

Research Article

A New Essential Norm Estimate of Composition Operators from Weighted Bloch Space into μ -Bloch Spaces

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Let μ be any weight function defined on the unit disk \mathbb{D} and let ϕ be an analytic self-map of \mathbb{D} . In the present paper, we show that the essential norm of composition operator C_ϕ mapping from the weighted Bloch space to μ -Bloch space \mathcal{B}^μ is comparable to $\limsup_{|a| \rightarrow 1^-} \|\sigma_a \circ \phi\|_{\mathcal{B}^\mu}$, where for $a \in \mathbb{D}$, σ_a is a certain special function in the weighted Bloch space. As a consequence of our estimate, we extend the results about the compactness of composition operators due to Tjani (2003).

1. Introduction

The weighted Bloch space appears in the literature when we study properties of certain operators acting on certain spaces of analytic functions on the unit disk \mathbb{D} of the complex plane \mathbb{C} . For instance, in 1991, Brown and Shields [1] showed that an analytic function u is a multiplier on the Bloch space \mathcal{B} if and only if

$$\|u\|_* = \sup_{z \in \mathbb{D}} (1 - |z|^2) \log \left(\frac{1}{1 - |z|^2} \right) |u'(z)| < \infty. \quad (1)$$

Also, in 1992, Attele [2] showed that Hankel operator induced by a function $f \in H(\mathbb{D})$ in the Bergman space L_a^1 (for the definition of Bergman space and the Hankel operator, see [3]) is bounded if and only if $\|f\|_* < \infty$, where $H(\mathbb{D})$ is the space of all holomorphic functions on \mathbb{D} with the topology of uniform convergence on compact subsets of \mathbb{D} . The set of all functions $f \in H(\mathbb{D})$ such that (1) holds is denoted by \mathcal{B}^{\log} and it is known that \mathcal{B}^{\log} is a Banach space with the norm $\|f\|_{\mathcal{B}^{\log}} = |f(0)| + \|f\|_*$.

In the last decade, many authors have studied different classes of Bloch-type spaces, where the weight function, $\nu(z) = (1 - |z|^2) \log(1/(1 - |z|^2))$, ($z \in \mathbb{D}$), is replaced by a bounded, continuous, and positive function μ defined on \mathbb{D} .

More precisely, a function $f \in H(\mathbb{D})$ is called a μ -Bloch function, denoted as $f \in \mathcal{B}^\mu$, if

$$\|f\|_\mu := \sup_{z \in \mathbb{D}} \mu(z) |f'(z)| < \infty. \quad (2)$$

If $\mu_1(z) = 1 - |z|^2$, we get the classical Bloch space \mathcal{B} (see [3]), while if $\mu_2(z) = \mu_1^\alpha(z)$ with $\alpha > 0$, \mathcal{B}^μ is just the α -Bloch space (see [4]). It is readily seen that \mathcal{B}^μ is a Banach space with the norm $\|f\|_{\mathcal{B}^\mu} := |f(0)| + \|f\|_\mu$.

A holomorphic function ϕ from the unit disk \mathbb{D} into itself induces a linear operator C_ϕ , defined by $C_\phi(f) = f \circ \phi$, where $f \in H(\mathbb{D})$. C_ϕ is the so-called *composition operator* with symbol ϕ . Composition operators continue to be widely studied on many subspaces of $H(\mathbb{D})$ and particularly in Bloch-type spaces.

The study of the properties of composition operators on Bloch-type spaces began with the celebrated work of Madigan and Matheson in [5], where they characterized the continuity and compactness of composition operators acting on the Bloch space \mathcal{B} . Many extensions of Madigan and Matheson's results have appeared (see, e.g., [6] and a lot of references therein). An important reference on this subject is the work of Montes-Rodríguez in [7] (see also [8]); here the author supposes that weight μ is radial ($\mu(|z|) = \mu(z)$ for all $z \in \mathbb{D}$), typical ($\lim_{|z| \rightarrow 1^-} \mu(z) = 0$), and decreasing function of $|z|$.

defined on \mathbb{D} . In particular, Yoneda in [9] has extended the results by Madigan and Matheson in [5] to composition operators C_ϕ acting between weighted Bloch spaces. The weight considered by Yoneda in [9] was $\mu_{\log}(z) = (1 - |z|^2) \log(2/(1 - |z|^2))$ which is not decreasing on \mathbb{D} .

Recently, many authors have found new criteria for the continuity and compactness of composition operators acting on Bloch-type spaces in terms of the n th power of the symbol ϕ and the norm of the n th power of the identity function on \mathbb{D} . The first result of this kind appears in 2009 and it is due to Wulan et al. [10]; in turn, their result was extended to α -Bloch spaces by Zhao in [11]. Another criterion for the continuity and compactness of composition operators on Bloch space is due to Tjani in [12] (see also [13] or more recently [10]); she showed the following result.

Theorem 1 (see [12]). *The composition operator C_ϕ is compact on \mathcal{B} if and only if $\phi \in \mathcal{B}$ and*

$$\lim_{|a| \rightarrow 1^-} \|\varphi_a \circ \phi\|_{\mathcal{B}} = 0, \quad (3)$$

where φ_a is a Möbius transformation from the unit disk onto itself; that is, $\varphi_a(z) = (a - z)/(1 - \bar{a}z)$, with $z \in \mathbb{D}$.

This last result has been recently extended to α -Bloch spaces by Malavé-Ramírez and Ramos-Fernández in [14].

The essential norm of a continuous linear operator T between normed linear spaces X and Y is its distance from the compact operators; that is, $\|T\|_e^{X \rightarrow Y} = \inf\{\|T - K\|^{X \rightarrow Y} : K : X \rightarrow Y \text{ is compact}\}$, where $\|\cdot\|^{X \rightarrow Y}$ denotes the operator norm. Notice that $\|T\|_e^{X \rightarrow Y} = 0$ if and only if T is compact, so that estimates on $\|T\|_e^{X \rightarrow Y}$ lead to conditions for T to be compact. The essential norm of a composition operator on \mathcal{B} was calculated by Montes-Rodríguez in [15]. He obtained similar results for essential norms of weighted composition operators between weighted Banach spaces of analytic functions in [7]. Recently, many extensions of the above results have appeared in the literature; for instance, Zhao in [11] gave a formula for the essential norm of $C_\phi : \mathcal{B}^\alpha \rightarrow \mathcal{B}^\beta$ in terms of an expression involving norms of powers of ϕ . More precisely, he showed that

$$\|C_\phi\|_e^{\mathcal{B}^\alpha \rightarrow \mathcal{B}^\beta} = \left(\frac{e}{2\alpha}\right)^\alpha \limsup_{j \rightarrow \infty} j^{\alpha-1} \|\phi^j\|_{\mathcal{B}^\beta}. \quad (4)$$

It follows from the discussion at the beginning of this paragraph that $C_\phi : \mathcal{B}^\alpha \rightarrow \mathcal{B}^\beta$ is compact if and only if

$$\lim_{j \rightarrow \infty} j^{\alpha-1} \|\phi^j\|_{\mathcal{B}^\beta} = 0. \quad (5)$$

Zhao's results in [11] have been extended recently to the weighted Bloch spaces by Castillo et al. in [16]. Also, Hyvärinen et al. in [17] obtained necessary and sufficient conditions for boundedness and an expression characterizing the essential norm of a weighted composition operator between general weighted Bloch spaces \mathcal{B}^μ , under the technical requirements that μ is radial, and that it is nonincreasing and tends to zero toward the boundary of \mathbb{D} .

The goal of the present paper is to give a new estimate of the essential norm of composition C_ϕ mapping from \mathcal{B}^{\log} to \mathcal{B}^μ . More precisely, in the next section we will show the following result.

Theorem 2. *Let ϕ be an analytic self-map of the unit disk \mathbb{D} . Then for the essential norm of the composition operator $C_\phi : \mathcal{B}^{\log} \rightarrow \mathcal{B}^\mu$, one has*

$$\|C_\phi\|_e^{\mathcal{B}^{\log} \rightarrow \mathcal{B}^\mu} : \limsup_{|a| \rightarrow 1^-} \|\sigma_a \circ \phi\|_{\mathcal{B}^\mu}. \quad (6)$$

The relation (6) means that there is a positive constant M such that

$$\frac{1}{M} \|C_\phi\|_e^{\mathcal{B}^{\log} \rightarrow \mathcal{B}^\mu} \leq \limsup_{|a| \rightarrow 1^-} \|\sigma_a \circ \phi\|_{\mathcal{B}^\mu} \leq M \|C_\phi\|_e^{\mathcal{B}^{\log} \rightarrow \mathcal{B}^\mu}, \quad (7)$$

and the functions σ_a with $a \in \mathbb{D}$ will be defined at the beginning of the next section. As a consequence of our estimate, we extend recent results, about the compactness of composition operators, due to Tjani in [12] (see Theorem 1) and Malavé-Ramírez and Ramos-Fernández in [14].

2. A Class of Special Functions for \mathcal{B}^{\log}

For convenience, throughout this paper we consider the following weight:

$$v_{\log}(z) = (1 - |z|^2) \log\left(\frac{e}{1 - |z|^2}\right), \quad (z \in \mathbb{D}), \quad (8)$$

in place of the one used by Brown and Shields in (1). It is not hard to see that with this new weight we have the same space \mathcal{B}^{\log} , and the seminorm

$$\|f\|_{\log} := \sup_{z \in \mathbb{D}} v_{\log}(z) |f'(z)| \quad (9)$$

is comparable to $\|f\|_*$ defined in (1). Moreover, this weight v_{\log} satisfies the following property which will be of main relevance in the proof of our result.

Lemma 3. *For all $r \in (0, 1)$ and all $z \in \mathbb{D}$, one has*

$$rv_{\log}(z) \leq v_{\log}(rz). \quad (10)$$

Proof. The result follows from the fact that the function $h(t) = t \log(e/t)$ with $t \in (0, 1)$ is concave and satisfies $h(t) \rightarrow 0$ as $t \rightarrow 0^+$. \square

The key to our results lies in considering the following family of functions: for $a \in \mathbb{D}$ fixed, we define

$$\sigma_a(z) = (1 - |a|)(H(\bar{a}z) - 1), \quad (z \in \mathbb{D}), \quad (11)$$

where for $a \in \mathbb{D}$, the function H is defined by

$$H(z) = \frac{1}{(1 - z) \log(e/(1 - z))}. \quad (12)$$

We have the following result.

Lemma 4. *The family $\{\sigma_a\}_{a \in \mathbb{D}}$ satisfies the following properties.*

- (1) σ_a converges to zero uniformly on compact subsets of \mathbb{D} as $|a| \rightarrow 1^-$.
- (2) There exists a constant $C_\sigma > 0$ such that

$$v_{\log}(a) |\sigma'_a(a)| \geq C_\sigma \quad (13)$$

for all $a \in \mathbb{D}$ such that $(1/2) < |a| < 1$.

- (3) There exists a constant $M_\sigma > 0$ such that

$$\sup_{a \in \mathbb{D}} \|\sigma_a\|_{\mathcal{B}^{\log}} \leq M_\sigma. \quad (14)$$

Proof. The property (1) is clear since the function H is bounded on compact subsets of \mathbb{D} .

- (2) If $(1/2) < |a| < 1$, then

$$|\sigma'_a(a)| = |a| \frac{(1 - |a|) \log(1 - |a|^2)}{(1 - |a|^2)^2 \log^2(e/(1 - |a|^2))} \quad (15)$$

$$\begin{aligned} &\geq \frac{1}{2} \frac{\log(1 - |a|^2)}{1 - \log(1 - |a|^2)} \\ &\quad \times \frac{1}{(1 - |a|^2) \log(1 - |a|^2)}, \end{aligned} \quad (16)$$

and the property follows since

$$\lim_{t \rightarrow 0^+} \frac{\log(t)}{1 - \log(t)} = 1. \quad (17)$$

- (3) Indeed, for $a \in \mathbb{D}$, we have

$$\begin{aligned} &\|\sigma_a\|_{\mathcal{B}^{\log}} \\ &= \sup_{z \in \mathbb{D}} |a| \frac{(1 - |z|^2) \log(e/(1 - |z|^2)) (1 - |a|) |\log(1 - \bar{a}z)|}{|(1 - \bar{a}z) \log(e/(1 - \bar{a}z))|^2} \\ &\leq \sup_{z \in \mathbb{D}} \frac{(1 - |z|^2) \log(e/(1 - |z|^2)) (1 - |a|) |\log(1 - \bar{a}z)|}{|1 - \bar{a}z|^2 \log^2(e/|1 - \bar{a}z|)} \end{aligned} \quad (18)$$

since, for each $w \in \mathbb{C}$, we have $|\log(w)| \geq \log|w|$. Note that the last expression is uniformly bounded if $1 \leq |1 - \bar{a}z| < 2$, and thus it will be enough to consider the case when $|1 - \bar{a}z| < 1$. Furthermore, since the function $h(t) = t \log(e/t)$ with $t \in (0, 1)$ is increasing, concave and satisfy $h(t) \rightarrow 0$ as $t \rightarrow 0^+$, then we can use the fact that $1 > |1 - \bar{a}z| \geq (1/2)(1 - |z|^2)$ and conclude that

$$\begin{aligned} h(|1 - \bar{a}z|) &\geq h\left(\frac{1}{2}(1 - |z|^2)\right) \\ &\geq \frac{1}{2} h(1 - |z|^2). \end{aligned} \quad (19)$$

Hence

$$\frac{(1 - |z|^2) \log(e/(1 - |z|^2))}{|1 - \bar{a}z| \log(e/|1 - \bar{a}z|)} \leq 2. \quad (20)$$

Also, it is clear that

$$\frac{1 - |a|}{|1 - \bar{a}z|} \leq 1. \quad (21)$$

Finally, since

$$\lim_{t \rightarrow 0^+} \frac{\sqrt{\log^2 t + 4\pi^2}}{1 - \log t} = 1, \quad (22)$$

then we can deduce that there exists a constant $M > 0$ such

$$\frac{|\log(1 - \bar{a}z)|}{\log(e/|1 - \bar{a}z|)} \leq \frac{\sqrt{\log^2 |1 - \bar{a}z| + 4\pi^2}}{1 - \log|1 - \bar{a}z|} \leq M. \quad (23)$$

Now the proof is complete. \square

3. A Criterion for the Compactness of Operators Acting on Spaces of Analytic Functions

We will need to prove three auxiliary facts. The first of which is the following lemma that appears in [18] and also in [12]. In both of these places, we point out a typographical error (“point evaluation functionals on X ” there, as one can see from Relation (16) in [12], should be “point evaluations on Y ”).

Lemma 5 (Tjani). *Let X, Y be two Banach spaces of analytic functions on \mathbb{D} . Suppose that*

- (1) *point evaluation functionals on Y are bounded;*
- (2) *the closed unit ball of X is a compact subset of X in the topology of uniform convergence on compact sets;*
- (3) *$T : X \rightarrow Y$ is continuous when X and Y are equipped with the topology of uniform convergence on compact sets.*

Then T is a compact operator if and only if given a bounded sequence $(f_j)_{j \in \mathbb{N}}$ in X such that $f_j \rightarrow 0$ uniformly on compact sets, $(Tf_j)_{j \in \mathbb{N}} \rightarrow 0$ in Y .

The following facts about μ -Bloch space are well known.

- (i) For each compact $K \subset \mathbb{D}$, there is a $C_K \geq 0$ such that for all $f \in \mathcal{B}^\mu$ and $z \in K$, we have that

$$|f(z)| \leq C_K \|f\|_{\mathcal{B}^\mu}. \quad (24)$$

- (ii) Every point evaluation functional on \mathcal{B}^μ is bounded.
- (iii) The closed unit ball of \mathcal{B}^μ is compact in the topology of uniform convergence on compact subsets of \mathbb{D} .
- (iv) If γ is a weight on \mathbb{D} and $\phi : \mathbb{D} \rightarrow \mathbb{D}$ is holomorphic, with $C_\phi : \mathcal{B}^\mu \rightarrow \mathcal{B}^\gamma$, then C_ϕ is continuous with respect to the compact-open subspace topologies on \mathcal{B}^μ and \mathcal{B}^γ .

Thus, combining the above and Lemma 5, we obtain the following auxiliary result.

Lemma 6. Let μ_1 and μ_2 be weights on \mathbb{D} and suppose that $\phi : \mathbb{D} \rightarrow \mathbb{D}$ is holomorphic. Then $C_\phi : \mathcal{B}^{\mu_1} \rightarrow \mathcal{B}^{\mu_2}$ is compact if and only if given a bounded sequence $(f_j)_{j \in \mathbb{N}}$ in \mathcal{B}^{μ_1} such that $f_j \rightarrow 0$ uniformly on compact subsets of \mathbb{D} , then $\|C_\phi(f_j)\|_{\mathcal{B}^{\mu_2}} \rightarrow 0$ as $j \rightarrow \infty$.

As a consequence of Lemma 5, we can see that the dilatation operator is compact on \mathcal{B}^{\log} . Recall that for $r \in [0, 1]$, the linear dilation operator $K_r : H(\mathbb{D}) \rightarrow H(\mathbb{D})$ is defined by $K_r f = f_r$, where f_r , for each $f \in H(\mathbb{D})$, is given by $f_r(z) = f(rz)$. It is clear that if $f \in H(\mathbb{D})$ then $rf_r \rightarrow f$ uniformly on compact subsets of \mathbb{D} as $r \rightarrow 1^-$.

Lemma 7. The following statements hold:

- (1) for $r \in [0, 1)$, the operator K_r is compact on \mathcal{B}^{\log} ;
- (2) for each $r \in [0, 1]$

$$\|K_r\|_{\mathcal{B}^{\log} \rightarrow \mathcal{B}^{\log}} \leq 1. \quad (25)$$

Proof. If $\{f_n\}$ is a bounded sequence in \mathcal{B}^{\log} such that $f_n \rightarrow 0$ uniformly on compact subsets of \mathbb{D} , then

$$\begin{aligned} 0 &\leq \lim_{n \rightarrow \infty} \|K_r f_n\|_{\log} \\ &= \sup_{z \in \mathbb{D}} v_{\log}(z) r |f'(rz)| \\ &\leq \|v_{\log}\|_{\infty} \lim_{n \rightarrow \infty} \sup_{s \in \mathbb{D}(0, r)} |f'(s)| = 0, \end{aligned} \quad (26)$$

and the item (1) follows by Lemma 5.

On the other hand, if $f \in \mathcal{B}^{\log}$ then

$$\begin{aligned} \|K_r f\|_{\mathcal{B}^{\log}} &= |f(0)| + \sup_{z \in \mathbb{D}} \frac{r v_{\log}(z)}{v_{\log}(rz)} v_{\log}(rz) |f'(rz)| \\ &\leq |f(0)| + \|f\|_{\log} = \|f\|_{\mathcal{B}^{\log}}, \end{aligned} \quad (27)$$

where we have used Lemma 3. This ends the proof of item (2). \square

4. Proof of Theorem 2

Now we can show Theorem 2. We set

$$L = \limsup_{|a| \rightarrow 1^-} \|\sigma_a \circ \phi\|_{\mathcal{B}^\mu}. \quad (28)$$

Let $K : \mathcal{B}^{\log} \rightarrow \mathcal{B}^\mu$ be any compact operator, $a \in \mathbb{D}$ fixed and define

$$f_a(z) = \frac{1}{M_\sigma} \sigma_a(z), \quad (z \in \mathbb{D}), \quad (29)$$

where M_σ is the constant in the item (3) of Lemma 4. Then f_a goes to zero uniformly on compact subsets of \mathbb{D} as $|a| \rightarrow 1^-$, $\|f_a\|_{\mathcal{B}^{\log}} \leq 1$ for all $a \in \mathbb{D}$, and

$$\begin{aligned} \|C_\phi - K\|_{\mathcal{B}^{\log} \rightarrow \mathcal{B}^\mu} &\geq \|(C_\phi - K) f_a\|_{\mathcal{B}^\mu} \\ &\geq \frac{1}{M_\sigma} \|\sigma_a \circ \phi\|_{\mathcal{B}^\mu} - \|K f_a\|_{\mathcal{B}^\mu}. \end{aligned} \quad (30)$$

Hence, taking $\limsup_{|a| \rightarrow 1^-}$ and using Lemma 5, we obtain

$$\|C_\phi\|_{\mathcal{B}^{\log} \rightarrow \mathcal{B}^\mu} \geq \frac{1}{M_\sigma} \limsup_{|a| \rightarrow 1^-} \|\sigma_a \circ \phi\|_{\mathcal{B}^\mu}. \quad (31)$$

Now, we go to show that there exists a constant $C > 0$ such that

$$\|C_\phi\|_{\mathcal{B}^{\log} \rightarrow \mathcal{B}^\mu} \leq C \limsup_{|a| \rightarrow 1^-} \|\sigma_a \circ \phi\|_{\mathcal{B}^\mu}. \quad (32)$$

Hence, if we consider a sequence $\{r_n\} \subset (0, 1)$ such that $r_n \rightarrow 1$ as $n \rightarrow \infty$ and we consider the dilatation operators $K_n = K_{r_n}$, then by Lemma 7, for all $n \in \mathbb{N}$, the operator $C_\phi K_n$ is a compact from \mathcal{B}^{\log} into \mathcal{B}^μ , and by definition of the essential norm, we have

$$\|C_\phi\|_{\mathcal{B}^{\log} \rightarrow \mathcal{B}^\mu} \leq \limsup_{n \rightarrow \infty} \|C_\phi - C_\phi K_n\|_{\mathcal{B}^{\log} \rightarrow \mathcal{B}^\mu}. \quad (33)$$

Thus, we have to show that there exists a constant $M > 0$ such that

$$\limsup_{n \rightarrow \infty} \|C_\phi - C_\phi K_n\|_{\mathcal{B}^{\log} \rightarrow \mathcal{B}^\mu} \leq ML. \quad (34)$$

To see this, consider any $f \in \mathcal{B}^{\log}$ such that $\|f\|_{\mathcal{B}^{\log}} = 1$; then since

$$\begin{aligned} \|(C_\phi - C_\phi K_n) f\|_{\mathcal{B}^\mu} &= |f(\phi(0)) - f(r_n \phi(0))| \\ &\quad + \|(f - f_{r_n}) \circ \phi\|_{\mu} \end{aligned} \quad (35)$$

and $|f(\phi(0)) - f(r_n \phi(0))| \rightarrow 0$ as $n \rightarrow \infty$, it is enough to show that there exists a constant $M > 0$ such that

$$\limsup_{n \rightarrow \infty} \|(f - f_{r_n}) \circ \phi\|_{\mu} \leq ML. \quad (36)$$

Furthermore, since $r_n(f')_{r_n} \rightarrow f'$ uniformly on compact subsets of \mathbb{D} as $n \rightarrow \infty$, we have

$$\limsup_{n \rightarrow \infty} \sup_{|\phi(z)| \leq r_N} \mu(z) |(f - f_{r_n})'(\phi(z))| |\phi'(z)| = 0, \quad (37)$$

where $N \in \mathbb{N}$ is large enough such that $r_n \geq (1/2)$ for all $n \geq N$. Hence we only have to show that there exists a constant $M > 0$ such that

$$\begin{aligned} S &:= \limsup_{n \rightarrow \infty} \sup_{|\phi(z)| > r_N} \mu(z) |(f - f_{r_n})'(\phi(z))| |\phi'(z)| \\ &\leq ML. \end{aligned} \quad (38)$$

Indeed, we write $S \leq \limsup_{n \rightarrow \infty} (S_1 + S_2)$, where

$$\begin{aligned} S_1 &= \sup_{|\phi(z)| > r_N} \mu(z) |f'(\phi(z))| |\phi'(z)|, \\ S_2 &= \sup_{|\phi(z)| > r_N} \mu(z) r_n |f'(r_n \phi(z))| |\phi'(z)|. \end{aligned} \quad (39)$$

Then we have

$$\begin{aligned}
 S_1 &= \sup_{|\phi(z)| > r_N} \mu(z) |f'(\phi(z))| |\phi'(z)| \\
 &\quad \times \frac{\nu_{\log}(\phi(z)) |\sigma'_{\phi(z)}(\phi(z))|}{\nu_{\log}(\phi(z)) |\sigma'_{\phi(z)}(\phi(z))|} \\
 &\leq \frac{1}{C_\sigma} \|f\|_{\mathcal{B}^{\log}} \sup_{|\phi(z)| > r_N} \mu(z) |\sigma'_{\phi(z)}(\phi(z))| |\phi'(z)| \quad (40) \\
 &\leq \frac{1}{C_\sigma} \sup_{|\phi(z)| > r_N} \sup_{|a| > r_N} \mu(z) |\sigma'_a(\phi(z))| |\phi'(z)| \\
 &\leq \frac{1}{C_\sigma} \sup_{|a| > r_N} \|\sigma_a \circ \phi\|_{\mathcal{B}^\mu},
 \end{aligned}$$

where we have used the item (2) of Lemma 4 in the first inequality and the fact that $\|f\|_{\mathcal{B}^{\log}} \leq 1$ in the second one. Taking limit as $N \rightarrow \infty$, we obtain

$$\limsup_{n \rightarrow \infty} S_1 \leq \frac{1}{C_\sigma} L. \quad (41)$$

In a similar way, we have

$$\begin{aligned}
 S_2 &\leq \frac{1}{C_\sigma} \|f\|_{\mathcal{B}^{\log}} \sup_{|\phi(z)| > r_N} \mu(z) |\sigma'_{\phi(z)}(\phi(z))| \\
 &\quad \times |\phi'(z)| \frac{r_n \nu_{\log}(\phi(z))}{\nu_{\log}(r_n \phi(z))} \quad (42) \\
 &\leq \frac{1}{C_\sigma} \sup_{|a| > r_N} \|\sigma_a \circ \phi\|_{\mathcal{B}^\mu},
 \end{aligned}$$

where we have used Lemma 3 in the last inequality. Therefore

$$\|C_\phi\|_{\mathcal{B}^{\log} \rightarrow \mathcal{B}^\mu} \leq \frac{2}{C_\sigma} \limsup_{|a| \rightarrow 1^-} \|\sigma_a \circ \phi\|_{\mathcal{B}^\mu}. \quad (43)$$

This completes the proof of the theorem.

As an immediate consequence of Theorem 2, we have the following corollary which extends the result due to Tjani in [12].

Corollary 8. *The composition operator C_ϕ is compact from \mathcal{B}^{\log} into \mathcal{B}^μ if and only if $\phi \in \mathcal{B}^\mu$ and*

$$\lim_{|a| \rightarrow 1^-} \|\sigma_a \circ \phi\|_\mu = 0. \quad (44)$$

We want to finish this paper with the following question: Is the above corollary true for the composition operator C_ϕ from \mathcal{B}^{μ_1} into \mathcal{B}^{μ_2} , where μ_1 and μ_2 are weights defined on \mathbb{D} ?

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