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Research Article

Generalized Quasilinearization for the System of Fractional Differential Equations

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This paper considers the initial value problems of the system of fractional differential equations and constructs two monotone sequences of upper and lower solutions. By using quasilinearization technique, monotone sequences of approximate solutions that converge quadratically to a solution are obtained.

1. Introduction

In this paper, we consider the system of Caputo fractional differential equations:

$$ {}^{c}D^{q}x = f(t,x), \qquad x(t_0) = x_0, \tag{1} $$

where $0 < q < 1$, $f \in C[\Omega, R^n]$, $\Omega = [(t,x) : \alpha(t) \le x \le \beta(t)$, $\alpha, \beta \in C^q[J, R^n], t \in J]$, and $J = [t_0, T]$.

The theories and properties of fractional differential equations have received attention from some researchers because many mathematical modeling appeared in the fields of physics, chemistry, engineering and biological sciences and so on. For examples and details, we can refer to the, monographs of Miller and Ross [1], Podlubny [2], Kilbas et al. [3], and West et al [4] and the papers of Debnath [5], Rossikhin and Shitikova [6], and Ferreira et al [7]. There are many results on the basic theory of initial value and boundary value problems for fractional differential equations, which can be found in [8–10]. Meanwhile, there are some qualitative and numerical solutions for various fractional equations with delay and impulsive effects. For details, see some recent papers [11–18] and the references therein.

It is well known that the monotone iterative technique is an ingenious method providing a constructive approach to find solutions for the nonlinear problem via linear iterates. Lakshmikantham and Vatsala [19], and McRae [20] investigated the existence of minimal and maximal solutions of fractional differential equations by establishing a comparison result and using the monotone method, respectively; Benchohra and Hamani [21] used a monotone iterative technique in the presence of lower and upper solutions to discuss the existence of solutions to impulsive fractional differential inclusions.

Quasilinearization [22] provides an elegant and easier approach to obtain a sequence of approximate solutions with quadratic convergence, and the method has been extended to fractional differential equations in [23, 24]. However, to the best of our knowledge, there are few results for the system of fractional differential equations, especially results on the convergence of the system. In the present paper, we will discuss the approximate solutions of the system of fractional differential equations through the application of quasilinearization. The significance of this work lies in the fact that the system of fractional differential equations can also obtain a monotone sequence of approximate solutions converging uniformly to the solution of the problem and possessing quadratic convergence.

The nonhomogeneous linear system of Caputo fractional differential equations is given by

$$ {}^{c}D^{q}x = Bx + g(t,y), \quad x(t_0) = x_0, \tag{2} $$

where B is an nth order matrix over complex field, and g is an n-dimensional locally integrable column vector function on J.

Using the method of successive approximations, we get the solution of (2) as

$$x(t) = x_0 E_q\left(B(t-t_0)^q\right) + (t-t_0)^{q-1} E_{q,q} \times \left(B(t-t_0)^q\right)^k g(t,y), \quad t \in J, \tag{3}$$

where

$$E_q(B) = \sum_{k=0}^{\infty} \frac{B^k}{\Gamma(qk+1)}, \qquad E_{q,q}(B) = \sum_{k=0}^{\infty} \frac{B^k}{\Gamma(qk+q)} \tag{4}$$

are Mittag-Leffler functions of one parameter and two parameters, respectively.

2. Preliminaries

Now, we present the following definition and lemma which help to prove our main result.

Definition 1. Let $\alpha, \beta \in C^q[J, R^n]$ be lower and upper solutions of (1) if they satisfy the inequalities

$$\begin{aligned} {}^cD^q\alpha &\le f(t,\alpha), \qquad \alpha_0 \le x_0, \\ {}^cD^q\beta &\ge f(t,\beta), \qquad \beta_0 \ge x_0, \end{aligned} \tag{5}$$

respectively, for $t \in J$.

Lemma 2. *Suppose that α, $\beta \in C^q[J, R^n]$ are lower and upper solutions of* (1)*, and*

(H_1) *$f \in C[\Omega, R^n]$ are quasimonotone nondecreasing in x for each $t \in J$ and for each $i \in I = \{1, 2, \ldots n\}$,*

$$f_i(t,x) - f_i(t,y) \le L\sum_{j=1}^{n}\left(x_j - y_j\right), \tag{6}$$

where $x \ge y$, $L \ge 0$ is a constant.

Then, $\alpha(t_0) \le \beta(t_0)$ implies that $\alpha(t) \le \beta(t)$.

Proof. Firstly, suppose that

$${}^cD^q\beta > f(t,\beta), \quad {}^cD^q\alpha \le f(t,\alpha), \quad t \in J \tag{7}$$

and $\alpha(t_0) < \beta(t_0)$. We will prove that $\alpha(t) < \beta(t)$, $t \in J$. Suppose the conclusion is not true, then the set

$$\mathbb{Z} = \bigcup_{i=1}^{n}\left[t \in J, \beta_i(t) \ge \alpha_i(t)\right] \tag{8}$$

is nonempty.

Let $t_1 = \inf \mathbb{Z}$. Certainly, $t_1 > 0$. Since the set $\mathbb{Z}$ is closed, $t_1 \in \mathbb{Z}$ and consequently there exists a j such that $\beta_j(t_1) = \alpha_j(t_1)$. Moreover, $\beta_i(t_1) \ge \alpha_i(t_1)$, for $i \ne j$, and

$$\alpha_j(t) < \beta_j(t), \quad t \in [0, t_1). \tag{9}$$

Hence, it easily follows that

$${}^cD^q\alpha_j(t_1) \ge {}^cD^q\beta_j(t_1). \tag{10}$$

This together with the quasimonotonicity of f yields

$$\begin{aligned} f_j(t_1, \alpha(t_1)) &\ge {}^cD^q\alpha_j(t_1) \ge {}^cD^q x\beta_j(t_1) \\ &> f_j\left(t_1, \beta_j(t_1)\right) \ge f_j(t_1, \alpha(t_1)), \end{aligned} \tag{11}$$

which leads to a contradiction.

In order to prove the case of nonstrict inequalities, consider the functions

$$\widetilde{\beta}_i(t) = \beta_i(t) + \varepsilon e^{(n+1)L_i t}, \tag{12}$$

where $\varepsilon > 0$ is sufficiently small constant. Then using (6), we have

$$\begin{aligned} {}^cD^q\widetilde{\beta}_i(t) &= {}^cD^q\beta_i(t) + (n+1)\varepsilon L_i e^{(n+1)L_i t} \\ &\ge f_i(t, \beta(t)) + (n+1)\varepsilon L_i e^{(n+1)L_i t} \\ &\ge f_i\left(t, \widetilde{\beta}(t)\right) - n\varepsilon L_i e^{(n+1)L_i t} + (n+1)\varepsilon L_i e^{(n+1)L_i t} \\ &> f_i\left(t, \widetilde{\beta}(t)\right). \end{aligned} \tag{13}$$

Also $\widetilde{\beta}_i(t_0) > \alpha_i(t_0)$. Now using the result corresponding to strict inequalities, we get

$$\alpha_i(t) < \widetilde{\beta}_i(t) = \beta_i(t) + \varepsilon e^{(n+1)L_i t}, \quad t \in J. \tag{14}$$

Letting $\varepsilon \to 0$, we obtain the required result and the proof is complete. □

Corollary 3. *The function $f(t,\alpha) = \sigma(t)\alpha$ where $\sigma(t) \le L$ is admissible in Lemma 2 to yield $\alpha(t) \le 0$, $t \in J$.*

3. Main Result

Theorem 4. *Assume that α_0, $\beta_0 \in C^q[J, R^n]$ are lower and upper solutions of* (1) *such that $\alpha_0(t) \le \beta_0(t)$, $t \in J$, and*

(H_1) *$f, \phi \in C[\Omega, R^n]$ are quasimonotone nondecreasing in x for $t \in J$, f_x, ϕ_x, f_{xx}, ϕ_{xx} exist and are continuous on Ω, satisfying $f_{xx} + \phi_{xx} \ge 0$, $\phi_{xx} \ge 0$;*

(H_2) *$a_{ij}(t, \alpha_0, \beta_0) \ge 0$, $i \ne j$, where $A(t, \alpha_0, \beta_0) = [a_{ij}(t, \alpha_0, \beta_0)]$ is an $n \times n$ matrix given by*

$$\begin{aligned} A(t, \alpha_0, \beta_0) &\equiv f_x(t, \alpha_0) + \phi_x(t, \alpha_0) - \phi_x(t, \beta_0) \\ &= F_x(t, \alpha_0) - \phi_x(t, \beta_0), \end{aligned} \tag{15}$$

where $F_x = f_x + \phi_x$.

Then, there exist monotone sequences $\{\alpha_n(t)\}$, $\{\beta_n(t)\}$ which converge uniformly to the solution of (1) *and the convergence is quadratic.*

Proof. (H_1) in Theorem 4 implies for any $x \ge y$, $x, y \in \Omega$,

$$\begin{aligned} f(t,x) &\ge f(t,y) + \left[f_x(t,y) + \phi_x(t,y)\right] \\ &\quad \times (x-y) - \left[\phi(t,x) - \phi(t,y)\right]. \end{aligned} \tag{16}$$

And for any

$$\alpha_0(t) \le y \le x \le \beta_0(t), \tag{17}$$

we have

$$a_{ij}(t, y, x) \ge 0, \quad i \ne j, \tag{18}$$

since $a_{ij}(t, y, x) \ge a_{ij}(t, \alpha_0, \beta_0)$ by assumption (H_1).

It is also clear that for $\alpha_0(t) \le y \le x \le \beta_0(t)$,

$$\begin{aligned} &f_i(t, x_1, \ldots, x_n) - f_i(t, y_1, \ldots, y_n) \\ &\quad \le L_i \sum_{i=1}^{n} (x_i - y_i). \end{aligned} \tag{19}$$

Let α_1, β_1 be the solutions of IVPs:

$$\begin{aligned} {}^cD^q\alpha_1 &= f(t, \alpha_0) + [A(t, \alpha_0, \beta_0)] \\ &\quad \times (\alpha_1 - \alpha_0), \quad \alpha_1(0) = x_0, \\ {}^cD^q\beta_1 &= f(t, \beta_0) + [A(t, \alpha_0, \beta_0)] \\ &\quad \times (\beta_1 - \beta_0), \quad \beta_1(0) = x_0, \end{aligned} \tag{20}$$

where $\alpha_0(0) \le x_0 \le \beta_0(0)$. We will prove that $\alpha_0 \le \alpha_1$, $t \in J$. To do this, let $p = \alpha_0 - \alpha_1$, so that $p(0) \le 0$. Then using (20), we obtain

$$\begin{aligned} {}^cD^q p &= {}^cD^q\alpha_0 - {}^cD^q\alpha_1 \\ &\le f(t, \alpha_0) - [f(t, \alpha_0) + A(t, \alpha_0, \beta_0)(\alpha_1 - \alpha_0)] \\ &= A(t, \alpha_0, \beta_0)\, p. \end{aligned} \tag{21}$$

Since $A(t, \alpha_0, \beta_0)$ is quasimonotone nondecreasing by assumption (H_1) to (H_2) and it follows from Corollary 3 that $p(t) \le 0$, $t \in J$, proving that $\alpha_0 \le \alpha_1$, $t \in J$.

Now we let $p = \alpha_1 - \beta_0$ and note that $p(0) \le 0$. Also,

$$\begin{aligned} {}^cD^q p &= {}^cD^q\alpha_1 - {}^cD^q\beta_0 \\ &\le f(t, \alpha_0) + A(t, \alpha_0, \beta_0)(\alpha_1 - \alpha_0) - f(t, \beta_0); \end{aligned} \tag{22}$$

since $\beta_0 \ge \alpha_0$, using (16), we get

$$\begin{aligned} f(t, \beta_0) &\ge f(t, \alpha_0) + F_x(t, \alpha_0)(\beta_0 - \alpha_0) \\ &\quad - [\phi(t, \beta_0) - \phi(t, \alpha_0)]. \end{aligned} \tag{23}$$

In view of (H_1), we have

$$\phi(t, \beta_0) - \phi(t, \alpha_0) \le \phi_x(t, \beta_0)(\beta_0 - \alpha_0), \tag{24}$$

which yields

$$f(t, \beta_0) \ge f(t, \alpha_0) + [F_x(t, \alpha_0) - \phi_x(t, \beta_0)](\beta_0 - \alpha_0). \tag{25}$$

Hence, we obtain

$${}^cD^q p \le A(t, \alpha_0, \beta_0)\, p; \tag{26}$$

this implies that $\alpha_1(t) \le \beta_0(t)$, $t \in J$, using Corollary 3. As a result, we have

$$\alpha_0(t) \le \alpha_1(t) \le \beta_0(t), \quad t \in J. \tag{27}$$

In a similar way, we can prove that

$$\alpha_0(t) \le \beta_1(t) \le \beta_0(t), \quad t \in J. \tag{28}$$

To show $\alpha_1(t) \le \beta_1(t)$, we use (16), (19), and (H_1):

$$\begin{aligned} {}^cD^q\alpha_1 &= f(t, \alpha_0) + [F_x(t, \alpha_0) - \phi_x(t, \beta_0)](\alpha_1 - \alpha_0) \\ &\le f(t, \alpha_1) + \phi(t, \alpha_1) - \phi(t, \alpha_0) \\ &\quad - \phi_x(t, \beta_0)(\alpha_1 - \alpha_0) \\ &\le f(t, \alpha_1) + \left[\int_0^1 \phi_x(t, s\alpha_1 + (1-s)\alpha_0)\, ds\right] \\ &\quad \times (\alpha_1 - \alpha_0) - \phi_x(t, \beta_0)(\alpha_1 - \alpha_0) \\ &\le f(t, \alpha_1) + \phi_x(t, \alpha_1)(\alpha_1 - \alpha_0) \\ &\quad - \phi_x(t, \beta_0)(\alpha_1 - \alpha_0) \\ &\le f(t, \alpha_1). \end{aligned} \tag{29}$$

Using a similar argument, it is easy to show that

$${}^cD^q\beta_1 \ge f(t, \beta_1), \tag{30}$$

And, therefore, by Lemma 2 and (19), it follows that

$$\alpha_1(t) \le \beta_1(t), \quad t \in J; \tag{31}$$

by (19), we have f is Lipschitzian in x on Ω. This proves that

$$\alpha_0(t) \le \alpha_1(t) \le \beta_1(t) \le \beta_0(t), \quad t \in J. \tag{32}$$

Now assume that for some $k > 1$,

$$\alpha_{k-1}(t) \le \alpha_k(t) \le \beta_k(t) \le \beta_{k-1}(t), \quad t \in J. \tag{33}$$

We now aim to show that

$$\alpha_k(t) \le \alpha_{k+1}(t) \le \beta_{k+1}(t) \le \beta_k(t), \quad t \in J, \tag{34}$$

where $\alpha_{k+1}(t)$ and $\beta_{k+1}(t)$ are the solutions of linear IVPs:

$$\begin{aligned} {}^cD^q\alpha_{k+1} &= f(t, \alpha_k) + [F_x(t, \alpha_k) - \phi_x(t, \beta_k)] \\ &\quad \times (\alpha_{k+1} - \alpha_k), \quad \alpha_{k+1}(0) = x_0, \\ {}^cD^q\beta_{k+1} &= f(t, \beta_k) + [F_x(t, \alpha_k) - \phi_x(t, \beta_k)] \\ &\quad \times (\beta_{k+1} - \beta_k), \quad \beta_{k+1}(0) = x_0. \end{aligned} \tag{35}$$

Now, set $p = \alpha_k - \alpha_{k+1}$ so that

$$\begin{aligned} {}^cD^q p &= {}^cD^q\alpha_k - {}^cD^q\alpha_{k+1} \\ &\le f(t, \alpha_k) - [f(t, \alpha_k) + (F_x(t, \alpha_k) - \phi_x(t, \beta_k)) \\ &\quad \times (\alpha_{k+1} - \alpha_k)] \\ &= F_x(t, \alpha_k) - \phi_x(t, \beta_k)\, p \\ &= A(t, \alpha_k, \beta_k)\, p \end{aligned} \tag{36}$$

and $p(0) = 0$. It follows from Corollary 3 and using (16) that

$$\alpha_k(t) \le \alpha_{k+1}(t), \quad t \in J. \tag{37}$$

On the other hand, letting $p = \alpha_{k+1} - \beta_k$ yields

$$\begin{aligned} {}^cD^q p &= {}^cD^q\alpha_{k+1} - {}^cD^q\beta_k \\ &\le f(t,\alpha_k) + [F_x(t,\alpha_k) - \phi_x(t,\beta_k)] \\ &\quad \times (\alpha_{k+1} - \alpha_k) - f(t,\beta_k). \end{aligned} \tag{38}$$

Since $\beta_k \ge \alpha_k$, (16) and (H_1) give, as before,

$$f(t,\beta_k) \ge f(t,\alpha_k) + [F_x(t,\alpha_k) - \phi_x(t,\beta_k)](\beta_k - \alpha_k), \tag{39}$$

which shows that

$${}^cD^q p \le [F_x(t,\alpha_k) - \phi_x(t,\beta_k)]\, p = A(t,\alpha_k,\beta_k)\, p. \tag{40}$$

This proves that $p(t) \le 0$ using Corollary 3 and (16), since $p(0) = 0$. Hence, we get

$$\alpha_k(t) \le \alpha_{k+1}(t) \le \beta_k(t), \quad t \in J. \tag{41}$$

Similarly, we can prove that

$$\alpha_k(t) \le \beta_{k+1}(t) \le \beta_k(t), \quad t \in J. \tag{42}$$

Also, by (16), (35), and the fact that $\alpha_{k+1} \ge \alpha_k$, we obtain

$$\begin{aligned} {}^cD^q\alpha_{k+1} &\le f(t,\alpha_{k+1}) + \phi(t,\alpha_{k+1}) - \phi(t,\alpha_k) \\ &\quad - \phi_x(t,\beta_k)(\alpha_{k+1} - \alpha_k) \\ &\le f(t,\alpha_{k+1}) + \left[\int_0^1 \phi_x(t, s\alpha_{k+1} + (1-s)\alpha_k)\, ds\right] \\ &\quad \times (\alpha_{k+1} - \alpha_k) - \phi_x(t,\beta_k)(\alpha_{k+1} - \alpha_k) \\ &\le f(t,\alpha_{k+1}) + \phi_x(t,\alpha_{k+1})(\alpha_{k+1} - \alpha_k) \\ &\quad - \phi_x(t,\beta_k)(\alpha_{k+1} - \alpha_k) \\ &\le f(t,\alpha_{k+1}). \end{aligned} \tag{43}$$

Using a similar argument, we have, as before, ${}^cD^q\beta_{k+1} \ge f(t,\beta_{k+1})$, and hence Lemma 2 shows that

$$\alpha_{k+1}(t) \le \beta_{k+1}(t), \quad t \in J. \tag{44}$$

By (19), we have that f is Lipschitzian in x on Ω and is quasimonotone nondecreasing in x. This proves (34). Therefore, by induction, we have for all n:

$$\begin{aligned} \alpha_0(t) &\le \alpha_1(t) \le \cdots \le \alpha_n(t) \le \beta_n(t) \\ &\le \cdots \beta_1(t) \le \beta_0(t), \quad t \in J. \end{aligned} \tag{45}$$

Employing the standard procedure, it is now easy to prove that the sequences $\{\alpha_n(t)\}$ and $\{\beta_n(t)\}$ converge uniformly and monotonically to the unique solution of (1) on J.

We will now show that the convergence of $\{\alpha_n(t)\}$ and $\{\beta_n(t)\}$ to $x(t)$ is quadratic. First set

$$p_{n+1} = x - \alpha_{n+1}, \quad r_{n+1} = \beta_{n+1} - x, \quad t \in J, \tag{46}$$

so that $p_{n+1}(0) = r_{n+1}(0) = 0$. Then using integral mean value theorem and the fact that $x \ge \alpha_{n+1}$ and (H_1), we get

$$\begin{aligned} {}^cD^q p_{n+1} &= {}^cD^q x - {}^cD^q\alpha_{n+1} \\ &= f(t,x) - [f(t,\alpha_n) + (F_x(t,\alpha_n) - \phi_x(t,\beta_n)) \\ &\qquad \times (\alpha_{n+1} - \alpha_n)] \\ &= F(t,x) - F(t,\alpha_n) - [F_x(t,\alpha_n) - \phi_x(t,\beta_n)] \\ &\quad \times (\alpha_{n+1} - \alpha_n) - [\phi(t,x) - \phi(t,\alpha_n)] \\ &= \left(\int_0^1 F_x(t, sx + (1-s)\alpha_n)\, ds\right) p_n \\ &\quad - [F_x(t,\alpha_n) - \phi_x(t,\beta_n)](-p_{n+1} + p_n) \\ &\quad - \left[\int_0^1 \phi_x(t, sx + (1-s)\alpha_n)\, ds\right](x - \alpha_n) \\ &\le (F_x(t,x) - F_x(t,\alpha_n))\, p_n + (\phi_x(t,\beta_n) \\ &\quad - \phi_x(t,\alpha_n))\, p_n + f_x(t,\alpha_n)\, p_{n+1}. \end{aligned} \tag{47}$$

So, we get

$$\begin{aligned} {}^cD^q p_{n+1} &\le p_n^T \left[\int_0^1 F_{xx}(t, sx + (1-s)\alpha_n)\, ds\right] p_n \\ &\quad + (r_n + p_n)^T \left[\int_0^1 \phi_{xx}(t, s\beta_n + (1-s)\alpha_n)\, ds\right] \\ &\quad + f_x(t,\alpha_n)\, p_{n+1} \\ &\le M p_{n+1} + W(p_n \cdot p_n) + N(r_n + p_n)\, p_n \\ &\le M p_{n+1} + \left(W + \frac{3N}{2}\right)|p_n|^2 + \frac{N}{2}|r_n|^2, \end{aligned} \tag{48}$$

where $|p_n|^2 = (|p_{n1}|^2, |p_{n2}|^2, \ldots, |p_{nn}|^2)$, $|r_n| = (|r_{n1}|^2, |r_{n2}|^2, \ldots, |r_{nn}|^2)$, $f_x \le M$, $\sum_{i=1}^n f_{xx}^i \le W$ and $\sum_{i=1}^n \phi_{xx}^i \le N$ in Ω, and M, W, and N are $n \times n$ positive matrices. Using (2) to (4), we can get

$$\begin{aligned} p_{n+1}(t) &\le \left[\left(W + \frac{3N}{2}\right)|p_n|^2 + \frac{N}{2}|r_n|^2\right] \\ &\quad \times (T - t_0)^{q-1} E_{q,q}\left(M(T - t_0)^q\right)^k \\ &\le V\left[\left(W + \frac{3N}{2}\right)|p_n|^2 + \frac{N}{2}|r_n|^2\right], \end{aligned} \tag{49}$$

where $V = (T - t_0)^{q-1} E_{q,q}(M(T - t_0)^q)^k$. Thus, we have

$$|p_{n+1}| \le V\left[\left(W + \frac{3N}{2}\right)|p_n|^2 + \frac{N}{2}|r_n|^2\right]. \tag{50}$$

Similarly, we have

$$|r_{n+1}| \le V\left[\frac{W}{2}|p_n|^2 + \left(\frac{3W}{2} + \frac{5N}{2}\right)|r_n|^2\right]. \tag{51}$$

The proof is complete. □

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