$\begin{aligned} \left(\int_B |f - E_{n-1} f|^p dP \right)^{1/p} &\le \left(\int_B |f - E_n f|^p dP \right)^{1/p} \\ &\quad + \left(\int_B |d_n f|^p dP \right)^{1/p} \\ &\le c_p^{-1} \left(\int_B S^{(n)}(f)^p dP \right)^{1/p} \\ &\quad + \left(\int_B |d_n f|^p dP \right)^{1/p} \\ &\le c_p^{-1} \left(\int_B S^{(n-1)}(f)^p dP \right)^{1/p} \\ &\quad + \left(\int_B S^{(n-1)}(f)^p dP \right)^{1/p}. \end{aligned} \tag{75}$$

Therefore,

$$\frac{c_p}{1 + c_p} \| f \|_{\mathscr{L}^-_{p,\phi}} \le \| f \|_{\mathscr{H}^{S-}_{p,\phi}}. \tag{76}$$

For the converse part, using the inequality

$$\left(\int_B |f - E_n f|^p dP \right)^{1/p} \le 2 \left(\int_B |f - E_{n-1} f|^p dP \right)^{1/p}, \tag{77}$$

which we have mentioned in Remark 4, we obtain

$$\begin{aligned} \left(\int_B S^{(n-1)}(f)^p dP \right)^{1/p} &\le \left(\int_B S^{(n)}(f)^p dP \right)^{1/p} \\ &\quad + \left(\int_B |d_n f|^p dP \right)^{1/p} \\ &\le C_p \left(\int_B |f - E_n f|^p dP \right)^{1/p} \\ &\quad + \left(\int_B |d_n f|^p dP \right)^{1/p} \\ &\le (2C_p + 1) \left(\int_B |f - E_{n-1} f|^p dP \right)^{1/p}. \end{aligned} \tag{78}$$

That is,

$$\|f\|_{\mathcal{L}^{-}_{p,\phi}} \le \left(2C_p + 1\right) \|f\|_{\mathcal{H}^{S-}_{p,\phi}}. \tag{79}$$

□

6.2. Proof of Theorem 12. We next show Theorem 12, a relation between $\mathcal{L}_{p,\phi}$ and $\mathcal{L}^{-}_{p,\phi}$, $\mathcal{H}^{S}_{p,\phi}$, and $\mathcal{H}^{S-}_{p,\phi}$.

Proof of Theorem 12. Inequality (24) was mentioned in Remark 4. Inequality (25) is deduced from the inequality $S(f)^2 - S_n(f)^2 \le S(f)^2 - S_{n-1}(f)^2$.

We now show (26). Let $B \in \mathcal{B}_n$ and $B' = \{\omega \in \Omega : E_{n-1}[\chi_B](\omega) > 0\}$. Since $B' \in \mathcal{F}_{n-1}$, we have

$$E\left[\chi_{\Omega\setminus B'}\chi_B\right] = E\left[\chi_{\Omega\setminus B'}E_{n-1}\left[\chi_B\right]\right] = 0; \tag{80}$$

that is, $B \subset B'$.

Suppose that $\{\mathcal{F}_n\}_{n\ge0}$ is regular. To show (26), we first prove

$$B' = \left\{\omega \in \Omega : E_{n-1}\left[\chi_B\right](\omega) \ge \frac{1}{R}\right\}, \tag{81}$$

where R is the regularity constant. By the definition of B', we have $B' \supset \{\omega \in \Omega : E_{n-1}[\chi_B](\omega) \ge 1/R\}$. We will show the converse. By the regularity, we have $\chi_B \le RE_{n-1}[\chi_B]$. This implies $B \subset \{\omega \in \Omega : E_{n-1}[\chi_B](\omega) \ge 1/R\}$, or equivalently,

$$\chi_B \le \chi_{\{E_{n-1}[\chi_B]\ge 1/R\}}. \tag{82}$$

Operating E_{n-1}, we have

$$E_{n-1}\left[\chi_B\right] \le \chi_{\{E_{n-1}[\chi_B]\ge 1/R\}}. \tag{83}$$

We have obtained (81).

From (81), we deduce that

$$P\left(B'\right) = E\left[\chi_{\{E_{n-1}[\chi_B]\ge 1/R\}}\right] \le E\left[RE_{n-1}\left[\chi_B\right]\right] = RP\left(B\right). \tag{84}$$

Hence, using the assumption (23) and the doubling condition on ϕ with (84), we have

$$\begin{aligned} &\frac{1}{P(B)}\int_B \left|f - E_{n-1}f\right|^p dP \\ &\quad\le \frac{1}{P(B)}\int_{B'} \left|f - E_{n-1}f\right|^p dP \\ &\quad\lesssim \frac{1}{P(B')}\int_{B'} \left|f - E_{n-1}f\right|^p dP \\ &\quad\le \phi\left(P\left(B'\right)\right)^p \|f\|^p_{\mathcal{L}_{p,\phi}} \\ &\quad\lesssim \phi(P(B))^p \|f\|^p_{\mathcal{L}_{p,\phi}}. \end{aligned} \tag{85}$$

We have obtained (26).

By the same way as above, we have (27). The proof is completed. □

6.3. Proof of Theorem 13. We now prove Theorem 13, a relation between martingale Morrey spaces and martingale Campanato spaces.

Proof of Theorem 13. The part $\|f\|_{\mathcal{L}_{p,\phi}} \le 2\|f\|_{L_{p,\phi}}$ was shown in Remark 4, and the part $\|f\|_{\mathcal{H}^{S-}_{p,\phi}} \le \|f\|_{H^S_{p,\phi}}$ is obvious. We now show the part $\|f\|_{L_{p,\phi}} \lesssim \|f\|_{\mathcal{L}^{-}_{p,\phi}}$.

Let $B \in A(\mathcal{F}_n)$. We take $B_k \in A(\mathcal{F}_k)$ such that $B = B_n \subset B_{n-1} \subset \cdots \subset B_0$. Let n_j be the decreasing sequence of integers defined in Lemma 4, with convention $n_0 = n$. Since $n_{j-1} \ge n_j$, the function $E_{n_{j-1}}f - E_{n_j}f$ is constant on $B_{n_{j-1}}$. Therefore, on the set B, we have

$$\begin{aligned} &\left|E_{n_{j-1}}f - E_{n_j}f\right| \\ &\quad= \left(\frac{1}{P\left(B_{n_{j-1}}\right)}\int_{B_{n_{j-1}}}\left|E_{n_{j-1}}f - E_{n_j}f\right|^p dP\right)^{1/p} \\ &\quad\le 2^{1/p}\left(\frac{1}{P\left(B_{n_j^+}\right)}\int_{B_{n_j^+}}\left|E_{n_{j-1}}f - E_{n_j}f\right|^p dP\right)^{1/p} \\ &\quad\le 2^{1/p}\|f\|_{\mathcal{L}^{-}_{p,\phi}}\phi\left(P\left(B_{n_j^+}\right)\right), \end{aligned} \tag{86}$$

where n_j^+ is the same as in (62).

Let J be the same as in Lemma 4 and let $m = \max J$. Using Lemma 4 and the assumption (28), we have

$$\begin{aligned} \left|E_n f - E_{n_m}f\right| &\le \sum_{j\in J}\left|E_{n_{j-1}}f - E_{n_j}f\right| \\ &\lesssim \|f\|_{\mathcal{L}^{-}_{p,\phi}}\sum_{j\in J}\phi\left(P\left(B_{n_j^+}\right)\right) \\ &\lesssim \|f\|_{\mathcal{L}^{-}_{p,\phi}}\int_{P(B)}^1\frac{\phi(t)}{t}dt \\ &\lesssim \|f\|_{\mathcal{L}^{-}_{p,\phi}}\phi\left(P\left(B\right)\right) \quad \text{on } B. \end{aligned} \tag{87}$$

For $|E_{n_m}f|$, we may assume that $n_m > 0$. By the definition of n_m, we have $P(B_1) \le P(B_0) < 2P(B_{n_m}) \le 2P(B_1) \le 2P(B_0)$. Therefore,

$$\begin{aligned} \phi\left(P\left(B_1\right)\right) &\lesssim \int_{P(B_0)/2}^{P(B_0)}\frac{\phi(t)}{t}dt \\ &\lesssim \int_{P(B)}^1\frac{\phi(t)}{t}dt \lesssim \phi\left(P\left(B\right)\right). \end{aligned} \tag{88}$$

Hence, on the set B, the constant $E_{n_m}f$ has the following bound:

$$\begin{aligned} &\left|E_{n_m}f\right| = \left|E_{n_m}f - E_0 f\right| \\ &\quad= \left(\frac{1}{P(B_{n_m})}\int_{B_{n_m}}\left|E_{n_m}f - E_0 f\right|^p dP\right)^{1/p} \end{aligned}$$

$$\le 2^{1/p}\left(\frac{1}{P(B_1)}\int_{B_1}\left|E_{n_m}f-E_0f\right|^p dP\right)^{1/p}$$
$$\le 2^{1/p}\|f\|_{\mathscr{L}^-_{p,\phi}}\phi\left(P\left(B_1\right)\right)\lesssim\|f\|_{\mathscr{L}^-_{p,\phi}}\phi\left(P\left(B\right)\right). \tag{89}$$

Combining (87) and (89), we have

$$\left|E_nf\right|\lesssim\|f\|_{\mathscr{L}^-_{p,\phi}}\phi\left(P\left(B\right)\right)\quad\text{on } B. \tag{90}$$

Using (24) in Theorem 12, we have

$$\begin{aligned}\left(\int_B|f|^p dP\right)^{1/p}&\le\left(\int_B\left|f-E_nf\right|^p dP\right)^{1/p}\\&\quad+P(B)^{1/p}\left|E_nf\right|\\&\lesssim P(B)^{1/p}\phi\left(P\left(B\right)\right)\|f\|_{\mathscr{L}^-_{p,\phi}}\\&\quad+P(B)^{1/p}\phi\left(P\left(B\right)\right)\|f\|_{\mathscr{L}^-_{p,\phi}}\\&\sim P(B)^{1/p}\phi\left(P\left(B\right)\right)\|f\|_{\mathscr{L}^-_{p,\phi}};\end{aligned}\tag{91}$$

that is,

$$\|f\|_{L_{p,\phi}}\lesssim\|f\|_{\mathscr{L}^-_{p,\phi}}. \tag{92}$$

We can show $\|f\|_{H^s_{p,\phi}}\lesssim\|f\|_{\mathscr{H}^{s-}_{p,\phi}}$ by the same way. Indeed, in (86), we can replace $|E_{n_{j-1}}f-E_{n_j}f|$ and $\|f\|_{\mathscr{L}^-_{p,\phi}}$ by $\{S_{n_{j-1}}(f)^2-S_{n_j}(f)^2\}^{1/2}$ and $\|f\|_{\mathscr{H}^{s-}_{p,\phi}}$, respectively. The rest is similar and we can obtain $\|f\|_{H^s_{p,\phi}}\lesssim\|f\|_{\mathscr{H}^{s-}_{p,\phi}}$.

If $\mathscr{B}=\mathscr{A}$, then $\mathscr{B}$ satisfies (23). Therefore, if $\mathscr{F}$ is regular, we can apply Theorem 12 to obtain the equivalence of $\|f\|_{L_{p,\phi}}$, $\|f\|_{\mathscr{L}_{p,\phi}}$, and $\|f\|_{\mathscr{L}^-_{p,\phi}}$ and the equivalence of $\|f\|_{H^s_{p,\phi}}$, $\|f\|_{\mathscr{H}^s_{p,\phi}}$, and $\|f\|_{\mathscr{H}^{s-}_{p,\phi}}$. □

6.4. Proof of Theorem 15. We will now prove Theorem 15, the John-Nirenberg-type theorem for martingale Campanato-Hardy spaces. Following Weisz [19, Definition 2.45], we define

$$\|f\|_{\mathrm{BMO}^s_p}=\sup_{n\ge0}\left\|\left(E_n\left[\left\{S(f)^2-S_{n-1}(f)^2\right\}^{p/2}\right]\right)^{1/p}\right\|_{L_\infty}, \tag{93}$$

for $f\in\mathscr{M}$ and $p\in(0,\infty)$.

Proof of Theorem 15. We may assume that $\mathscr{B}=\mathscr{F}$ by Proposition 9.

By Hölder's inequality and Theorem 12, we only need to show that $\|f\|_{\mathscr{H}^s_{p,\phi}}\le C_{p,q,\phi}\|f\|_{\mathscr{H}^{s-}_{q,\phi}}$ for $0<q\le1<p$.

Recall the notation $S^{(n)}(f)^2=S(f)^2-S_n(f)^2$. Let $f\in\mathscr{H}^{s-}_{q,\phi}\cap L^0_1\subset\mathscr{M}$, $A\in\mathscr{F}_n$, and $m\ge n+1$. By (73), we have $S^{(m-1)}(f\chi_A-E_n[f\chi_A])=S^{(m-1)}(f)\chi_A$. Hence, for $B\in\mathscr{F}_m$, $m\ge n+1$, we have

$$\begin{aligned}&\frac{1}{P\left(B\right)}\int_B S^{(m-1)}(f\chi_A-E_n\left[f\chi_A\right])^q dP\\&\quad=\frac{1}{P\left(B\right)}\int_{A\cap B}S^{(m-1)}(f)^q dP\\&\quad\le\frac{1}{P\left(A\cap B\right)}\int_{A\cap B}S^{(m-1)}(f)^q dP\\&\quad\le\phi(P\left(A\cap B\right))^q\|f\|^q_{\mathscr{H}^{s-}_{q,\phi}}\\&\quad\lesssim\phi(P\left(A\right))^q\|f\|^q_{\mathscr{H}^{s-}_{q,\phi}}.\end{aligned}\tag{94}$$

Therefore, for the $\{\mathscr{F}_m\}_{m\ge n}$-martingale $(E_m[f\chi_A]-E_n[f\chi_A])_{m\ge n}$, we have

$$\begin{aligned}&\left\|\left(E_m\left[f\chi_A\right]-E_n\left[f\chi_A\right]\right)_{m\ge n}\right\|_{\mathrm{BMO}^s_q}\\&\quad\lesssim\phi\left(P\left(A\right)\right)\|f\|_{\mathscr{H}^{s-}_{q,\phi}}.\end{aligned}\tag{95}$$

By [19, Theorem 2.50], there exists a positive constant $C_{p,q,\phi}$ that depends only on p, q, and ϕ such that

$$\begin{aligned}&\left(E_{n+1}\left[S^{(n)}(f\chi_A-E_n\left[f\chi_A\right])^p\right]\right)^{1/p}\\&\quad\le C_{p,q,\phi}\phi\left(P\left(A\right)\right)\|f\|_{\mathscr{H}^{s-}_{q,\phi}}.\end{aligned}\tag{96}$$

Combining (96) and the fact that $S^{(n)}(f\chi_A-E_n[f\chi_A])=S^{(n)}(f)\chi_A$, we have

$$\left(E_{n+1}\left[S^{(n)}(f)^p\right]\right)^{1/p}\chi_A\le C_{p,q,\phi}\phi\left(P\left(A\right)\right)\|f\|_{\mathscr{H}^{s-}_{q,\phi}}. \tag{97}$$

Therefore, for $A\in\mathscr{F}_n$, we have

$$\begin{aligned}&\left(\frac{1}{P(A)}\int_A S^{(n)}(f)^p dP\right)^{1/p}\\&\quad=\left(\frac{1}{P(A)}\int_A E_{n+1}\left[S^{(n)}(f)^p\right]dP\right)^{1/p}\\&\quad\le C_{p,q,\phi}\phi\left(P\left(A\right)\right)\|f\|_{\mathscr{H}^{s-}_{q,\phi}};\end{aligned}\tag{98}$$

that is,

$$\|f\|_{\mathscr{H}^s_{p,\phi}}\le C_{p,q,\phi}\|f\|_{\mathscr{H}^{s-}_{q,\phi}} \tag{99}$$

for $f\in\mathscr{H}^{s-}_{q,\phi}\cap L^0_1$. For general $f\in\mathscr{H}^{s-}_{q,\phi}$, applying (99) to the martingale $f^{(m)}=(f_{\min(m,n)})_{n\ge0}$, we have

$$\left\|f^{(m)}\right\|_{\mathscr{H}^s_{p,\phi}}\le C_{p,q,\phi}\left\|f^{(m)}\right\|_{\mathscr{H}^{s-}_{q,\phi}}\le C_{p,q,\phi}\|f\|_{\mathscr{H}^{s-}_{q,\phi}}. \tag{100}$$

Taking $p=2$ in (100), we have that f is an L^2-bounded martingale. Therefore, we have (99) for all $f\in\mathscr{H}^{s-}_{q,\phi}$. The proof is completed. □

7. Proofs of the Results in Section 4

In this section, we prove the results in Section 4.

7.1. Proofs of Theorem 16 and Corollary 17. Recall that $S^{(n)}(f)^2 = S(f)^2 - S_n(f)^2$.

Proof of Theorem 16. Using the assumption that $(\gamma_n)_{n\geq 0}$ is almost decreasing, we have

$$\begin{aligned} S^{(n)}\left(I_\gamma f\right)^2 &= \sum_{k=n+1}^{\infty} \left|\gamma_{k-1} d_k f\right|^2 \\ &\leq A_\gamma^2 \gamma_n^2 \sum_{k=n+1}^{\infty} \left|d_k f\right|^2 \\ &= A_\gamma^2 \gamma_n^2 S^{(n)}(f)^2. \end{aligned} \tag{101}$$

Then, for $B \in \mathscr{B}_n$, using the assumption (39), we have

$$\begin{aligned} \int_B S^{(n)}\left(I_\gamma f\right)^p dP &\leq A_\gamma^p \int_B \gamma_n^p S^{(n)}(f)^p dP \\ &\leq A_\gamma^p \|\gamma_n \chi_B\|_{L_\infty}^p \int_B S^{(n)}(f)^p dP \\ &\leq A_\gamma^p \|\gamma_n \chi_B\|_{L_\infty}^p P(B)\, \phi(P(B))^p \|f\|_{\mathscr{H}_{p,\phi}^S}^p \\ &\leq A_\gamma^p C_{\gamma,\phi,\psi}^p P(B)\, \psi(P(B))^p \|f\|_{\mathscr{H}_{p,\phi}^S}^p. \end{aligned} \tag{102}$$

Therefore, we have

$$\left\|I_\gamma f\right\|_{\mathscr{H}_{p,\psi}^S} \leq A_\gamma C_{\gamma,\phi,\psi} \|f\|_{\mathscr{H}_{p,\phi}^S} \tag{103}$$

and $I_\gamma \in B(\mathscr{H}_{p,\phi}^S, \mathscr{H}_{p,\psi}^S)$. The proof is completed. □

Proof of Corollary 17. Let $\gamma_n = \rho(b_n)$. Then, we have that $(\gamma_n)_{n\geq 0}$ is almost decreasing and that $\|\gamma_n \chi_B\|_{L_\infty} = \rho(P(B))$ for $B \in A(\mathscr{F}_n)$. Hence,

$$\begin{aligned} C_{\gamma,\phi,\psi} &= \sup_{n\geq 0} \sup_{B\in A(\mathscr{F}_n)} \frac{\rho(P(B))\,\phi(P(B))}{\psi(P(B))} \\ &\leq \sup_{0<t\leq 1} \frac{\rho(t)\,\phi(t)}{\psi(t)} < \infty. \end{aligned} \tag{104}$$

Therefore, we can apply Theorem 16 to obtain $I_\gamma \in B(\mathscr{H}_{p,\phi}^S, \mathscr{H}_{p,\psi}^S)$. If we further assume that $\{\mathscr{F}_n\}_{n\geq 0}$ is regular and that ψ is almost increasing and satisfies the doubling condition, then, by the John-Nirenberg-type equivalence (Theorem 15), we have $I_\gamma \in B(\mathscr{H}_{p,\phi}^S, \mathscr{H}_{q,\psi}^S)$ for all $q \in (0,\infty)$. □

Remark 23. Assume that (39) holds. Suppose further that there exists a positive number C' such that $\sum_{k=n+1}^{\infty} \gamma_{k-1} \leq C'\gamma_n$ a.s. for all $n \geq 0$. Then, in the light of Remark 22, we see that

$$\begin{aligned} &\sup_{n\geq 0} \sup_{B\in\mathscr{B}_n} \frac{1}{\psi(P(B))} \left(\frac{1}{P(B)} \int_B \left(\sum_{k=n+1}^{\infty} |\gamma_{k-1} d_k f| \right)^p dP \right)^{1/p} \\ &\quad \leq C' C_{\gamma,\phi,\psi} \|f\|_{\mathscr{H}_{p,\phi}^S}. \end{aligned} \tag{105}$$

7.2. Proofs of Theorem 19 and Corollaries 20 and 21

Proof of Theorem 19. We first show the part $I_\gamma \in B(H_{p,\phi}^S, H_{q,\phi^{p/q}}^S)$. Assume (45). Let $f = (f_n)_{n\geq 0} \in \mathscr{M}$ such that $\|f\|_{H_{p,\phi}^S} = 1$. We need only to show that there exists $C > 0$ independent of f such that

$$\left\|I_\gamma f\right\|_{H_{q,\phi^{p/q}}^S} \leq C. \tag{106}$$

To obtain (106), we first show that

$$S\left(I_\gamma f\right) \leq C S(f)^{p/q}. \tag{107}$$

Let $N = \sum_{n=0}^{\infty} \chi_{\{\phi(b_n)\leq S(f)\}}$. We define measurable subsets Ω_1, Ω_2, and Ω_3 by

$$\begin{aligned} \Omega_1 &= \{N = \infty\}, \\ \Omega_2 &= \{N = 0\}, \\ \Omega_3 &= \{0 < N < \infty\}. \end{aligned} \tag{108}$$

Let $\omega \in \Omega_1$. Then, we can take infinitely many integers n such that $\phi(b_{n-1}(\omega)) \leq S(f)(\omega)$. For such n, we have

$$S_n\left(I_\gamma f\right)(\omega) \leq C\phi(b_{n-1}(\omega))^{p/q} \leq CS(f)(\omega)^{p/q} \tag{109}$$

by Lemma 5. Letting $n \to \infty$ along n that satisfies $\phi(b_{n-1}(\omega)) \leq S(f)(\omega)$, we have (107) on Ω_1.

On Ω_2, again by Lemma 5, we have

$$S\left(I_\gamma f\right) \leq C\phi(b_0)^{p/q-1} S(f) \leq CS(f)^{p/q-1} S(f) = CS(f)^{p/q}. \tag{110}$$

Let $\omega \in \Omega_3$. Then, we can take an integer n such that

$$\phi\left(b_{n-1}(\omega)\right) \leq S(f)(\omega), \qquad \phi\left(b_n(\omega)\right) > S(f)(\omega). \tag{111}$$

Hence, by Lemma 5, we have

$$\begin{aligned} S\left(I_\gamma f\right)(\omega) &\leq S_n\left(I_\gamma f\right)(\omega) \\ &\quad + S^{(n)}\left(I_\gamma f\right)(\omega) \\ &\lesssim \phi(b_{n-1}(\omega))^{p/q} \\ &\quad + \phi(b_n(\omega))^{p/q-1} S(f)(\omega) \\ &\leq S(f)(\omega)^{p/q} \\ &\quad + S(f)(\omega)^{p/q-1} S(f)(\omega) \\ &\lesssim S(f)(\omega)^{p/q}. \end{aligned} \tag{112}$$

We have obtained (107).

We now show (106). Let $B \in \cup_n A(\mathcal{F}_n)$. Using (107), we have

$$\begin{aligned} &\left(\int_B S\left(I_\gamma f\right)^q dP\right)^{1/q} \\ &\quad \lesssim \left\{\left(\int_B S(f)^p dP\right)^{1/p}\right\}^{p/q} \\ &\quad \leq P(B)^{1/q} \phi(P(B))^{p/q} \|f\|_{H^S_{p,\phi}}^{p/q} \\ &\quad = P(B)^{1/q} \phi(P(B))^{p/q}. \end{aligned} \tag{113}$$

We have obtained (106).

We now show the part $I_\gamma \in B(H^S_p, H^S_q)$. Let $\phi(t) = t^{-1/p}$. We simply denote $H^S_{p,\phi}$ by $H^S_{p,-1/p}$. Let $f = (f_n)_{n\geq 0} \in \mathcal{M}$ such that $\|f\|_{H^S_p} = 1$. Observe that $\|f\|_{H^S_{p,-1/p}} \leq \|f\|_{H^S_p}$. By the assumption that (45) holds for $\phi(t) = t^{-1/p}$, we can apply (107) to $f/\|f\|_{H^S_{p,-1/p}}$, and we have

$$S\left(I_\gamma f\right) \leq CS(f)^{p/q} \|f\|_{H^S_{p,-1/p}}^{1-p/q} \leq CS(f)^{p/q}. \tag{114}$$

Hence, we obtain

$$\left(\int_\Omega S\left(I_\gamma f\right)^q dP\right)^{1/q} \lesssim \left\{\left(\int_\Omega S(f)^p dP\right)^{1/p}\right\}^{p/q} = 1. \tag{115}$$

The proof is completed. □

Proof of Corollary 20. Let $\gamma_n = \rho(b_n)$. We only have to verify (45). Using Lemma 3 and the assumption (48), we have

$$\begin{aligned} &\sum_{k=0}^{n} \rho\left(b_{k-1}\right) \phi\left(b_{k-1}\right) \chi_{\{b_k \neq b_{k-1}\}} \\ &\quad + \phi\left(b_n\right) \sum_{k=n+1}^{\infty} \rho\left(b_{k-1}\right) \chi_{\{b_k \neq b_{k-1}\}} \\ &\quad \lesssim \int_{b_n}^1 \frac{\rho(t)\phi(t)}{t} dt \\ &\quad + \phi\left(b_n\right) \int_0^{b_n} \frac{\rho(t)}{t} dt \lesssim \phi(b_n)^{p/q}. \end{aligned} \tag{116}$$

By Theorem 19, we have the conclusion. □

Proof of Corollary 21. If $\rho(r) = r^\alpha$ and $\phi(t) = t^{-1/p}$, then

$$\begin{aligned} \int_r^1 \frac{\rho(t)\phi(t)}{t} dt + \phi(r) \int_0^r \frac{\rho(t)}{t} dt &\sim r^{\alpha-1/p} \\ &= r^{-1/q} = \phi(r)^{p/q}. \end{aligned} \tag{117}$$

Observing (116) and applying Theorem 19 to I_α, we have the conclusion. □

Remark 24. In the light of Remark 22, we see that

$$\begin{aligned} &\sup_{n\geq 0} \sup_{B\in A(\mathcal{F}_n)} \frac{1}{\phi(P(B))^{p/q}} \\ &\quad \times \left(\frac{1}{P(B)} \int_B \left(\sum_{k=1}^{\infty} \left|\gamma_{k-1} d_k f\right|\right)^q dP\right)^{1/q} \\ &\quad \leq C\|f\|_{H^S_{p,\phi}}, \end{aligned} \tag{118}$$

which is similar to (105).

In words of harmonic analysis, this corresponds to the embedding $F^{s+n/p_1-n/p_2}_{p_1q_1} \hookrightarrow F^s_{p_2q_2}$ for $0 < p_1 < p_2 < \infty$, $0 < q_1, q_2 \leq \infty$, and $s \in \mathbb{R}$; see [30, Section 2.3] and [30, page 129] for the definition of the space F^s_{pq} and the above embedding, respectively. It may be interesting to observe that this embedding is translated into the fact that I_α makes functions have bounded variation.

7.3. *Proof of Proposition 18.*

In this subsection, we prove Proposition 18.

Proof of Proposition 18. To prove $I_\gamma \notin B(H^S_{p,\phi}, H^S_{q,\phi^{p/q}}) \setminus \{0\}$, we only need to show the following for any $f = (f_n)_{n\geq 0} \in \mathcal{M}$:

$$\begin{aligned} &\text{if } I_\gamma \in B\left(H^S_{p,\phi}, H^S_{q,\phi^{p/q}}\right), \text{ then } \chi_{\{|\gamma_{k-1}|>0\}} d_k f = 0 \\ &\qquad (k = 1, 2, \ldots). \end{aligned} \tag{119}$$

We now show (119) for $f = (f_n)_{n\geq 0} \in \mathcal{M}$. We may assume that $P(|\gamma_{k-1}| > 0) > 0$. For γ_{k-1}, define

$$\begin{aligned} &\mathcal{F}^+_k\left(\{|\gamma_{k-1}| > 0\}\right) \\ &\quad = \left\{B \in \mathcal{F}_k \setminus \mathcal{F}_{k-1} : P\left(\{|\gamma_{k-1}| > 0\} \cap B\right) > 0\right\}. \end{aligned} \tag{120}$$

If $\mathcal{F}^+_k(\{|\gamma_{k-1}| > 0\}) = \emptyset$, then the function $\chi_{\{|\gamma_{k-1}|>0\}} d_k f$ is $\mathcal{F}_{k-1}$-measurable. Therefore, we have

$$\begin{aligned} \chi_{\{|\gamma_{k-1}|>0\}} d_k f &= E_{k-1}\left[\chi_{\{|\gamma_{k-1}|>0\}} d_k f\right] \\ &= \chi_{\{|\gamma_{k-1}|>0\}} E_{k-1}\left[d_k f\right] = 0. \end{aligned} \tag{121}$$

To complete the proof of (119), we only have to show the following:

$$\text{if } \mathscr{F}_k^+\left(\{|\gamma_{k-1}| > 0\}\right) \neq \emptyset, \text{ then } I_\gamma \notin B\left(H_{p,\phi}^S, H_{q,\phi^{p/q}}^S\right). \tag{122}$$

Assume that $\mathscr{F}_k^+(\{|\gamma_{k-1}| > 0\}) \neq \emptyset$. We fix $B \in \mathscr{F}_k \setminus \mathscr{F}_{k-1}$ and $\delta > 0$ such that $P(\{|\gamma_{k-1}| \geq \delta\} \cap B) > 0$. Let $B_1 = \{|\gamma_{k-1}| \geq \delta\}$ and $B_1' = \{|\gamma_{k-1}| \geq \delta\} \cap B$. Note that $B_1 \in \mathscr{F}_{k-1}$, $B_1' \in \mathscr{F}_k^+(\{|\gamma_{k-1}| > 0\})$, and $B_1' \subset B_1$.

To prove (122), we define two decreasing sequences of measurable sets $\{B_n\}_{n=1}^\infty$ and $\{B_n'\}_{n=1}^\infty$ that satisfy

$$\begin{gathered} B_n \in \mathscr{F}_{k-1}, \quad P(B_n) = \frac{P(B_1)}{2^{n-1}}, \\ B_n' \in \mathscr{F}_k^+\left(\{|\gamma_{k-1}| > 0\}\right), \quad B_n' \subset B_n \end{gathered} \tag{123}$$

for every $n \geq 1$, inductively as follows.

Suppose that we can choose B_{n-1} and B_{n-1}' that satisfy

$$\begin{gathered} B_{n-1} \in \mathscr{F}_{k-1}, \quad P(B_{n-1}) = \frac{P(B_1)}{2^{n-2}}, \\ B_{n-1}' \in \mathscr{F}_k^+\left(\{|\gamma_{k-1}| > 0\}\right), \quad B_{n-1}' \subset B_{n-1}. \end{gathered} \tag{124}$$

By the assumption that $\mathscr{F}_0$ is nonatomic, $\mathscr{F}_{k-1}$ is also nonatomic. Hence, there exists $B_n \in \mathscr{F}_{k-1}$ such that $B_n \subset B_{n-1}$, $P(B_n) = P(B_{n-1})/2$ and $P(B_n \cap B_{n-1}') > 0$. Let $B_n' = B_n \cap B_{n-1}'$. Then, we have B_n and B_n' with $B_n \subset B_{n-1}$ and $B_n' \subset B_{n-1}'$ that satisfy (123).

For the set B_n' defined above, let $g_n = \chi_{B_n'} - E_{k-1}[\chi_{B_n'}]$. Since B_n is $\mathscr{F}_{k-1}$-measurable, we have

$$\begin{aligned} g_n \chi_{\Omega \setminus B_n} &= \left(\chi_{B_n'} - E_{k-1}\left[\chi_{B_n'}\right]\right) \chi_{\Omega \setminus B_n} \\ &= \chi_{B_n'} \chi_{\Omega \setminus B_n} - E_{k-1}\left[\chi_{B_n'} \chi_{\Omega \setminus B_n}\right] \\ &= 0. \end{aligned} \tag{125}$$

By (125) and the assumption that $t^{1/p}\phi(t)$ is almost increasing, we have, for any $A \in \cup_{n=0}^\infty \mathscr{F}_n$,

$$\begin{aligned} &\frac{1}{\phi(P(A))}\left(\frac{1}{P(A)} \int_A |g_n|^p dP\right)^{1/p} \\ &\quad = \frac{1}{\phi(P(A)) P(A)^{1/p}} \left(\int_{A \cap B_n} |g_n|^p dP\right)^{1/p} \\ &\quad \lesssim \frac{1}{\phi(P(A \cap B_n)) P(A \cap B_n)^{1/p}} \\ &\quad\quad \times \left(\int_{A \cap B_n} |g_n|^p dP\right)^{1/p} \\ &\quad = \frac{1}{\phi(P(A \cap B_n))} \\ &\quad\quad \times \left(\frac{1}{P(A \cap B_n)} \int_{A \cap B_n} |g_n|^p dP\right)^{1/p}. \end{aligned} \tag{126}$$

We now show that $I_\gamma \notin B(H_{p,\phi}^S, H_{q,\phi^{p/q}}^S)$. Suppose that $I_\gamma \in B(H_{p,\phi}^S, H_{q,\phi^{p/q}}^S)$; that is, there exists a positive number C such that

$$\left\|I_\gamma f\right\|_{H_{q,\phi^{p/q}}^S} \leq C\|f\|_{H_{p,\phi}^S} \tag{127}$$

for all $f \in H_{p,\phi}^S$.

Since $B_n' \in \mathscr{F}_k \setminus \mathscr{F}_{k-1}$, we have $d_k g_n = g_n \neq 0$, and $d_j g_n = 0$ for $j \neq k$.

Therefore, we have

$$S(g_n) = g_n, \qquad S\left(I_\gamma g_n\right) = \gamma_{k-1} g_n. \tag{128}$$

For g_n, we take $D_n \in \cup_{n=0}^\infty \mathscr{F}_n$ such that

$$\frac{1}{\phi(P(D_n))}\left(\frac{1}{P(D_n)} \int_{D_n} |g_n|^p dP\right)^{1/p} \geq \frac{1}{2}\|g_n\|_{H_{p,\phi}^S}. \tag{129}$$

By (126), we may assume that $D_n \subset B_n$. As a consequence, we have $D_n \subset B_1 = \{|\gamma_{k-1}| \geq \delta\}$, and $\lim_{n\to\infty} P(D_n) = 0$ by (123). Then, using (127) with (128), we have

$$\begin{aligned} &\frac{1}{\phi(P(D_n))^{p/q}}\left(\frac{1}{P(D_n)} \int_{D_n} |g_n|^q dP\right)^{1/q} \\ &\quad \leq \frac{1}{\delta} \frac{1}{\phi(P(D_n))^{p/q}} \\ &\quad\quad \times \left(\frac{1}{P(D_n)} \int_{D_n} |\gamma_{k-1} g_n|^q dP\right)^{1/q} \\ &\quad \leq \frac{1}{\delta}\left\|I_\gamma g_n\right\|_{H_{q,\phi^{p/q}}^S} \\ &\quad \leq \frac{C}{\delta}\|g_n\|_{H_{p,\phi}^S} \\ &\quad \leq \frac{2C}{\delta} \frac{1}{\phi(P(D_n))} \\ &\quad\quad \times \left(\frac{1}{P(D_n)} \int_{D_n} |g_n|^p dP\right)^{1/p} \\ &\quad \leq \frac{2C}{\delta} \frac{1}{\phi(P(D_n))} \\ &\quad\quad \times \left(\frac{1}{P(D_n)} \int_{D_n} |g_n|^q dP\right)^{1/q}. \end{aligned} \tag{130}$$

Therefore, we have

$$\phi(P(D_n))^{1-p/q} \leq \frac{2C}{\delta}. \tag{131}$$

However, this contradicts $p < q$ and $\lim_{t\to 0}\phi(t) = \infty$. We have (122) and hence have (119). □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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