

Research Article

Controllability of Impulsive Fractional Functional Integro-Differential Equations in Banach Spaces

C. Ravichandran¹ and J. J. Trujillo²

¹ *Department of Mathematics, RVS Faculty of Engineering, RVS Technical Campus, Coimbatore 641 402, Tamil Nadu, India*

² *Departamento de Análisis Matemático, University of La Laguna, 38271 La Laguna, Tenerife, Spain*

Correspondence should be addressed to C. Ravichandran; ravibirthday@gmail.com

Received 13 May 2013; Revised 6 August 2013; Accepted 14 August 2013

Academic Editor: Mitsuru Sugimoto

Copyright © 2013 C. Ravichandran and J. J. Trujillo. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

This paper is concerned with the controllability problem for a class of mixed type impulsive fractional integro-differential equations in Banach spaces. Sufficient conditions for the controllability result are established by using suitable fixed point theorem combined with the fractional calculus theory and solution operator under some weak conditions. The example is given to illustrate the theory. The results of this article are generalization and improved of the recent results on this issue.

1. Introduction

In the past few decades, the fractional calculus, that is, calculus of integrals and derivatives of any arbitrary real or complex order has gained considerable popularity and importance, based on the wide applications in engineering and sciences such as fluid flow, rheology, dynamical processes in self-similar and porous structures, diffusive transport akin to diffusion, and electrical networks. For more details about fractional calculus theory and fractional differential equations with applications see the monographs of Baleanu et al. [1, 2], Kilbas et al. [3], Lakshmikantham et al. [4], Miller and Ross [5], Podlubny [6], and the papers of [7–14].

Differential equations with impulsive conditions constitute an important field of research due to their numerous applications in ecology, medicine biology, electrical engineering, and other areas of science and technology. There has been a significant development in impulsive theory especially in the area of impulsive differential equations with fixed moments, see for instance the monographs by Lakshmikantham et al. [15], Bainov and Simeonov [16], Samoilenko and Perestyuk [17] and the papers of [18–24].

On the other hand, the controllability for fractional dynamical system has become an interesting research area to this field and one of the fundamental concepts in modern mathematical control theory. Very recently, the authors Shu et al. [25] studied the existence of solutions for impulsive fractional differential equations, assuming the operator A to be a sectorial, and the results are obtained by using Banach contraction theorem and Leray-Schauder's alternative fixed point theorem. Shu et al. [26] established the existence and uniqueness of solutions for class of fractional partial semilinear functional differential equations with finite delay, here assuming A is the infinitesimal generator of an analytic semigroup and by using Banach fixed point theorem.

Tomar and Dabas [27] extended the results of [25] into a controllability of impulsive fractional semilinear evolution equations with nonlocal conditions with A as the α -resolvent family and the results are obtained by using Banach contraction principle. Many researchers [28–39] investigated the existence and controllability problem combined with fractional derivative with (or without) impulsive conditions. From above the collection of the literature survey, up till now, there is no work reported on this topic, and inspired by the above mentioned works [25–27, 40] we will establish

the controllability of impulsive fractional mixed type functional integro-differential equations with finite delay of the form

$$\begin{aligned} D^\alpha x(t) &= Ax(t) + f\left(t, x_t, \int_0^t h(t, s, x_s) ds, \int_0^T k(t, s, x_s) ds\right) \\ &\quad + Bu(t), \quad t \in J := [0, T], \quad t \neq t_k, \quad k = 1, 2, \dots, m, \end{aligned} \quad (1)$$

$$\Delta x|_{t=t_k} = I_k(x(t_k^-)), \quad k = 1, 2, \dots, m, \quad (2)$$

$$x(t) = \phi(t), \quad t \in [-r, 0], \quad (3)$$

where D^α is Caputo fractional derivative of order $0 < \alpha < 1$, $A : D(A) \subset X \rightarrow X$ is the bounded linear operator of an α -resolvent family $\{S_\alpha(t) : t \geq 0\}$ defined on a Banach space X , $B : U \rightarrow X$ is a bounded linear operator, $0 = t_0 < t_1 < t_2 < \dots < t_m < t_{m+1} = T$, $h : J \times J \times D \rightarrow X$, and $f : J \times D \times X \times X \rightarrow X$ are given functions, where $D = \{\phi : [-r, 0] \rightarrow X \text{ such that } \phi \text{ is continuous everywhere except for a finite number of points } s \text{ at which } \phi(s^-) \text{ and } \phi(s^+) \text{ exists and } \phi(s^-) = \phi(s), \phi \in D (0 < r < \infty), 0 = t_0 < t_1 < \dots < t_k < \dots < t_m < t_{m+1} = T, \Delta x|_{t=t_k} = x(t_k^+) - x(t_k^-), x(t_k^+) = \lim_{h \rightarrow 0^+} x(t_k + h) \text{ and } x(t_k^-) = \lim_{h \rightarrow 0^-} x(t_k + h) \text{ represent the right and left limits of } x(t) \text{ at } t = t_k, \text{ respectively.}\}$

For any continuous function, x is defined on the interval $[-r, T] - \{t_1, t_2, \dots, t_m\}$ and any $t \in J$. We denote by x_t be the element of D defined by

$$x_t(\theta) = x(t + \theta), \quad \theta \in [-r, 0]. \quad (4)$$

Here, $x_t(\cdot)$ represents the history of the time $t - r$, upto the present time t . For $\phi \in D$, then $\|\phi\|_D = \sup\{|\phi(\theta)| : \theta \in [-r, 0]\}$.

We consider the problems (1)–(3) to study the controllability results using the solution operator and fixed-point theorems. The rest of this paper is organized as follows. In Section 2, we present some necessary definitions and preliminary results that will be used to prove our main results. The proof of our main results is given in Section 3. Finally, an example is included in Section 4.

2. Preliminaries

In this section, we mention some definitions and properties required for establishing our results. Let X be a complex Banach space with its norm denoted as $\|\cdot\|_X$, and $L(X)$ represents the Banach space of all bounded linear operators from X into X , and the corresponding norm is denoted by $\|\cdot\|_{L(X)}$. Let $C(J, X)$ denote the space of all continuous functions from J into X with supremum norm denoted by $\|\cdot\|_{C(J, X)}$. In addition, $B_r(x, X)$ represents the closed ball in X with the center at x and the radius r .

A two-parameter function of the Mittag-Leffler type is defined by the series expansion

$$E_{\alpha, \beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} = \frac{1}{2\pi i} \int_{\text{Ha}} \frac{\mu^{\alpha-\beta} e^\mu}{\mu^\alpha - z} d\mu, \quad (5)$$

$\alpha, \beta > 0, z \in \mathbb{C},$

where Ha is a Hankel path, that is, a contour which starts and ends at $-\infty$ and encircles the disc $|\mu| \leq |z|^{1/\alpha}$ contour clockwise. For short, $E_\alpha(z) = E_{\alpha,1}(z)$. It is an entire function which provides a simple generalization of the exponent function: $E_1(z) = e^z$ and the cosine function: $E_2(-z^2) = \cos(z)$ and plays an important role in the theory of fractional differential equations. The most interesting properties of the Mittag-Leffler functions are associated with their Laplace integral

$$\begin{aligned} L(t^{\beta-1} E_{\alpha, \beta}(-w^\alpha t^\alpha))(\lambda) \\ = \frac{\lambda^{\alpha-\beta}}{\lambda^\alpha + w^\alpha}, \quad \text{Re } \lambda > w^{1/\alpha}, \quad w > 0, \end{aligned} \quad (6)$$

see [3, 6, 41] for more details.

Definition 1 (see [40]). Caputo derivative of order α for a function $f : [0, \infty) \rightarrow \mathbb{R}$ is defined as

$$\frac{d^\alpha}{dt^\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s) ds, \quad (7)$$

for $n-1 < \alpha < n, n \in \mathbb{N}$. If $0 < \alpha \leq 1$, then

$$\frac{d^\alpha}{dt^\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} f^{(1)}(s) ds. \quad (8)$$

The Laplace transform of the Caputo derivative of order $\alpha > 0$ is given as

$$\begin{aligned} L\{D_t^\alpha f(t) : \lambda\} \\ = \lambda^\alpha \hat{f}(\lambda) - \sum_{k=0}^{n-1} \lambda^{\alpha-k-1} f^{(k)}(0); \quad n-1 < \alpha \leq n. \end{aligned} \quad (9)$$

Definition 2 (see [42]). Let A be a closed and linear operator with domain $D(A)$ defined on a Banach space X and $\alpha > 0$. Let $\rho(A)$ be the resolvent set of A . We call A the generator of an α -resolvent family if there exists $w \geq 0$ and a strongly continuous function $S_\alpha : \mathbb{R}_+ \rightarrow L(X)$ such that $\{\lambda^\alpha : \text{Re } \lambda > w\} \subset \rho(A)$ and

$$\begin{aligned} (\lambda^\alpha I - A)^{-1} x &= \int_0^\infty e^{-\lambda t} S_\alpha(t) x dt, \\ \text{Re } \lambda &> w, \quad x \in X. \end{aligned} \quad (10)$$

In this case, $S_\alpha(t)$ is called the α -resolvent family generated by A .

Definition 3 (see [43]). Let A be a closed and linear operator with domain $D(A)$ defined on a Banach space X and $\alpha > 0$. Let $\rho(A)$ be the resolvent set of A . We call A the generator of an α -resolvent family if there exists $w \geq 0$ and a strongly continuous function $T_\alpha : R_+ \rightarrow L(X)$ such that $\{\lambda^\alpha : \operatorname{Re} \lambda > w\} \subset \rho(A)$ and

$$\lambda^{\alpha-1}(\lambda^\alpha I - A)^{-1}x = \int_0^\infty e^{-\lambda t} T_\alpha(t) x dt, \quad \operatorname{Re} \lambda > w, \quad x \in X. \quad (11)$$

In this case, $T_\alpha(t)$ is called the solution operator generated by A .

The concept of the solution operator is closely related to the concept of a resolvent family [44, Chapter 1]. For more details on α -resolvent family and solution operators, we refer to [44, 45] and the references therein.

3. Controllability Results

In this section, we present and prove the controllability for the system (1)–(3). In order to prove the controllability results, we need the following results which are taken from [25, 41]. If $\alpha \in (0, 1)$ and $A \in A^\alpha(\theta_0, w_0)$, then for any $x \in X$ and $t > 0$, we have

$$\|S_\alpha(t)\| \leq M e^{wt}, \quad \|T_\alpha(t)\| \leq C e^{wt} (1 + t^{\alpha-1}), \quad t > 0, \quad w > w_0. \quad (12)$$

Let $\widetilde{M}_S := \sup_{0 \leq t \leq T} \|S_\alpha(t)\|_{L(X)}$, $\widetilde{M}_T := \sup_{0 \leq t \leq T} C e^{wt} (1 + t^{\alpha-1})$, where $L(X)$ is the Banach space of bounded linear operators from X into X equipped with its natural topology. So, we have

$$\|S_\alpha(t)\|_{L(X)} \leq \widetilde{M}_S, \quad \|T_\alpha(t)\|_{L(X)} \leq t^{\alpha-1} \widetilde{M}_T. \quad (13)$$

Let us consider the set functions $\operatorname{PC}([-r, T], X) = \{x : [-r, T] \rightarrow X : x \in C((t_k, t_{k+1}], X), k = 0, 1, 2, \dots, m, \text{ and there exist } x(t_k^-) \text{ and } x(t_k^+), k = 1, 2, \dots, m \text{ with } x(t_k^-) = x(t_k^+)\}$. Endowed with the norm

$$\|x\|_{\operatorname{PC}} = \sup_{t \in [-r, T]} \|x(t)\|_X, \quad (14)$$

the space $(\operatorname{PC}([-r, T], X), \|\cdot\|_{\operatorname{PC}})$ is a Banach space.

Lemma 4 (see [25, 27, 40]). If f satisfies the uniform Hölder condition with the exponent $\beta \in (0, 1)$ and A is a sectorial operator, then the unique solution of the Cauchy problem

$$\begin{aligned} D^\alpha x(t) &= Ax(t) \\ &+ f\left(t, x_t, \int_0^t h(t, s, x_s) ds, \int_0^T k(t, s, x_s) ds\right) \\ &+ Bu(t), \quad t > t_0, \quad t_0 \in R, \quad 0 < \alpha < 1 \\ x(t) &= \phi(t), \quad t \in [-r, 0] \end{aligned} \quad (15)$$

is given by

$$\begin{aligned} x(t) &= S_\alpha(t - t_0)(x(t_0^+)) \\ &+ \int_{t_0}^t T_\alpha(t - s) f\left(s, x_s, \int_0^s h(s, \tau, x_\tau) d\tau, \int_0^T k(s, \tau, x_\tau) d\tau\right) ds \\ &+ \int_0^t T_\alpha(t - s) Bu(s) ds, \end{aligned} \quad (16)$$

where

$$\begin{aligned} S_\alpha(t) &= E_{\alpha,1}(At^\alpha) = \frac{1}{2\pi i} \int_{\widehat{B}_r} e^{\lambda t} \frac{\lambda^{\alpha-1}}{\lambda^\alpha - A} d\lambda, \\ T_\alpha(t) &= t^{\alpha-1} E_{\alpha,\alpha}(At^\alpha) = \frac{1}{2\pi i} \int_{\widehat{B}_r} e^{\lambda t} \frac{1}{\lambda^\alpha - A} d\lambda, \end{aligned} \quad (17)$$

\widehat{B}_r denotes the Bromwich path, $S_\alpha(t)$ is called the α -resolvent family, and $T_\alpha(t)$ is the solution operator generated by A .

Now, we define the mild solution of a system (1)–(3).

Definition 5. A function $x(\cdot) \in \operatorname{PC}$ is called a mild solution of the system (1)–(3) if $x(t) = \phi(t)$ on $[-r, 0]$; $\Delta x|_{t=t_k} = I_k(x(t_k^-))$, $k = 1, 2, \dots, m$ and satisfies the following integral equation:

$$x(t) = \begin{cases} S_\alpha(t) \phi(0) + \int_0^t T_\alpha(t - s) \times f\left(s, x_s, \int_0^s h(s, \tau, x_\tau) d\tau, \int_0^T k(s, \tau, x_\tau) d\tau\right) ds + \int_0^t T_\alpha(t - s) Bu(s) ds, & t \in (0, t_1]; \\ S_\alpha(t - t_1)(x(t_1^-) + I_1(x(t_1^-))) + \int_{t_1}^t T_\alpha(t - s) \times f\left(s, x_s, \int_0^s h(s, \tau, x_\tau) d\tau, \int_0^T k(s, \tau, x_\tau) d\tau\right) ds + \int_{t_1}^t T_\alpha(t - s) Bu(s) ds, & t \in (t_1, t_2]; \\ \vdots \\ S_\alpha(t - t_m)(x(t_m^-) + I_m(x(t_m^-))) + \int_{t_m}^t T_\alpha(t - s) \times f\left(s, x_s, \int_0^s h(s, \tau, x_\tau) d\tau, \int_0^T k(s, \tau, x_\tau) d\tau\right) ds + \int_{t_m}^t T_\alpha(t - s) Bu(s) ds, & t \in (t_m, T]. \end{cases} \quad (18)$$

From Lemma 4 we can verify that definition.

Note that, mild solution depends on control functions $u(\cdot)$. The solution of (1)–(3) under a control $u(\cdot)$ denoted by $x(\cdot; u)$ is called the trajectory (state) function of (1) under $u(\cdot)$. The set of all possible terminal states, denoted by

$$[K_T(f) := \{x(T; u) \in X : u \in L^2([0, T]; U)\}], \quad (19)$$

is called the reachable set of system (1) at terminal time T .

Definition 6. The system (1)–(3) is said to be controllable on J if $K_T(f) = X$.

Now we list the following hypothesis:

(H₁) $f : J \times D \times X \times X \rightarrow X$ is continuous and there exist functions $L \in L^1(J, R^+)$ such that

$$\begin{aligned} & \|f(t, x_t, u_1, v_1) - f(t, y_t, u_2, v_2)\|_X \\ & \leq L[\|x - y\| + \|u_1 - u_2\| + \|v_1 - v_2\|], \quad (20) \\ & \text{for } x, y \in \text{PC}, \quad u_i, v_i \in X, \quad i = 1, 2. \end{aligned}$$

(H₂) $h : J \times J \times D \rightarrow X$ is continuous and there exists a constant $M_1 > 0$ such that for all $(t, s) \in J \times J$

$$\left\| \int_0^t [h(t, s, x_s) - h(t, s, y_s)] ds \right\|_X \leq M_1 \|x - y\|_{\text{PC}}. \quad (21)$$

(H₃) $k : J \times J \times D \rightarrow X$ is continuous and there exists a constant $M_2 > 0$ such that for all $(t, s) \in J \times J$

$$\left\| \int_0^T [k(t, s, x_s) - k(t, s, y_s)] ds \right\|_X \leq M_2 \|x - y\|_{\text{PC}}. \quad (22)$$

(H₄) The linear operators $W_k : L^2([t_{k-1}, t_k] : U) \rightarrow X$, defined by

$$W_k u = \int_{t_{k-1}}^{t_k} T_\alpha(t - s) B u(s) ds, \quad (23)$$

has an invertible operator W_k^{-1} taking values in $L^2([t_{k-1}, t_k] : U)/\text{Ker } W_k$ and there exists a positive constant M_k such that $\|B W_k^{-1}\| \leq \Omega_k$ and $\Omega = \max\{\Omega_k\}$ (For the construction of the operator W and its inverse, see [46]).

(H₅) The function $I_k : X \rightarrow X$ is continuous and there exists $\rho_k > 0$ such that

$$\|I_k(x) - I_k(y)\|_X \leq \rho_k \|x - y\|, \quad x, y \in X, \quad k = 1, 2, \dots, m. \quad (24)$$

Theorem 7. If the hypotheses (H₁)–(H₅) are satisfied, then the impulsive fractional integro-differential system (1)–(3) is controllable on J provided

$$\begin{aligned} & \left[\widetilde{M}_S(1 + \rho_m) + \frac{\widetilde{M}_T}{\alpha} T^\alpha \Omega_m \right. \\ & \times \left(\widetilde{M}_S(1 + \rho_m) + \widetilde{M}_T L(1 + M_1 + M_2) \frac{T^\alpha}{\alpha} \right) \\ & \left. + \frac{\widetilde{M}_T}{\alpha} T^\alpha L(1 + M_1 + M_2) \right] < 1. \end{aligned} \quad (25)$$

Proof. Let $Z \in \text{PC}(J, X)$ be any arbitrary function, now to transfer the system (1) from initial state to $Z(T)$ consider the control

$$u(t) = \begin{cases} W_1^{-1} \left[Z(t_1) - S_\alpha(t_1) \phi(0) - \int_0^{t_1} T_\alpha(t-s) \right. \\ \quad \times f\left(s, x_s, \int_0^s h(s, \tau, x_\tau) d\tau, \right. \\ \quad \left. \left. \int_0^T k(s, \tau, x_\tau) d\tau \right) ds \right](t), & t \in (0, t_1]; \\ W_2^{-1} \left[Z(t_2) - S_\alpha(t_2 - t_1) \right. \\ \quad \times [x(t_1^{-1}) + I_1 x(t_1^{-1})] \\ \quad - \int_{t_2}^{t_1} T_\alpha(t_2 - s) \\ \quad \times f\left(s, x_s, \int_0^s h(s, \tau, x_\tau) d\tau, \right. \\ \quad \left. \left. \int_0^T k(s, \tau, x_\tau) d\tau \right) ds \right](t), & t \in (t_1, t_2]; \\ \vdots \\ W_{m+1}^{-1} \left[Z(T) - S_\alpha(T - t_m) \right. \\ \quad \times [x(t_m^{-1}) + I_m x(t_m^{-1})] \\ \quad + \int_{t_m}^T T_\alpha(T - s) \\ \quad \times f\left(s, x_s, \int_0^s h(s, \tau, x_\tau) d\tau, \right. \\ \quad \left. \left. \int_0^T k(s, \tau, x_\tau) d\tau \right) ds \right](t), & t \in (t_m, T]. \end{cases} \quad (26)$$

We define the operator $N : PC([-r, T], X) \rightarrow PC([-r, T], X)$ by

$$Nx(t) = \begin{cases} S_\alpha(t) \phi(t) + \int_0^t T_\alpha(t-s) \\ \times f\left(s, x_s, \int_0^s h(s, \tau, x_\tau) d\tau, \right. \\ \left. \int_0^T k(s, \tau, x_\tau) d\tau\right) ds \\ + \int_0^t T_\alpha(t-s) Bu(s) ds, & t \in (0, t_1]; \\ S_\alpha(t-t_1) (x(t_1^-) + I_1(x(t_1^-))) \\ + \int_{t_1}^t T_\alpha(t-s) \\ \times f\left(s, x_s, \int_0^s h(s, \tau, x_\tau) d\tau, \right. \\ \left. \int_0^T k(s, \tau, x_\tau) d\tau\right) ds \\ + \int_{t_1}^t T_\alpha(t-s) Bu(s) ds, & t \in (t_1, t_2]; \\ \vdots \\ S_\alpha(t-t_m) (x(t_m^-) + I_m(x(t_m^-))) \\ + \int_{t_m}^t T_\alpha(t-s) \\ \times f\left(s, x_s, \int_0^s h(s, \tau, x_\tau) d\tau, \right. \\ \left. \int_0^T k(s, \tau, x_\tau) d\tau\right) ds \\ + \int_{t_m}^t T_\alpha(t-s) Bu(s) ds, & t \in (t_m, T]. \end{cases} \quad (27)$$

Note that N is well defined on $PC([-r, T], X)$.

For our convenience, let us take

$$\begin{aligned} C(s, x) \\ = BW^{-1} \left[Z(t_1) - S_\alpha(t_1) \phi(0) \right. \\ \left. - \int_0^{t_1} T_\alpha(t_1-s) f\left(s, x_s, \int_0^s h(s, \tau, x_\tau) d\tau, \right. \right. \\ \left. \left. \int_0^T k(s, \tau, x_\tau) d\tau\right) ds \right](s), \end{aligned} \quad (28)$$

and for $i = 2, 3, \dots, m+1$.

$$\begin{aligned} D_i(s, x) \\ = BW_i^{-1} \left[Z(t_i) - S_\alpha(t_i - t_{i-1}) \right. \end{aligned}$$

$$\begin{aligned} & \times [x(t_{i-1}^{-1}) + I_{i-1}(x(t_{i-1}^{-1}))] \\ & - \int_{t_{i-1}}^{t_i} T_\alpha(t_i-s) f\left(s, x_s, \int_0^s h(s, \tau, x_\tau) d\tau, \right. \\ & \left. \int_0^T k(s, \tau, x_\tau) d\tau\right) ds \Big](s). \end{aligned} \quad (29)$$

From our assumptions, we have

$$\begin{aligned} & \|C(s, x) - C(s, y)\| \\ & \leq \Omega_1 \frac{\widetilde{M}_T}{\alpha} T^\alpha [L(1 + M_1 + M_2)] \|x - y\|_{PC}, \\ & \|D_i(s, x) - D_i(s, y)\| \\ & \leq \Omega_i \left[\widetilde{M}_S(1 + \rho_{i-1}) \right. \\ & \quad \left. + \widetilde{M}_T L(1 + M_1 + M_2) \frac{T^\alpha}{\alpha} \right] \|x - y\|_{PC}. \end{aligned} \quad (30)$$

Let us take $t \in (0, t_1]$ and $x, y \in PC([-r, T], X)$. From (13) we have

$$\begin{aligned} & \|(Nx)(t) - (Ny)(t)\|_X \\ & \leq \int_0^t \|T_\alpha(t-s)\| \|C(s, x) - C(s, y)\| ds \\ & \quad + \int_0^t \|T_\alpha(t-s)\| \\ & \quad \times \left\| f\left(s, x_s, \int_0^s h(s, \tau, x_\tau) d\tau, \int_0^T k(s, \tau, x_\tau) d\tau\right) \right. \\ & \quad \left. - f\left(s, y_s, \int_0^s h(s, \tau, y_\tau) d\tau, \int_0^T k(s, \tau, y_\tau) d\tau\right) \right\| ds \\ & \leq \frac{\widetilde{M}_T}{\alpha} T^\alpha \Omega_1 \frac{\widetilde{M}_T}{\alpha} T^\alpha L [1 + M_1 + M_2] \|x - y\|_{PC} \\ & \quad + \frac{\widetilde{M}_T}{\alpha} T^\alpha L [1 + M_1 + M_2] \|x - y\|_{PC} \\ & \leq \left[1 + \Omega_1 \frac{\widetilde{M}_T}{\alpha} T^\alpha \right] \frac{\widetilde{M}_T}{\alpha} T^\alpha L [1 + M_1 + M_2] \|x - y\|_{PC}. \end{aligned} \quad (31)$$

For $t \in (t_1, t_2]$, and by using (13)–(29) we have

$$\begin{aligned} & \|(Nx)(t) - (Ny)(t)\|_X \\ & \leq \|S_\alpha(t-t_1)\| [\|x(t_1^-) - y(t_1^-)\| \\ & \quad + \|I_1(x(t_1^-)) - I_1(y(t_1^-))\|] \end{aligned}$$

$$\begin{aligned}
& + \int_0^t \|T_\alpha(t-s)\| \|D_2(s, x) - D_2(s, y)\| ds \\
& + \int_0^t \|T_\alpha(t-s)\| \\
& \times \left[\left\| f\left(s, x_s, \int_0^s h(s, \tau, x_\tau) d\tau, \int_0^T k(s, \tau, x_\tau) d\tau\right) \right. \right. \\
& \quad \left. \left. - f\left(s, y_s, \int_0^s h(s, \tau, y_\tau) d\tau, \int_0^T k(s, \tau, y_\tau) d\tau\right) \right\| \right] ds \\
& \leq \left[\widetilde{M}_S(1 + \rho_1) + \frac{\widetilde{M}_T}{\alpha} T^\alpha \Omega_2 \right. \\
& \quad \times \left(\widetilde{M}_S(1 + \rho_1) + \widetilde{M}_T L(1 + M_1 + M_2) \frac{T^\alpha}{\alpha} \right) \\
& \quad \left. + \frac{\widetilde{M}_T}{\alpha} T^\alpha L(1 + M_1 + M_2) \right] \|x - y\|_{PC}. \tag{32}
\end{aligned}$$

Similarly, for $t \in (t_i, t_{i+1}]$

$$\begin{aligned}
& \|(Nx)(t) - (Ny)(t)\|_X \\
& \leq \left[\widetilde{M}_S(1 + \rho_i) + \frac{\widetilde{M}_T}{\alpha} T^\alpha \Omega_i \right. \\
& \quad \times \left(\widetilde{M}_S(1 + \rho_i) + \widetilde{M}_T L(1 + M_1 + M_2) \frac{T^\alpha}{\alpha} \right) \\
& \quad \left. + \frac{\widetilde{M}_T}{\alpha} T^\alpha L(1 + M_1 + M_2) \right] \|x - y\|_{PC}, \tag{33}
\end{aligned}$$

and for $t \in (t_m, T]$

$$\begin{aligned}
& \|(Nx)(t) - (Ny)(t)\|_X \\
& \leq \left[\widetilde{M}_S(1 + \rho_m) + \frac{\widetilde{M}_T}{\alpha} T^\alpha \Omega_m \right. \\
& \quad \times \left(\widetilde{M}_S(1 + \rho_m) + \widetilde{M}_T L(1 + M_1 + M_2) \frac{T^\alpha}{\alpha} \right) \\
& \quad \left. + \frac{\widetilde{M}_T}{\alpha} T^\alpha L(1 + M_1 + M_2) \right] \|x - y\|_{PC}. \tag{34}
\end{aligned}$$

Thus, for all $t \in [0, T]$, we have

$$\begin{aligned}
& \|(Nx) - (Ny)\|_{PC} \\
& \leq \max_{1 \leq i \leq m} \left[\widetilde{M}_S(1 + \rho_m) + \frac{\widetilde{M}_T}{\alpha} T^\alpha \Omega_m \right.
\end{aligned}$$

$$\begin{aligned}
& \times \left(\widetilde{M}_S(1 + \rho_m) + \widetilde{M}_T L(1 + M_1 + M_2) \frac{T^\alpha}{\alpha} \right) \\
& \quad \left. + \frac{\widetilde{M}_T}{\alpha} T^\alpha L(1 + M_1 + M_2) \right] \|x - y\|_{PC}. \tag{35}
\end{aligned}$$

Since $\max_{1 \leq i \leq m} [\widetilde{M}_S(1 + \rho_m) + (\widetilde{M}_T/\alpha) T^\alpha \Omega_m (\widetilde{M}_S(1 + \rho_m) + \widetilde{M}_T L(1 + M_1 + M_2)(T^\alpha/\alpha)) + (\widetilde{M}_T/\alpha) T^\alpha L(1 + M_1 + M_2)] < 1$, then N is a contraction, and so by Banach fixed point theorem there exists a unique fixed point $x \in PC(J, X)$ such that $(Nx)(t) = x(t)$. This fixed point is then a solution of the system (1)–(3), and clearly, $x(T) = (Nx)(T) = Z(T)$, which implies that the system is controllable on J . This completes the proof. \square

4. Example

Consider the following fractional partial functional integro-differential equations of the form

$$\begin{aligned}
D_t^\alpha z(t, \xi) &= \frac{\partial}{\partial \xi} z(t, \xi) + m(\xi) u(t, \xi) \\
&+ F\left(t, x(t, \xi), \int_0^t k_1(t, w(x, \xi - r)) ds, \right. \\
&\quad \left. \int_0^b h_1(t, w(x, \xi - r)) ds\right), \\
&\text{for } (t, \xi) \in [0, T] \times (0, \pi), \quad t \neq \frac{T}{2}, \tag{36}
\end{aligned}$$

$$z(t, 0) = z(t, \pi) = 0, \quad t \in [0, T],$$

$$z(0, \xi) = z_0(\xi), \quad 0 < \xi < \pi,$$

$$\Delta z|_{t=T/2} = I_1\left(\frac{T^-}{2}\right),$$

where $T > 0$, $0 < \alpha < 1$. The the above example resembles the control system (1)–(3), if we take

(i) $X = L^2([0, \pi])$ as the state space and $z(t, \cdot) = \{z(t, \xi) : 0 \leq \xi \leq \pi\}$ as the state.

(ii) Input trajectory $u(t, \cdot) \in U$ as the control, where U is any Banach space.

(iii) $A : D(A) \subset X \rightarrow X$ is defined by $D(A) = \{y \in X : \partial z/\partial x, \partial^2 z/\partial x^2 \in X \text{ are absolutely continuous and } y(0) = y(\pi) = 0\}$,

and $Au = \partial^2 u/\partial x^2$. Then $Az = -\sum_{n=1}^\infty n^2(y, y_n)y_n$, $y \in D(A)$, where $z_n(x) = \sqrt{2/\pi} \sin(nx)$, $n \in \mathbb{N}$ is the orthogonal set of eigen vectors of A . It is well known that A is the infinitesimal generator of an analytic semigroup $\{T(t)_{t \geq 0}\}$ in X and that is given by

$$T(t)y = \sum_{n=1}^\infty e^{-n^2 t} (y, y_n) y_n, \tag{37}$$

for all $y \in X$, and every $t > 0$. From these expression it follows that, $\{T(t)_{t \geq 0}\}$ is a uniformly bounded compact semigroup, so that, $R(\lambda, A) = (\lambda I - A)^{-1}$ is a compact operator for $\lambda \in \rho(A)$, that is, $A \in \mathcal{A}^\alpha(\theta_0, \omega_0)$.

(iv) $B : X \rightarrow X$ by $(Bu)(\xi) = m(\xi)u(\xi)$ for almost every $\xi \in [0, \pi]$.

(v) $I_1 : X \rightarrow X$ is any function satisfying assumption (H_3) .

Therefore, the the system (36) can be written to the abstract form (1)–(3), and all the conditions of Theorem 7 are satisfied. We can conclude that the system (36) is controllable on J .

5. Conclusions

In this article, abstract results concerning the controllability of impulsive fractional functional integro-differential equations involving Caputo fractional derivative in Banach spaces are obtained. By using fractional calculus theory and some standard fixed point theorem, we derived the controllability results. An example is provided to show the effectiveness of the proposed results.

Acknowledgment

The authors wish to thank the referees for their several detailed remarks that led to a substantial improvement in the content as well as in the presentation of the paper. The work is partially supported by the Government of Spain and the FEDER project MTM2010-16499. C. Ravichandran would like to thank Dr. K. V. Kupusamy, Chairman, and Dr. Y. Robinson, Director at RVS Technical Campus, Coimbatore, for their constant encouragement and support for this research work.

References

- [1] D. Baleanu, Z. B. Gunvenc, and J. A. T. Machado, *New Trends in Nanotechnology and Fractional Calculus Applications*, Springer, New York, NY, USA, 2010.
- [2] D. Baleanu, K. Diethelm, E. Scalas, and J. J. Trujillo, *Fractional Calculus Models and Numerical Methods*, vol. 3 of *Series on Complexity, Nonlinearity and Chaos*, World Scientific, 2012.
- [3] A. Kilbas, H. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, Amsterdam, The Netherlands, 2006.
- [4] V. Lakshmikantham, S. Leela, and J. Vasundhara Devi, *Theory of Fractional Dynamic Systems*, Cambridge Scientific, 2009.
- [5] K. S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, John Wiley & Sons, New York, NY, USA, 1993.
- [6] I. Podlubny, *Fractional Differential Equations*, Academic Press, New York, NY, USA, 1999.
- [7] M. Belmekki and M. Benchohra, "Existence results for fractional order semilinear functional differential equations with nondense domain," *Nonlinear Analysis*, vol. 72, no. 2, pp. 925–932, 2010.
- [8] Y. F. Luchko, M. Rivero, J. J. Trujillo, and M. P. Velasco, "Fractional models, non-locality, and complex systems," *Computers & Mathematics with Applications*, vol. 59, no. 3, pp. 1048–1056, 2010.
- [9] I. S. Jesus and J. A. Tenreiro Machado, "Application of integer and fractional models in electrochemical systems," *Mathematical Problems in Engineering*, vol. 2012, Article ID 248175, 17 pages, 2012.
- [10] E. Hernández, D. O'Regan, and K. Balachandran, "On recent developments in the theory of abstract differential equations with fractional derivatives," *Nonlinear Analysis*, vol. 73, no. 10, pp. 3462–3471, 2010.
- [11] E. Hernández, D. O'Regan, and K. Balachandran, "Existence results for abstract fractional differential equations with nonlocal conditions via resolvent operators," *Indagationes Mathematicae*, vol. 24, no. 1, pp. 68–82, 2013.
- [12] C. Ravichandran and D. Baleanu, "Existence results for fractional neutral functional integro-differential evolution equations with infinite delay in Banach spaces," *Advances in Difference Equations*, vol. 215, pp. 1–12, 2013.
- [13] J. P. C. dos Santos, V. Vijayakumar, and R. Murugesu, "Existence of mild solutions for nonlocal Cauchy problem for fractional neutral integro-differential equation with unbounded delay," *Communications in Mathematical Analysis*, vol. 14, no. 1, pp. 59–71, 2013.
- [14] J. R. Wang, W. Wei, and Y. Zhou, "Fractional finite time delay evolution systems and optimal controls in infinite-dimensional spaces," *Journal of Dynamical and Control Systems*, vol. 17, no. 4, pp. 515–535, 2011.
- [15] V. Lakshmikantham, D. D. Bainov, and P. S. Simeonov, *Theory of Impulsive Differential Equations*, World Scientific, Singapore, 1989.
- [16] D. D. Bainov and P. S. Simeonov, *Systems with Impulse Effect*, Ellis Horwood, Chichester, UK, 1989.
- [17] A. M. Samoilenko and N. A. Perestyuk, *Impulsive Differential Equations*, vol. 14, World Scientific, Singapore, 1995.
- [18] A. Anguraj, M. Mallika Arjunan, and E. Hernández, "Existence results for an impulsive neutral functional differential equation with state-dependent delay," *Applicable Analysis*, vol. 86, no. 7, pp. 861–872, 2007.
- [19] V. Vijayakumar, S. Sivasankaran, and M. Mallika Arjunan, "Existence of global solutions for second order impulsive abstract functional integrodifferential equations," *Dynamics of Continuous, Discrete & Impulsive Systems*, vol. 18, no. 6, pp. 747–766, 2011.
- [20] S. Sivasankaran, M. Mallika Arjunan, and V. Vijayakumar, "Existence of global solutions for second order impulsive abstract partial differential equations," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 74, no. 17, pp. 6747–6757, 2011.
- [21] V. Vijayakumar, S. Sivasankaran, and M. M. Arjunan, "Existence of solutions for second-order impulsive neutral functional integro-differential equations with infinite delay," *Nonlinear Studies*, vol. 19, no. 2, pp. 327–343, 2012.
- [22] V. Vijayakumar, K. Alagiri Prakash, and R. Murugesu, "Existence of global solutions for second order impulsive differential equations with nonlocal conditions," *Nonlinear Studies*, vol. 20, no. 3, pp. 1–13, 2013.
- [23] C. Cuevas, E. Hernández, and M. Rabelo, "The existence of solutions for impulsive neutral functional differential equations," *Computers & Mathematics with Applications*, vol. 58, no. 4, pp. 744–757, 2009.

- [24] E. Hernández and H. R. Henríquez, "Impulsive partial neutral differential equations," *Applied Mathematics Letters*, vol. 19, no. 3, pp. 215–222, 2006.
- [25] X. B. Shu, Y. Z. Lai, and Y. Chen, "The existence of mild solutions for impulsive fractional partial differential equations," *Nonlinear Analysis*, vol. 74, no. 5, pp. 2003–2011, 2011.
- [26] X. B. Shu, Y. Z. Lai, and F. Xu, "Existence and uniqueness of mild solution for abstract fractional functional differential equations," *Dynamics of Continuous, Discrete & Impulsive Systems*, vol. 18, no. 3, pp. 371–382, 2012.
- [27] N. K. Tomar and J. Dabas, "Controllability of impulsive fractional order semilinear evolution equations with nonlocal conditions," *Journal of Nonlinear Evolution Equations and Applications*, vol. 5, pp. 57–67, 2012.
- [28] A. Debbouche and D. Baleanu, "Controllability of fractional evolution nonlocal impulsive quasilinear delay integro-differential systems," *Computers & Mathematics with Applications*, vol. 62, no. 3, pp. 1442–1450, 2011.
- [29] K. Balachandran and J. Y. Park, "Controllability of fractional integrodifferential systems in Banach spaces," *Nonlinear Analysis. Hybrid Systems*, vol. 3, no. 4, pp. 363–367, 2009.
- [30] F. Chen, A. Chen, and X. Wang, "On the solutions for impulsive fractional functional differential equations," *Differential Equations and Dynamical Systems*, vol. 17, no. 4, pp. 379–391, 2009.
- [31] Y. Q. Chen, H. S. Ahn, and D. Xue, "Robust controllability of interval fractional order linear time invariant systems," *Signal Processing*, vol. 86, no. 10, pp. 2794–2802, 2006.
- [32] J. Dabas, A. Chauhan, and M. Kumar, "Existence of the mild solutions for impulsive fractional equations with infinite delay," *International Journal of Differential Equations*, vol. 2011, Article ID 793023, 20 pages, 2011.
- [33] T. L. Guo, "Controllability and observability of impulsive fractional linear time-invariant system," *Computers & Mathematics with Applications*, vol. 64, no. 10, pp. 3171–3182, 2012.
- [34] K. Balachandran and J. Kokila, "On the controllability of fractional dynamical systems," *International Journal of Applied Mathematics and Computer Science*, vol. 22, no. 3, pp. 523–531, 2012.
- [35] K. Balachandran and J. Kokila, "Controllability of non-linear implicit fractional dynamical systems," *IMA Journal of Applied Mathematics*, pp. 1–9, 2013.
- [36] K. Balachandran, V. Govindaraj, L. Rodriguez-Germa, and J. J. Trujillo, "Controllability results for nonlinear fractional order dynamical systems," *Journal of Optimization Theory and Applications*, vol. 156, no. 1, pp. 33–44, 2013.
- [37] K. Balachandran, V. Govindaraj, L. Rodriguez-Germa, and J. J. Trujillo, "Controllability of nonlinear higher order fractional dynamical systems," *Nonlinear Dynamics*, vol. 71, no. 4, pp. 605–612, 2013.
- [38] J. A. Machado, C. Ravichandran, M. Rivero, and J. J. Trujillo, "Controllability results for impulsive mixed-type functional integro-differential evolution equations with nonlocal conditions," *Fixed Point Theory and Applications*, vol. 66, pp. 1–16, 2013.
- [39] H. Wang, "Existence results for fractional functional differential equations with impulses," *Journal of Applied Mathematics and Computing*, vol. 38, no. 1-2, pp. 85–101, 2012.
- [40] A. Chauhan and J. Dabas, "Existence of mild solutions for impulsive fractional-order semilinear evolution equations with nonlocal conditions," *Electronic Journal of Differential Equations*, vol. 107, pp. 1–11, 2011.
- [41] E. Bazhiekova, *Fractional evolution equations in Banach spaces [Ph.D. thesis]*, Eindhoven University of Technology, 2001.
- [42] D. Araya and C. Lizama, "Almost automorphic mild solutions to fractional differential equations," *Nonlinear Analysis*, vol. 69, no. 11, pp. 3692–3705, 2008.
- [43] R. P. Agarwal, B. De Andrade, and G. Siracusa, "On fractional integro-differential equations with state-dependent delay," *Computers & Mathematics with Applications*, vol. 62, no. 3, pp. 1143–1149, 2011.
- [44] J. Pruss, *Evolutionary Integral Equations and Applications*, vol. 87 of *Monographs in Mathematics*, Birkhäuser, Basel, Switzerland, 1993.
- [45] C. Lizama, "Regularized solutions for abstract Volterra equations," *Journal of Mathematical Analysis and Applications*, vol. 243, no. 2, pp. 278–292, 2000.
- [46] M. D. Quinn and N. Carmichael, "An approach to nonlinear control problems using fixed-point methods, degree theory and pseudo-inverses," *Numerical Functional Analysis and Optimization*, vol. 7, no. 2-3, pp. 197–219, 1985.

