

Research Article **Nontrivial Solutions for a Modified Capillary Surface Equation**

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A negative solution and a positive solution are obtained for a modified capillary surface equation by variational methods.

1. Introduction

In this paper, we study the existence of nontrivial solutions to the following quasilinear elliptic equation:

$$-\operatorname{div}\left(\frac{|\nabla u|^{2p-2}\nabla u}{\sqrt{1+|\nabla u|^{2p}}}\right) = f(x,u) \quad \text{in } \Omega,$$

$$u = 0 \quad \text{on } \partial\Omega,$$
(1)

where p > 1, and Ω is a bounded domain in \mathbb{R}^N with smooth boundary. The function $f \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$ with the subcritical growth

(*f*)

$$\left|f\left(x,t\right)\right| \le c\left(1+\left|t\right|^{q-1}\right), \quad t \in \mathbb{R}, \ x \in \Omega,$$
(2)

where $q \in [1, Np/(N - p))$ if $1 or <math>q \in [1, +\infty)$ if $1 < N \le p$, and *c* is a positive constant.

In the case that p = 1, (1) is the mean curvature equation or the capillary surface equation; when $f(x, u) \equiv u$, it describes the equilibrium shape of a liquid surface with constant surface tension in a uniform gravity field, and this is the shape of a pendent drop [1]. When p > 1, one calls (1) a modified capillary surface equation which is also worth considering even though it is not exactly the capillary surface equation [2]. For the capillary surface equation, radially symmetric solutions in the case that Ω is a ball or entire space have been investigated precisely; See, for example, [3–5] and the references therein. In [2], by minimization sequence method and the Ambrosetti-Rabinowitz mountain pass lemma without Palais-Smale condition, positive solutions were obtained to nonlinear eigenvalue problem for the modified capillary surface equation which is of the form

$$-\operatorname{div}\left(\frac{|\nabla u|^{2p-2}\nabla u}{\sqrt{1+|\nabla u|^{2p}}}\right) = \lambda f(x,u) \quad \text{in } \Omega,$$

$$u \ge 0 \quad \text{in } \Omega,$$

$$u = 0 \quad \text{on } \partial\Omega,$$
(3)

where λ is a positive parameter. In the proof of the main results of [2], λ is crucial not only to the existence of global or local minimizer but also to the construction of mountain pass geometry. In our paper, one object is to find existence conditions of solutions to (1) without the constraint of λ . Since

$$\sqrt{1 + |\nabla u|^{2p}} - 1 \sim |\nabla u|^p$$
 as $|\nabla u| \longrightarrow \infty$, (4)

the other object is to investigate the probability to present the property of f by the eigenvalue of the problem

$$-\Delta_{p}u = \lambda |u|^{p-2}u \quad \text{in } \Omega,$$

$$u = 0 \quad \text{on } \partial\Omega,$$
 (5)

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$.

In the following, we recall some known facts about problem (5). Let $\lambda_1 > 0$ be the first eigenvalue of the problem (5). It is known that λ_1 is characterized by

$$\lambda_{1} := \inf\left\{\int_{\Omega} |\nabla u|^{p} dx : \int_{\Omega} |u|^{p} dx = 1, u \in W_{0}^{1,p}(\Omega) \setminus \{0\}\right\},$$
(6)

where $W_0^{1,p}(\Omega)$ is the reflexive Banach space defined as the completion of $C_0^{\infty}(\Omega)$ with respect to the norm ||u|| := $\left(\int_{\Omega} |\nabla u|^p dx\right)^{1/p}$. Also, λ_1 is single and has an associated eigenfunction $\varphi_1 > 0$ in Ω and $\|\varphi_1\| = 1$. The reader is referred to [6, 7] for details.

By a solution u of (1), we mean that u satisfies (1) in the weak sense; that is, for all $\varphi \in W_0^{1,p}(\Omega)$,

$$\int_{\Omega} \frac{|\nabla u|^{2p-2} \nabla u \nabla \varphi}{\sqrt{1+|\nabla u|^{2p}}} dx = \int_{\Omega} f(x, u) \varphi dx.$$
(7)

A solution such that $u(x) \ge 0$ in Ω and $u \ne 0$, respectively, $u(x) \leq 0$ in Ω and $u \neq 0$, is a positive, respectively, negative, solution.

Define

$$J(u) = \frac{1}{p} \int_{\Omega} \left(\sqrt{1 + |\nabla u|^{2p}} - 1 \right) dx, \quad u \in W_0^{1,p}(\Omega),$$

$$K(u) = \int_{\Omega} F(x, u) dx, \quad u \in W_0^{1,p}(\Omega),$$

$$I(u) = J(u) - K(u), \quad u \in W_0^{1,p}(\Omega),$$
(8)

where $F(x,t) = \int_0^t f(x,s) ds$. From a variational stand point, finding solutions of (1) in $W_0^{1,p}(\Omega)$ is equivalent to finding critical points of the C^1 functional *I*. As to the differentiability of the functional I, one can consult [2] for details. Since fsatisfies the subcritical growth condition (f_0) , stand proofs show that *K* is weakly continuous. Since the function $\varphi(t) =$ $\sqrt{1+t^{2p}}$ is convex, the functional *J* is also convex. In addition, J belongs to C^1 . Hence, J is weakly lower semicontinuous. Thus, we have shown that I is weakly lower semicontinuous.

Now, let us state the main results of this paper.

Theorem 1. Let (f) hold. Furthermore, assume that f satisfies the following conditions.

$$(f_0)$$
 There is some $r > 0$ small such that
 $pF(x,t) \ge \lambda_1 |t|^p, \quad |t| \le r, \ x \in \Omega,$
(9)

 $(f_1) \limsup_{|t| \to \infty} (pF(x,t)/|t|^p) < \lambda_1 \text{ uniformly for } x \in \Omega.$

Then, (1) has at least a negative solution and a positive solution which correspond to negative critical values of the associated *functional given by* (8).

Theorem 2. Let (f) and (f_0) hold. Furthermore, assume that *f* satisfies the following conditions.

$$\begin{array}{l} (f_2) \lim_{|t| \to \infty} (pF(x,t)/|t|^p) = \lambda_1 \ uniformly \ for \ x \in \Omega, \\ (f_3) \lim_{|t| \to \infty} (f(x,t) - pF(x,t)) = +\infty \ uniformly \ for \ x \in \Omega. \end{array}$$

Then, (1) has at least a negative solution and a positive solution which correspond to negative critical values of the associated functional given by (8).

Remark 3. With the conditions $(f_0)-(f_3)$, Liu and Su in [8] have studied the existence of solutions to *p*-Laplacian quasilinear elliptic equation

$$-\Delta_p u = f(x, u) \quad \text{in } \Omega,$$

$$u = 0 \quad \text{on } \partial\Omega.$$
 (10)

Under the conditions (f_0) and (f_1) , (10) may be resonant at the eigenvalue λ_1 near the origin. With the conditions (f_2) and (f_3) , it may be resonant at λ_1 both near the origin and near infinity. In fact, the condition (f_0) allows (10) to be resonant near the origin from the right side of λ_1 , while the conditions (f_2) and (f_3) allow it to be resonant at infinity from the left side of λ_1 .

Remark 4. Theorems 1 and 2 have shown a new fact that the interaction between the first eigenvalue of $-\Delta_p$ with zero Dirichlet boundary data and nonlinearity f can influence the existence of nontrivial solutions to (1).

Before concluding this section, we explain some notations used in the paper. $|\Omega|$ is the Lebesgue measure of Ω . c_i ($i \in \mathbb{N}$) is always a positive constant independent of functions. $\langle\cdot,\cdot\rangle$ is the duality between $(W_0^{1,p}(\Omega))^*$ and $W_0^{1,p}(\Omega)$. In addition, we use $|\cdot|$ to denote the usual norm of \mathbb{R}^N

2. The Proof of the Main Results

In this section, we prove Theorems 1 and 2.

Proof of Theorem 1. The proof consists of two steps.

(i) To obtain a positive solution, cut-off techniques are used. Define

$$\widehat{f}(x,t) = \begin{cases} f(x,t), & t \ge 0, \\ 0, & t < 0, \end{cases}$$

$$\widehat{F}(x,t) = \int_0^t \widehat{f}(x,s) \, ds, \qquad (11)$$

$$\widehat{I}(u) = J(u) - \int_{\Omega} \widehat{F}(x, u) \, dx, \quad u \in W_0^{1, p}(\Omega).$$

Since $f \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$ and (f_1) holds, for any given $\varepsilon > 0$, there exists $c_1 > 0$ such that

$$\widehat{F}(x,t) \le \frac{1}{p} \left(\lambda_1 - \varepsilon\right) |t|^p + c_1, \quad t \in \mathbb{R}, \ x \in \Omega.$$
(12)

By the Poincaré inequality, for $u \in W_0^{1,p}(\Omega)$,

$$\begin{split} \widehat{I}(u) &= \frac{1}{p} \int_{\Omega} \sqrt{1 + |\nabla u|^{2p}} dx - \int_{\Omega} \widehat{F}(x, u) \, dx - \frac{|\Omega|}{p} \\ &\geq \frac{1}{p} \int_{\Omega} |\nabla u|^{p} dx - \frac{1}{p} \left(\lambda_{1} - \varepsilon\right) \int_{\Omega} |u|^{p} dx \\ &- \left(c_{1} + \frac{1}{p}\right) |\Omega| \\ &\geq \frac{1}{p} \left(1 - \frac{\lambda_{1} - \varepsilon}{\lambda_{1}}\right) \|u\|^{p} - \left(c_{1} + \frac{1}{p}\right) |\Omega| \\ &= \frac{\varepsilon}{p\lambda_{1}} \|u\|^{p} - \left(c_{1} + \frac{1}{p}\right) |\Omega| \,. \end{split}$$

$$(13)$$

Hence, \hat{I} is coercive; that is, $\hat{I}(u) \to \infty$ as $n \to \infty$. In addition, since \hat{f} also satisfies the condition (f), \hat{I} is weakly lower semi-continuous. So, it has a global minimizer.

Take a number $t_0 > 0$ such that $0 < t_0 \varphi_1 \le r$ in Ω . By the condition (f_0) , we have that

$$\begin{split} \widehat{I}\left(t_{0}\varphi_{1}\right) &= \frac{1}{p}\int_{\Omega}\left(\sqrt{1+t_{0}^{2p}}\left|\nabla\varphi_{1}\right|^{2p}-1\right)dx\\ &-\int_{\Omega}F\left(x,t_{0}\varphi_{1}\right)dx\\ &< \frac{1}{p}t_{0}^{p}\int_{\Omega}\left|\nabla\varphi_{1}\right|^{p}dx-\frac{1}{p}\lambda_{1}t_{0}^{p}\int_{\Omega}\varphi_{1}^{p}dx\\ &= 0. \end{split}$$
(14)

Thus, the global minimizer of \hat{I} is a nontrivial critical point, denoted by u_1 which satisfies $\hat{I}(u_1) < 0$. Putting $u_1^-(x) = \min\{u_1(x), 0\}$, we have that

$$\left\langle \widehat{I}'(u_1), u_1^{-} \right\rangle = \int_{\Omega} \frac{\left| \nabla u_1^{-} \right|^{2p}}{\sqrt{1 + \left| \nabla u_1^{-} \right|^{2p}}} dx = 0.$$
 (15)

Hence, $u_1^- = 0$. So, u_1 is a positive solution of (1), and $I(u_1) < 0$.

(ii) To obtain a negative solution, we only need to replace \widehat{f} with

$$\tilde{f}(x,t) = \begin{cases} 0, & t > 0, \\ f(x,t), & t \le 0. \end{cases}$$
(16)

Similar to step (i), it is shown that (1) has a negative solution u_2 with $I(u_2) < 0$.

The proof is completed. \Box

Proof of Theorem 2. We adopt the notations in the proof of Theorem 1.

First of all, we show that the functional \hat{I} is also coercive under the conditions (f_2) and (f_3) . Write

$$\widehat{F}(x,t) = \frac{1}{p}\lambda_{1}(t^{+})^{p} + \widehat{G}(x,t),$$

$$\widehat{f}(x,t) = \lambda_{1}|t|^{p-2}t^{+} + \widehat{g}(x,t),$$
(17)

where $t^+ = \max\{t, 0\}$. Given $x \in \Omega$, we have that

$$\lim_{t \to +\infty} \frac{p\widehat{G}(x,t)}{t^{p}} = 0,$$

$$\lim_{t \to +\infty} \left(\widehat{g}(x,t)t - p\widehat{G}(x,t)\right) = +\infty.$$
(18)

Thus, for every M > 0, there exists $R_M > 0$ such that

$$\widehat{g}(x,t)t - p\widehat{G}(x,t) \ge M, \quad t \ge R_M, \ x \in \Omega.$$
(19)

Integrating the equality

$$\frac{d}{dt}\left(\frac{\widehat{G}(x,t)}{t^{p}}\right) = \frac{\widehat{g}(x,t)t - p\widehat{G}(x,t)}{t^{p+1}}$$
(20)

over the interval $[t, T] \in [R_M, +\infty)$,

$$\frac{\widehat{G}(x,T)}{T^p} - \frac{\widehat{G}(x,t)}{t^p} \ge \frac{M}{p} \left(\frac{1}{t^p} - \frac{1}{T^p}\right).$$
(21)

Letting $T \to +\infty$, we show that $\widehat{G}(x, t) \leq -M/p, t \geq R_M$.

Suppose that $\{u_n\} \in W_0^{1,p}(\Omega)$ satisfies $||u_n|| \to \infty$ and $\widehat{I}(u_n) \leq C$ for some constant $C \in \mathbb{R}$. Let $v_n = u_n/||u_n||$. Up to subsequence if necessary, we may assume that there exists $v_0 \in W_0^{1,p}(\Omega)$ such that

$$v_n \rightarrow v_0$$
 in E ,
 $v_n \rightarrow v_0$ in $L^p(\Omega)$, (22)
 $v_n(x) \rightarrow v_0(x)$ a.e. $x \in \Omega$.

Given M = 1 in (19), we have that

$$\widehat{G}(x,t) \le -\frac{1}{p}, \quad t \ge R_1.$$
(23)

Let $c_2 = \max_{(x,t)\in\overline{\Omega}\times[-R_1,R_1]}|\widehat{G}(x,t)|$. Thus,

$$\frac{C}{\|u_{n}\|^{p}} \geq \frac{1}{p\|u_{n}\|^{p}} \left(\int_{\Omega} \sqrt{1 + |\nabla u|^{2p}} dx - \lambda_{1} \int_{\Omega} |u_{n}|^{p} dx \right)
- \frac{1}{\|u_{n}\|^{p}} \int_{\Omega} \widehat{G}(x, u_{n}) dx - \frac{|\Omega|}{p\|u_{n}\|^{p}}
\geq \frac{1}{p} \int_{\Omega} \left(|\nabla v_{n}|^{p} - \lambda_{1}|v_{n}|^{p} \right) dx - \frac{|\Omega|}{p\|u_{n}\|^{p}}
- \frac{1}{\|u_{n}\|^{p}} \int_{|u_{n}| \geq R_{1}} \widehat{G}(x, u_{n}) dx
- \frac{1}{\|u_{n}\|^{p}} \int_{|u_{n}| \leq R_{1}} \widehat{G}(x, u_{n}) dx
\geq \frac{1}{p} \int_{\Omega} \left(|\nabla v_{n}|^{p} - \lambda_{1}|v_{n}|^{p} \right) dx - \frac{|\Omega|}{p\|u_{n}\|^{p}} - \frac{c_{2} |\Omega|}{\|u_{n}\|^{p}}
= \frac{1}{p} \int_{\Omega} \left(|\nabla v_{n}|^{p} - \lambda_{1}|v_{n}|^{p} \right) dx + \frac{c_{3}}{\|u_{n}\|^{p}},$$
(24)

where $c_3 = (1/p + c_2)|\Omega|$. It follows from (22) and the previous inequality that

$$\limsup_{n \to \infty} \int_{\Omega} |\nabla v_n|^p dx \le \lambda_1 \int_{\Omega} |v_0|^p dx.$$
 (25)

Because the norm is weakly lower semi-continuous, using Poincaré inequality, we get that

$$\begin{split} \limsup_{n \to \infty} \int_{\Omega} |\nabla v_n|^p dx &\leq \lambda_1 \int_{\Omega} |v_0|^p dx \leq \int_{\Omega} |\nabla v_0|^p dx \\ &\leq \liminf_{n \to \infty} \int_{\Omega} |\nabla v_n|^p dx \qquad (26) \\ &\leq \limsup_{n \to \infty} \int_{\Omega} |\nabla v_n|^p dx. \end{split}$$

Hence, $\int_{\Omega} |\nabla v_0|^p dx = \lambda_1 \int_{\Omega} |v_0|^p dx$ and $v_n \to v_0$ in $W_0^{1,p}(\Omega)$ with $||v_0|| = 1$. So, v_0 is the corresponding eigenfunction to λ_1 . Without loss of generality, we may assume that $v_0 = \varphi_1$. Thus, $u_n \to +\infty$ a.e. $x \in \Omega$. Consequently, $\widehat{G}(x, u_n(x)) \to -\infty$ a.e. $x \in \Omega$. Therefore,

$$C \ge -\int_{\Omega} \widehat{G}(x, u_n) \, dx \longrightarrow +\infty, \tag{27}$$

which contradicts the fact that $C \in \mathbb{R}$. From the fact that \widehat{I} is weakly low semi-continuous, we know that it has a global minimizer u_1 . As in the proof of Theorem 1, u_1 is a positive solution of (1) with $I(u_1) < 0$. In a similar way, we can obtain a negative solution with negative critical value.

The proof is completed.

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