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Research Article

Bounded Domains of Generalized Riesz Methods with the Hahn Property

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In 2002 Bennett et al. started the investigation to which extent sequence spaces are determined by the sequences of 0s and 1s that they contain. In this relation they defined three types of Hahn properties for sequence spaces: the Hahn property, separable Hahn property, and matrix Hahn property. In general all these three properties are pairwise distinct. If a sequence space E is solid and $(\{0,1\}^{\mathbb{N}} \cap E)^{\beta} = E^{\beta} = \ell^1$ then the two last properties coincide. We will show that even on these additional assumptions the separable Hahn property and the Hahn property still do not coincide. However if we assume E to be the bounded summability domain of a regular Riesz matrix R_p or a regular nonnegative Hausdorff matrix H_p, then this assumption alone guarantees that E has the Hahn property. For any (infinite) matrix A the Hahn property of its bounded summability domain is related to the strongly nonatomic property of the density d_A defined by A. We will find a simple necessary and sufficient condition for the density d_A defined by the generalized Riesz matrix $R_{p,m}$ to be strongly nonatomic. This condition appears also to be sufficient for the bounded summability domain of $R_{p,m}$ to have the Hahn property.

1. Preliminaries and Introduction

We start with some preliminaries. For other notations and preliminary results we refer the reader to [1–3].

Let χ denote the set of all sequences of 0s and 1s and let $\chi(E)$ denote the linear hull of $\chi \cap E$.

An FK-space is a sequence space endowed with a complete, metrizable, locally convex topology under which all coordinate mappings $x = (x_j) \to x_k$ $(k \in \mathbb{N})$ are continuous.

A sequence space E is said to have the *Hahn property*, the *separable Hahn property*, and the *matrix Hahn property*, if $\chi(E) \subset F$ implies $E \subset F$ whenever F is any FK-space, a separable FK-space, and a matrix domain c_B, respectively. Obviously, the Hahn property implies the separable Hahn property, and the latter implies the matrix Hahn property.

If E has the matrix Hahn property then $\chi(E)^{\beta} = E^{\beta}$ (cf. Theorem 5.1 in [1]), but, in general, the inverse implication does not hold even for monotone sequence spaces (see Theorem 1.1 in [4]). Still if we ask E to be a solid sequence space containing φ (the set of all finite sequences) and satisfying $E^{\beta} = \ell^1$, then $\chi(E)^{\beta} = E^{\beta}$ implies the separable Hahn property of E (see [2, Theorem 6]). This result suggests the following problem due to Boos and Leiger (cf. Problem 3 in [2]). Now, we formulate this question as a problem and give a negative answer to it.

Problem 1. Let E be a solid sequence space containing φ and satisfying $\chi(E)^{\beta} = E^{\beta} = \ell^1$. Then E has the separable Hahn property. Does it have the Hahn property?

To answer this problem we consider some facts from the theory of double sequence spaces.

A double sequence space is a linear subspace of Ω, the space of all real double sequences $x = (x_{kl})$. In particular, the following sets are double sequence spaces:

$$\mathscr{M}_u := \left\{ x \in \Omega \mid \|x\|_{\infty} := \sup_{k,l} |x_{k,l}| < \infty \right\},$$

$$\mathscr{C}_{\mathrm{be0}} := \left\{ x \in \Omega \mid (x_{kl})_k \in \ell^\infty \ (l \in \mathbb{N}), \left(\limsup_k |x_{k,l}| \right)_l \in c_0 \right\}. \tag{1}$$

We denote by ξ_1 the set of double sequences of zeros and ones in $\mathscr{C}_{\mathrm{be0}}$; that is,

$$\xi_1 := \{0,1\}^{\mathbb{N}\times\mathbb{N}} \cap \mathscr{C}_{\mathrm{be0}} = \{0,1\}^{\mathbb{N}\times\mathbb{N}} \cap (\mathscr{M}_u \cap \mathscr{C}_{\mathrm{be0}}). \tag{2}$$

The space of all double sequences Ω can be identified with the space of all sequences ω using a suitable isomorphism $T : \Omega \to \omega$ (see (2) in [5] for a possible definition of T).

Theorem 1. *There exists a solid sequence space E containing φ and satisfying $\chi(E)^\beta = E^\beta = \ell^1$ which does not have the Hahn property.*

Proof. Set $E := T(\mathscr{M}_u \cap \mathscr{C}_{\mathrm{be0}})$. Then E is a solid sequence space containing φ and failing to have the Hahn property (see Example 3.7 in [5]). In view of Theorem 3.2 in [5] we have also $\chi(E)^\beta = (T(\xi_1))^\beta = \ell^1$ as well as $E^\beta = \ell^1$. □

Taking the explained situation concerning the sufficiency of $\chi(E)^\beta = E^\beta = \ell^1$ for the Hahn property of E into consideration, it is mathematically interesting to research for classes of sequence spaces E with the property that $\chi(E)^\beta = E^\beta = \ell^1$ implies the Hahn property of E (or another equivalent condition). In that sense we consider on the base of related results in [6, 7] the bounded domains of Riesz methods in Section 2 and, more general, of the generalized Riesz methods in Section 3.

2. Bounded Domains of Riesz Methods

Let $p = (p_n)$ be a real sequence with

$$p_1 > 0, \qquad p_k \geq 0 \quad (k \in \mathbb{N}), \qquad P_n := \sum_{k=1}^{n} p_k \quad (n \in \mathbb{N}). \tag{3}$$

The Riesz matrix $R_p = (r_{nk})$ (associated with p) is defined by

$$r_{nk} := \begin{cases} \dfrac{p_k}{P_n} & \text{if } k \leq n, \\ 0 & \text{otherwise} \end{cases} \quad (n, k \in \mathbb{N}). \tag{4}$$

The summability method corresponding R_p is called *Riesz method*.

The Riesz matrix R_p is conservative, and it is either regular (being equivalent to $p \notin \ell^1$) or coercive. If $p \in \ell^1$, then $\ell^\infty \cap c_{R_p}$ has the Hahn property since $\ell^\infty \cap c_{R_p} = \ell^\infty$. If $p \notin \ell^1$, then $\ell^\infty \cap c_{R_p}$ has the Hahn property if and only if $(p_n/P_n) \in c_0$ (cf. Corollary 3.9 in [6]).

In Section 1 we have seen that the relation $\chi(E)^\beta = E^\beta = \ell^1$ does not imply in general the Hahn property of E even for a solid space E. This result suggests the following problem due to Boos and Leiger (cf. Problem 3 in [7, Section 4]).

Problem 2. Does the relation $\chi(\ell^\infty \cap c_{R_p})^\beta = \ell^1$ imply the Hahn property of $\ell^\infty \cap c_{R_p}$?

Aiming to a positive result we first prove the following more general theorem.

Theorem 2. *Let $B = (b_{nk})$ be an infinite matrix such that there exists a subset F of χ with $\varphi \subset \langle F \rangle \subset \ell_B^\infty$ and $F^\beta (= \langle F \rangle^\beta) = \ell^1$. Then $\ell^\infty \subset \ell_B^\infty$ (being equivalent to $\sup_n \sum_k |b_{nk}| < \infty$).*

Proof. In view of $F \subset \ell_B^\infty$ we have $\sup_n |\sum_k b_{nk} x_k| < \infty$ for each $x \in F$, so in particular $(b_{nk})_k \in F^\beta = \ell^1$ for each $n \in \mathbb{N}$.

Assume on contrary that there exists an $x \in \ell^\infty \setminus \ell_B^\infty$. Hence $\sup_n \sum_k |b_{nk}| = \infty$ since

$$\left| \sum_k b_{nk} x_k \right| \leq \|x\|_\infty \sum_k |b_{nk}| \quad (x \in \ell^\infty, \ n \in \mathbb{N}). \tag{5}$$

Aiming to a contradiction we will construct by induction two index sequences (k_i) and (n_i), and then with the help of them we will define $w \in F^\beta \setminus \ell^1$. First of all set $k_1 := 0$ and $A_1 := 0$. Now choose $n_1 \in \mathbb{N}$ such that

$$\sum_k |b_{n_1 k}| > 1^2 (A_1 + 1) + \frac{1}{2}. \tag{6}$$

Then choose $k_2 > k_1$ such that

$$\sum_{k=k_2+1}^{\infty} |b_{n_1 k}| < \frac{1}{2}, \tag{7}$$

and set

$$B_1 := \sum_{k=k_1+1}^{k_2} |b_{n_1 k}|. \tag{8}$$

Now suppose that $k_1, \ldots, k_i$ and $n_1, \ldots, n_{i-1}$ are already chosen for $i > 1$. Since $\varphi \subset \ell_B^\infty$, then

$$A_i := \sup_n \sum_{k=1}^{k_i} |b_{nk}| < \infty. \tag{9}$$

We choose $n_i > n_{i-1}$ such that

$$\sum_k |b_{n_i k}| > i^2 (A_i + 1) + \frac{1}{2^i} \tag{10}$$

and then we take $k_{i+1} > k_i$ such that

$$\sum_{k=k_{i+1}+1}^{\infty} |b_{n_i k}| < \frac{1}{2^i}. \tag{11}$$

So

$$B_i := \sum_{k=k_i+1}^{k_{i+1}} |b_{n_i k}| > i^2 (A_i + 1), \qquad \frac{A_i}{B_i} < \frac{1}{i^2}. \tag{12}$$

Now we define a $w = (w_k)$ by

$$w_k := \frac{b_{n_i k}}{B_i i} \quad \text{for } k_i + 1 \le k \le k_{i+1},\ i \in \mathbb{N} \tag{13}$$

and note that $w \notin \ell^1$ because

$$\sum_k |w_k| = \sum_i \frac{1}{B_i i} \sum_{k=k_i+1}^{k_{i+1}} |b_{n_i k}| = \sum_i \frac{1}{i} = \infty. \tag{14}$$

We are going to verify $w \in F^\beta$ in contradiction to $F^\beta = \ell^1$. For that end let $u \in F$ and $s, t \in \mathbb{N}$ with $t > s$ be fixed. We choose $i_1, i_2 \in \mathbb{N}$ such that $k_{i_1-1} < s \le k_{i_1}$ and $k_{i_2} < t \le k_{i_2+1}$. If $k_{i_1-1} = k_{i_2}$ we have

$$\begin{aligned} \left|\sum_{k=s}^{t} w_k u_k\right| &= \left|\sum_{k=s}^{t} \frac{b_{n_{i_1-1}k} u_k}{B_{i_1-1}(i_1-1)}\right| \\ &\le \frac{1}{B_{i_1-1}(i_1-1)} \sum_{k=k_{i_1-1}+1}^{k_{i_1}} \left|b_{n_{i_1-1}k}\right| \\ &= \frac{1}{i_1-1} \longrightarrow 0, \quad \text{as } s, t \longrightarrow \infty. \end{aligned} \tag{15}$$

If $k_{i_1} \le k_{i_2}$ we get

$$\begin{aligned} \sum_{k=s}^{t} w_k u_k = &\sum_{k=s}^{k_{i_1}} \frac{b_{n_{i_1-1}k} u_k}{B_{i_1-1}(i_1-1)} + \sum_{i=i_1}^{i_2-1} \sum_{k=k_i+1}^{k_{i+1}} \frac{b_{n_i k} u_k}{B_i i} \\ &+ \sum_{k=k_{i_2}+1}^{t} \frac{b_{n_{i_2} k} u_k}{B_{i_2} i_2}. \end{aligned} \tag{16}$$

Since $u \in F \subset \ell_B^\infty$, we have $\sup_n |[Bu]_n| := M < \infty$; hence,

$$\begin{aligned} &\left|\sum_{k=k_i+1}^{k_{i+1}} \frac{b_{n_i k} u_k}{B_i i}\right| \\ &\quad \le \frac{1}{B_i i}\left(\left|[Bu]_{n_i}\right| + \sum_{k=1}^{k_i} \left|b_{n_i k}\right| + \sum_{k=k_{i+1}+1}^{\infty} \left|b_{n_i k}\right|\right) \\ &\quad \le \frac{1}{B_i i}\left(M + A_i + \frac{1}{2^i}\right) < \frac{1 + M + 2^{-i}}{i^3} \end{aligned} \tag{17}$$

for any $i = i_1, \ldots, i_2 - 1$. So for the second term on the right-hand side of (16) we have the estimate

$$\left|\sum_{i=i_1}^{i_2-1} \sum_{k=k_i+1}^{k_{i+1}} \frac{b_{n_i k} u_k}{B_i i}\right| \le \sum_{i=i_1}^{i_2-1} \frac{1 + M + 2^{-i}}{i^3}. \tag{18}$$

For the first and the third term on the right-hand side of (16) we have the estimates

$$\begin{gathered} \left|\sum_{k=s}^{k_{i_1}} \frac{b_{n_{i_1-1}k} u_k}{B_{i_1-1}(i_1-1)}\right| \le \frac{1}{B_{i_1-1}(i_1-1)} \sum_{k=k_{i_1-1}+1}^{k_{i_1}} \left|b_{n_{i_1-1}k}\right| = \frac{1}{i_1-1}, \\ \left|\sum_{k=k_{i_2}+1}^{t} \frac{b_{n_{i_2}k} u_k}{B_{i_2} i_2}\right| \le \frac{1}{i_2}. \end{gathered} \tag{19}$$

Hence

$$\begin{aligned} &\left|\sum_{k=s}^{t} w_k u_k\right| \\ &\quad \le \frac{1}{i_1-1} + \sum_{i=i_1}^{i_2-1} \frac{1 + M + 2^{-i}}{i^3} + \frac{1}{i_2} \longrightarrow 0, \quad \text{as } s, t \longrightarrow \infty; \end{aligned} \tag{20}$$

so the series $\sum_k w_k u_k$ converges and $w \in F^\beta$. □

The last theorem provides us an alternative way (cf. Theorem 2.3.8 in [3]) to show that every matrix summing all thin sequences is conservative for null sequences. Let $\mathcal{T}$ denote the set of all thin sequences. Then we have the following.

Corollary 3. *Let B be a matrix satisfying $\mathcal{T} \subset c_B$, then $c_0 \subset c_B$. Moreover, if $\mathcal{T} \cup \{e\} \subset c_B$; then B is conservative and $B \in KG$ (i.e., each matrix A with $\chi(\ell^\infty \cap c_B) \subset c_A$ is conservative).*

Proof. Taking $F = \mathcal{T}$ in Theorem 2 in view of $\mathcal{T}^\beta = \ell^1$ we have $\ell^\infty \subset \ell_B^\infty$, so $\sup_n \sum_k |b_{nk}| < \infty$. Since $e^k \in \langle \mathcal{T} \rangle \subset c_B$ it follows that $c_0 \subset c_B$.

The second part of the corollary follows immediately from the first one. □

The following two corollaries are basic for the answer to Problem 2.

Corollary 4. *Let $B = (b_{nk})$ be an infinite matrix, and let F be a subset of χ with $\varphi \subset \langle F \rangle \subset \ell_B^\infty$ and $F^\beta = \ell^1$. Let E be a subset of ℓ^∞ such that $E^\beta = \ell^1$ and $F \cap E \subset c_B$. Furthermore, let (E, τ_E) be an F-space with a dense subset $F \cap E$. Then $E \subset c_B$.*

Proof. We consider the functionals $f_n : E \to \mathbb{R}$ with $f_n(x) = \sum_k b_{nk} x_k$ $(n \in \mathbb{N})$. By Theorem 2 the sequence (f_n) is pointwise bounded on $E \subset \ell^\infty$, and by our assumptions it converges pointwise on the dense subset $F \cap E$ of E. Hence by generalized version of Banach-Steinhaus theorem (cf. Theorem 6.8.6 in [3]) the sequence (f_n) converges pointwise on E. So $E \subset c_B$. □

Corollary 5. *Let A be a conservative matrix. Then $A \in KG$ if and only if $\chi(c_A \cap \ell^\infty)^\beta = \ell^1$.*

Proof. Suppose $A \in$ KG and let $u \in \chi(c_A \cap \ell^\infty)^\beta$. We set $b_{nk} := u_k$ $(k, n \in \mathbb{N})$. Then $\chi(c_A \cap \ell^\infty) \subset c_B$; hence, B is conservative. Therefore $u \in c^\beta = \ell^1$. Hence $\chi(c_A \cap \ell^\infty)^\beta \subset \ell^1$. On the other hand in view of conservativity of A we have $\chi(c_A \cap \ell^\infty)^\beta \supset (c_A \cap \ell^\infty)^\beta = \ell^1$.

Now let B be a matrix satisfying $\chi \cap c_A \subset c_B$. Take $E = c_0$ and $F = \chi(c_A \cap \ell^\infty)$ in Corollary 4. Then since $\chi(c_A \cap \ell^\infty) \cap c_0 = \varphi$ is dense in $(c_0, \| \|_\infty)$, we have $c_0 \subset c_B$. Moreover, since $e \in \chi \cap c_A \subset c_B$, it follows that $c \subset c_B$. Hence $A \in$ KG. □

Now we give a positive answer to Problem 2.

Proposition 6. *Let R_p be regular. Then $\ell^\infty \cap c_{R_p}$ has the Hahn property if and only if $\chi(\ell^\infty \cap c_{R_p})^\beta = \ell^1$.*

Proof. In view of Corollary 5 we have $\chi(\ell^\infty \cap c_{R_p})^\beta = \ell^1$ if and only if $R_p \in$ KG, but the last condition is equivalent to the Hahn property of $\ell^\infty \cap c_{R_p}$ (cf. Corollary 3.9 in [6]). □

In the same way as for Riesz matrices we obtain a similar result for nonnegative Hausdorff matrices, where we make use that a nonnegative regular Hausdorff matrix H_p is KG if and only if $\ell^\infty \cap c_{H_p}$ has the Hahn property (Theorem 3.2.1 in [7]).

Proposition 7. *Let H_p be a nonnegative regular Hausdorff matrix. Then $\ell^\infty \cap c_{H_p}$ has the Hahn property if and only if $\chi(\ell^\infty \cap c_{H_p})^\beta = \ell^1$.*

3. Bounded Domains of Generalized Riesz Methods

In the previous section we demonstrated that for $A = R_p$ the relation $\chi(\ell^\infty \cap c_A)^\beta = \ell^1$ is equivalent to the Hahn property of $\ell^\infty \cap c_A$. Now we consider two more conditions being equivalent to these properties for $A = R_p$ (cf. [6, Corollary 3.9 (ii)], [7, Theorem 3.1.1]):

(a) $A = (a_{nk})$ has spreading rows; that is, $(\sup_{k\in\mathbb{N}} a_{nk}) \in c_0$;

(b) there exists a sequence

$$(\mathcal{N}_t) = \left(\left\{N_{\nu t} \mid \nu \in \mathbb{N}_{\nu_t}\right\}\right), \quad \nu_t \in \mathbb{N} \quad (t \in \mathbb{N}) \tag{21}$$

of partitions of $\mathbb{N}$ (called an *admissible partition sequence*) such that

$$\limsup_{t\to\infty} \limsup_{n\to\infty} \sup_{1\le\nu\le\nu_t} A_{n\nu t} = 0 \quad \text{where } A_{n\nu t} := \sum_{k\in N_{\nu t}} a_{nk}. \tag{22}$$

Recall that $\mathcal{N}_t = \{N_{\nu t} \mid \nu \in \mathbb{N}_{\nu_t}\}$ is called a partition of $\mathbb{N}$ if $\mathbb{N} = \bigcup_{\nu=1}^{\nu_t} N_{\nu t}$ and $N_{\nu t} \cap N_{\mu t} = \emptyset$ if $\nu \ne \mu$.

So in fact conditions (a) and (b) are equivalent for $A = R_p$. Note that for any nonnegative and regular for null sequences matrix A condition (b) is equivalent to the following condition (see [7] for the corresponding definitions).

(b') Density d_A defined by the matrix A is strongly nonatomic on the power set $P(\mathbb{N})$ of $\mathbb{N}$.

Now we replace a Riesz matrix R_p with a more general matrix obtained as a row submatrix of R_p. More precisely, we consider the matrix $R_{p,m} = (a_{n,k})$, called *generalized Riesz matrix*, defined by

$$a_{nk} := \begin{cases} \dfrac{p_k}{P_{m_n}} & \text{if } 1 \le k \le m_n, \\ 0 & \text{otherwise,} \end{cases} \quad (n, k \in \mathbb{N}), \tag{23}$$

where (m_n) is any fixed index sequence and $p = (p_i)$ is a sequence of positive reals with $p \notin \ell^1$.

Boos and Leiger showed (cf. Theorem 2.1 in [8]) that in the case of some sufficient conditions on terms of $R_{p,m}$ the properties (a) and (b) are equivalent for $A = R_{p,m}$. These conditions covered all possible cases except

$$\limsup_n \frac{P_{m_n}}{P_{m_{n+1}}} = 1, \qquad \liminf_n \frac{P_{m_n}}{P_{m_{n+1}}} = 0. \tag{24}$$

Note that even in this case the implication (b)⇒(a) holds (cf. Theorem 2.1 in [8]). So only the implication (a)⇒(b) in case (24) was under the question.

In relation with this in [8], Problem 2.3 Boos and Leiger posed the following.

Problem 3. Complete the distinction of cases in [8, Theorem 2.1]. May be the relation $\mathcal{T} \subset c_{0R_{p,m}}$ hold?

First we answer the first question in Problem 3 and then demonstrate that the relation $\mathcal{T} \subset c_{0R_{p,m}}$ does not hold in general for generalized Riesz matrices $R_{p,m}$ satisfying (24).

For our first aim we apply—in the same way as Boos and Leiger in [8]—the following lemma due to Kuttner and Parameswaran (cf. [9, Lemma 2]).

Lemma 8. *Let $\{z_1, z_2, \ldots, z_n\}$ for any $n \in \mathbb{N}$ be a set of nonnegative real numbers. Let $Z_n := \sum_{k=1}^n z_k$ and suppose that $B_n > 0$ with $z_k \le B_n$ $(k \in \mathbb{N}_n)$, and let $t \in \mathbb{N}$ be arbitrarily given. Then one can divide the set $\mathbb{N}_n$ into t (pairwise disjoint) subsets $N_1, N_2, \ldots, N_t$ (some of them may be empty), such that*

$$\sum_{k\in N_s} z_k \le \frac{1}{t} Z_n + B_n \quad (s = 1, 2, \ldots, t). \tag{25}$$

Proposition 9. *Suppose that $R_{p,m} = (a_{nk})$ has spreading rows; that is,*

$$\lim_n \frac{1}{P_{m_n}} \sup_{1\le k\le m_n} p_k = 0, \tag{26}$$

$$\liminf_n \frac{P_{m_n}}{P_{m_{n+1}}} = 0; \tag{27}$$

then there exists an admissible partition sequence of $\mathbb{N}$ satisfying (22).

Proof. We set $A_n := P_{m_n}$. For a given $t \in \mathbb{N}$ in view of (27) we can choose the minimal index n_1 such that $A_{n_1}/A_1 > (t+1)/t$. Afterwards (27) we choose the minimal index $n_2 > n_1$ such that $A_{n_2}/A_{n_1} > (t+1)/t$. Continuing in the same way we choose the minimal index $n_i > n_{i-1}$ such that $A_{n_i}/A_{n_{i-1}} > (t+1)/t$. In view of (26) it follows that $m_{n_{i+1}} - m_{n_i} \to \infty$. Again by (26) we can choose $\alpha_t \in \mathbb{N}$, $\alpha_t \ge n_1$ such that

$$\frac{p_k}{A_n} < \frac{1}{t^2} \quad \text{for each } k \in \mathbb{N}_{m_n},\ n \ge \alpha_t; \tag{28}$$

also we can find $\beta_t \in \mathbb{N}$, $\beta_t > \alpha_t$ such that

$$\frac{A_{\alpha_t}}{A_n} < \frac{1}{t^2} \quad \text{for } n \geq \beta_t. \tag{29}$$

By Lemma 8 for any positive integer $s \geq \alpha_t$ we can divide the set $I_s := (m_{n_s}, m_{n_{s+1}}] \cap \mathbb{N}$ into t^2 disjoint subsets (some of them may be empty), say $N_{st}(1), N_{st}(1), \ldots, N_{st}(t^2)$ such that (cf. (28))

$$\sum_{k \in N_{st}(\nu)} p_k \leq \frac{1}{t^2} \sum_{k \in I_s} p_k + \frac{1}{t^2} A_{n_{s+1}} \leq \frac{2}{t^2} A_{n_{s+1}}. \tag{30}$$

Now, setting

$$N_{\nu t} := \bigcup_s N_{st}(\nu), \qquad N_t := \{N_{\nu t} \mid \nu \in \mathbb{N}_{t^2}\}, \tag{31}$$

we get an admissible partition sequence $\mathcal{N} := (\mathcal{N}_t)_t$ of $\mathbb{N}$. We verify that this partition satisfies (22). Given $n \geq \beta_t$ we find $r \in \mathbb{N}$ such that $n_r \leq n < n_{r+1}$. By construction of (n_i) it follows that $A_{n_r} \geq tA_n/(t+1)$. Therefore

$$\begin{aligned}
\frac{1}{A_n} \sum_{k \in N_{\nu t}, k \leq m_n} p_k &\leq \frac{A_{\alpha_t}}{A_n} + \frac{1}{A_n} \sum_{s=\alpha_t}^{r-1} \sum_{k \in N_{st}(\nu)} p_k + \frac{A_n - A_{n_r}}{A_n} \\
&\leq \frac{1}{t^2} + \frac{2}{t^2} \frac{1}{A_n} \sum_{s=1}^{r-1} A_{n_{s+1}} + \frac{1}{t+1} \\
&\leq \frac{1}{t^2} + \frac{2}{t^2} \frac{A_{n_r}}{A_n} \sum_{s=0}^{\infty} \left(\frac{t}{t+1}\right)^s + \frac{1}{t+1} \\
&\leq \frac{1}{t^2} + \frac{2}{t^2} \frac{1}{1 - (t/(t+1))} + \frac{1}{t+1} \\
&= \frac{2t+3}{t^2} + \frac{1}{t+1}.
\end{aligned} \tag{32}$$

Therefore

$$\limsup_{t \to \infty} \limsup_{n \to \infty} \sup_{\nu \in \mathbb{N}_{t^2}} \frac{1}{A_n} \sum_{k \in N_{\nu t}, k \leq m_n} p_k = 0. \tag{33}$$

□

Now combining the obtained result with Theorem 2.1 in [8] we get the following.

Theorem 10. *Let $p = (p_i)$ be a sequence of positive reals with $p \notin \ell^1$. Then the following conditions are equivalent:*

(i) *$R_{p,m}$ has spreading rows; that is, $\lim_n (1/P_{m_n}) \sup_{1 \leq k \leq m_n} p_k = 0$;*

(ii) *$d_{R_{p,m}}$ is strongly nonatomic.*

The following example demonstrates that the assumption of Problem 3 that $\mathcal{T} \subset c_{0R_{p,m}}$ in the case of (24) is not true in general.

Example 11. We set $m_1 := 1$, $p_{m_1} := 1$, $m_2 := m_1 + 1$, $p_{m_2} := 2^{-m_2}$, $m_3 := m_2 + 2^2[P_{m_2}]$, $p_{m_3} := 1$, $p_{m_2+2k} := 1$ for $k = 1, \ldots, 2[P_{m_2}]-1$, and $p_k := 2^{-k}$ for $m_2 < k < m_3$, $k \notin \{m_2+2k \mid k = 1, \ldots, 2[P_{m_2}] - 1\}$.

Suppose that m_{2n-2}, m_{2n-1}, and p_k for $k \leq m_{2n-1}$ are defined. We set $m_{2n} := m_{2n-1} + 1$ and $m_{2n+1} := m_{2n} + (n+1)^2[P_{m_{2n}}]$. Then we set $p_{m_{2n}} := 2^{-m_{2n}}$, $p_{m_{2n+1}} := 1$, $p_{m_{2n}+k(n+1)} := 1$ $(k = 1, \ldots, [P_{m_{2n}}](n+1) - 1)$, and $p_k := 2^{-k}$ for $m_{2n} < k < m_{2n+1}$, $k \notin \{m_{2n} + k(n+1) \mid k = 1, \ldots, [P_{m_{2n}}](n+1) - 1\}$. Then

$$\frac{P_{m_{2n+1}}}{P_{m_{2n}}} \geq \frac{P_{m_{2n}} + (n+1)\left[P_{m_{2n}}\right]}{P_{m_{2n}}} \longrightarrow \infty, \qquad \frac{P_{m_{2n-1}}}{P_{m_{2n}}} \longrightarrow 1. \tag{34}$$

Consequently the matrix $R_{p,m}$ satisfies condition (24).

On the other hand for $x \in \mathcal{T}$ having $x_n := 1$ when $p_n = 1$ and $x_n := 0$ otherwise $(n \in \mathbb{N})$ we have

$$\begin{aligned}
[Ax]_{m_{2n+1}} &\geq \frac{1}{P_{m_{2n+1}}} \sum_{k=1}^{[P_{m_{2n}}](n+1)} p_{m_{2n}+k(n+1)} \\
&\geq \frac{(n+1)\left[P_{m_{2n}}\right]}{P_{m_{2n}} + (n+1)\left[P_{m_{2n}}\right] + \sum_i 2^{-i}} \longrightarrow 1.
\end{aligned} \tag{35}$$

Hence $x \notin c_{0R_{p,m}}$.

As well as for Riesz matrices in the case of a generalized Riesz matrix the assumption that the matrix has spreading rows implies that its bounded summability domain has the Hahn property. To prove it we will adjust the methods developed in [6, Theorem 3.8].

Let I be an at most countable set, and let $N = \{N_i \mid i \in I\}$ be a partition of $\mathbb{N}$. Let $(n_{ik})_k$ be the sequence of all elements of N_i arranged in the ascending order $(i \in I)$. We introduce the notation

$$bs(N) := \left\{ x \in \omega \mid \forall j \in I : \|x\|_{bs(N)} := \sup_j \left\| (x_k)_{k \in N_j} \right\|_{bs} < \infty \right\}. \tag{36}$$

For the proof of the main result we need two lemmas.

Lemma 12 (cf. Lemma 3.6 in [6]). *Let $n \in \mathbb{N}$ and $p_1, \ldots, p_n$ be a (finite) sequence of numbers. Then there exists a partition $M_1, \ldots, M_t$ of $\{1, \ldots, n\}$ such that $t \leq 2\sqrt{n}$ and $(p_i)_{i \in M_j}$ is monotone $(j = 1, \ldots, t)$.*

The second required lemma generalizes Lemma 3.7 in [6] concerning Riesz matrices.

Lemma 13. *Let $N = \{N_i \mid i \in I\}$ be a partition of $\mathbb{N}$. If*

$$\frac{1}{P_{m_n}} \sum_{\substack{s\in\mathbb{N} \\ N_s\cap[1,m_n]\neq\emptyset}} \left(\sum_{\substack{j\in\mathbb{N} \\ n_{s,j+1}\le m_n}} \left| p_{n_{sj}} - p_{n_{s,j+1}} \right| + \sum_{\substack{j\in\mathbb{N} \\ n_{sj}\le m_n < n_{s,j+1}}} p_{n_{sj}} \right) \longrightarrow 0 \quad (n \longrightarrow \infty), \tag{37}$$

then $bs(N_i, I) \subset c_{0R_{p,m}} \subset c_{R_{p,m}}$.

Proof. To get a proof of the statement just replace everywhere n with m_n in the proof of Lemma 3.7 [6], the case of R_p. □

In the next theorem we generalize Theorem 3.8 in [6] in the case of Riesz matrices R_p. The proof requires nontrivial refinements of the methods used in the corresponding part of the proof in [6].

Theorem 14. *Let (p_n) be a positive sequence such that $P_n \to \infty$ and $R_{p,m} = (a_{nk})$ has spreading rows. Then $c_{R_{p,m}} \cap \ell^\infty$ has the Hahn property.*

Proof. By Lemma 13 it is sufficient to verify the existence of a partition $N = \{N_i \mid i \in I\}$ of $\mathbb{N}$ satisfying the condition in the lemma. Aiming to that we set $\widetilde{k}_0 := k_0 := 1$ and suppose that $k_1, \dots, k_i, \widetilde{k}_1, \dots, \widetilde{k}_i$ are already chosen. Since $P_n \to \infty$, we can choose

$$\widetilde{k}_{i+1} := \min\left\{ k \in \mathbb{N} \mid \frac{P_{m_k}}{P_{m_{k_i}}} \ge 2 \right\}. \tag{38}$$

If $P_{m_{\widetilde{k}_{i+1}}}/P_{m_{k_i}} \le 8$ or $\widetilde{k}_{i+1} = k_i + 1$ we set $k_{i+1} := \widetilde{k}_{i+1}$. Otherwise we set $k_{i+1} := \widetilde{k}_{i+1} - 1$ and $k_{i+2} := \widetilde{k}_{i+2} := \widetilde{k}_{i+1}$. In the latter case we have $P_{m_{k_{i+1}}}/P_{m_{k_i}} < 2$ and $k_{i+2} = k_{i+1} + 1$. Hence in both cases

$$\frac{P_{m_{k_{i+2}}}}{P_{m_{k_i}}} \ge 2, \quad \frac{P_{m_{k_{i+1}}}}{P_{m_{k_i}}} \le 8 \quad \text{or } k_{i+1} = k_i + 1. \tag{39}$$

Since $R_{p,m} = (a_{nk})$ has spreading rows, then

$$\varepsilon_i := \frac{\max\left\{ p_j \mid j \le m_{k_i} \right\}}{P_{m_{k_i}}} \longrightarrow 0. \tag{40}$$

Let $i_0 \in \mathbb{N}$ be such that $\varepsilon_i < 1$ for $i \ge i_0$.

For every $i \ge i_0$ we choose the minimal integer $j_i \in \mathbb{N}$ such that $m_{k_i}^{1/j_i} < \varepsilon_i^{-1/2}$. Since $\varepsilon_i = \max\{p_k \mid k \le m_{k_i}\}/P_{m_{k_i}} \ge 1/m_{k_i}$, then $m_{k_i} \ge \varepsilon_i^{-1}$ $(i \ge i_0)$. Hence $j_i > 2$ $(i \ge i_0)$. Note that $m_{k_i}^{1/j_i} \ge \varepsilon_i^{-1/4}$ $(i \ge i_0)$. (Otherwise $m_{k_i}^{1/[(j_i+2)/2]} \le m_{k_i}^{2/j_i} < \varepsilon_i^{-1/2}$, so $j_i/2 + 1 \ge [j_i/2] + 1 \ge j_i$, which is equivalent to $j_i \le 2$ and contradicts $j_i > 2$).

Set

$$\alpha_i := \frac{P_{m_{k_{i+1}}}}{m_{k_{i+1}}}, \quad \beta_i := \max\left\{ p_k \mid k \le m_{k_{i+1}} \right\} \quad (i \in \mathbb{N}). \tag{41}$$

For every $i \in \mathbb{N}$ and $s = 2, \dots, j_{i+1} - 2$ we use the notations

$$\begin{aligned} M_{i1} &:= \left\{ m_{k_i} < k \le m_{k_{i+1}} \mid p_k \le m_{k_{i+1}}^{1/j_{i+1}} \alpha_i \right\}, \\ M_{is} &:= \left\{ m_{k_i} < k \le m_{k_{i+1}} \mid m_{k_{i+1}}^{(s-1)/j_{i+1}} \alpha_i < p_k \le m_{k_{i+1}}^{s/j_{i+1}} \alpha_i \right\}, \\ M_{i,j_{i+1}-1} &:= \left\{ m_{k_i} < k \le m_{k_{i+1}} \mid m_{k_{i+1}}^{(j_{i+1}-2)/j_{i+1}} \alpha_i < p_k \right\}. \end{aligned} \tag{42}$$

Set $\nu_{is} := |M_{is}|$ $(s = 1, \dots, j_{i+1} - 1)$. Note that

$$\nu_{is} \le \frac{P_{m_{k_{i+1}}}}{m_{k_{i+1}}^{(s-1)/j_{i+1}} \alpha_i} = \frac{m_{k_{i+1}} \alpha_i}{m_{k_{i+1}}^{(s-1)/j_{i+1}} \alpha_i} = m_{k_{i+1}}^{(j_{i+1}+1-s)/j_{i+1}}. \tag{43}$$

By Lemma 12 for every $i \in \mathbb{N}$ and $s = 1, \dots, j_{i+1} - 1$ we may find a partition $(S_{is1}, \dots, S_{isr_{is}})$ of M_{is} with $r_{is} \le 2\sqrt{\nu_{is}}$ such that $(p_\nu)_{\nu \in S_{ist}}$ is monotone $(t = 1, \dots, r_{is})$. Let $e_{isk} := |S_{isk}|$ and let $(\xi_{iskl})_{l=1}^{e_{isk}}$ be the finite sequence of all elements of S_{isk} arranged in the ascending order $(i \in \mathbb{N},\ s = 1, \dots, j_{i+1} - 1,\ k = 1, \dots, r_{is})$. Then

$$\begin{aligned} &\sum_{k=1}^{r_{is}} \left(p_{\xi_{isk1}} + \sum_{l=1}^{e_{isk}-1} \left| p_{\xi_{iskl}} - p_{\xi_{i,s,k,l+1}} \right| + p_{\xi_{iske_{isk}}} \right) \\ &= \sum_{k=1}^{r_{is}} 2 \max\left\{ p_{\xi_{isk1}}, p_{\xi_{iske_{isk}}} \right\} \le 2 r_{is} m_{k_{i+1}}^{s/j_{i+1}} \alpha_i \le 4\sqrt{\nu_{is}} m_{k_{i+1}}^{s/j_{i+1}} \alpha_i \\ &\le 4 m_{k_{i+1}}^{(j_{i+1}+1-s)/2j_{i+1}} m_{k_{i+1}}^{s/j_{i+1}} \alpha_i = 4 m_{k_{i+1}}^{(j_{i+1}+1+s)/2j_{i+1}} \alpha_i \end{aligned} \tag{44}$$

for $s = 1, \dots, j_{i+1} - 2$ and

$$\begin{aligned} &\sum_{k=1}^{r_{is}} \left(p_{\xi_{isk1}} + \sum_{l=1}^{e_{isk}-1} \left| p_{\xi_{iskl}} - p_{\xi_{i,s,k,l+1}} \right| + p_{\xi_{iske_{isk}}} \right) \\ &\quad = \sum_{k=1}^{r_{is}} 2 \max\left\{ p_{\xi_{isk1}}, p_{\xi_{iske_{isk}}} \right\} \le 4\sqrt{\nu_{is}} \beta_i \le 4 m_{k_{i+1}}^{1/j_{i+1}} \beta_i \end{aligned} \tag{45}$$

for $s = j_{i+1} - 1$ $(i \in \mathbb{N})$. Hence for every $i \ge i_0$ we get

$$\begin{aligned} &\frac{1}{P_{m_{k_{i+1}}}} \sum_{s=1}^{j_{i+1}-1} \sum_{k=1}^{r_{is}} \left(p_{\xi_{isk1}} + \sum_{l=1}^{e_{isk}-1} \left| p_{\xi_{iskl}} - p_{\xi_{i,s,k,l+1}} \right| + p_{\xi_{iske_{isk}}} \right) \\ &\quad \le \frac{1}{P_{m_{k_{i+1}}}} \left(\sum_{s=1}^{j_{i+1}-2} 4 m_{k_{i+1}}^{(j_{i+1}+1+s)/2j_{i+1}} \alpha_i + 4 m_{k_{i+1}}^{1/j_{i+1}} \beta_i \right) \\ &\quad = \frac{4 m_{k_{i+1}}^{(j_{i+1}+1)/2j_{i+1}} \alpha_i}{m_{k_{i+1}} \alpha_i} \left(\frac{m_{k_{i+1}}^{(j_{i+1}-1)/2j_{i+1}} - 1}{m_{k_{i+1}}^{1/2j_{i+1}} - 1} - 1 \right) + \frac{4\beta_i m_{k_{i+1}}^{1/j_{i+1}}}{P_{m_{k_{i+1}}}} \\ &\quad \le 4 m_{k_{i+1}}^{(-j_{i+1}+1)/2j_{i+1}} \frac{m_{k_{i+1}}^{(j_{i+1}-1)/2j_{i+1}}}{4^{-1} m_{k_{i+1}}^{1/2j_{i+1}}} + 4\varepsilon_{i+1}\varepsilon_{i+1}^{-1/2} \\ &\quad = 16 m_{k_{i+1}}^{-1/2j_{i+1}} + 4\varepsilon_{i+1}^{-1/2} \le 16\varepsilon_{i+1}^{1/8} + 4\varepsilon_{i+1}^{1/2}. \end{aligned} \tag{46}$$

To define N_i $(i \in \mathbb{N})$ we use the notation $N|_n := \{k \in N \mid k \leq n\}$ for a subset $N \subset \mathbb{N}$ and $n \in \mathbb{N}$.

Set $N_1|_{m_{k_{i_0}}} := \{1, \ldots, m_{k_{i_0}}\}$ and $N_j|_{m_{i_0}} := \emptyset$ for $j > 1$. Let $s_1^0, \ldots, s_{t_0}^0$ be all indexes such that $M_{i_0 s_t^0} \neq \emptyset$ $(t = 1, \ldots, t_0)$. We set

$$N_j\big|_{m_{k_{i_0+1}}} := \begin{cases} N_j\big|_{m_{k_{i_0}}} \cup S_{i_0 s_t^0 k} & \text{for } j = \sum_{\tau=1}^{t-1} r_{i_0 s_\tau^0} + k \\ & \left(t = 1, \ldots, t_0,\ k = 1, \ldots, r_{i_0 s_t^0}\right), \\ N_j\big|_{m_{k_{i_0}}} & \text{for } j > \sum_{\tau=1}^{t_0} r_{i_0 s_\tau^0}. \end{cases} \tag{47}$$

Continuing inductively for $i > i_0$ let $s_1^{i-i_0}, \ldots, s_{t_{i-i_0}}^{i-i_0}$ be all indexes such that $M_{i s_t^{i-i_0}} \neq \emptyset$ $(t = 1, \ldots, t_{i-i_0})$. We set

$$N_j\big|_{m_{k_{i+1}}} := \begin{cases} N_j\big|_{m_{k_i}} \cup S_{i s_t^{i-i_0} k} & \text{for } j = \sum_{\tau=1}^{t-1} r_{i s_\tau^{i-i_0}} + k \\ & \left(t = 1, \ldots, t_{i-i_0},\ k = 1, \ldots, r_{i_0 s_t^{i-i_0}}\right), \\ N_j\big|_{m_{k_{i_0}}} & \text{for } j > \sum_{\tau=1}^{t_{i-i_0}} r_{i s_\tau^{i-i_0}}. \end{cases} \tag{48}$$

Let I be the set of all indexes $i \in \mathbb{N}$ such that $N_i \neq \emptyset$. If I is infinite, then by our construction every N_i is infinite. If I is finite, then without loss of generality we may assume that N_i is infinite $(i \in I)$.

Let $n \in \mathbb{N}$ with $k_i < n \leq k_{i+1}$ and $i \geq i_0$ be fixed. Then

$$\begin{aligned} B_n &:= \frac{1}{P_{m_n}} \times \sum_{\substack{s \in \mathbb{N} \\ N_s \cap [1, m_n] \neq \emptyset}} \left(\sum_{\substack{j \in \mathbb{N} \\ n_{s,j+1} \leq m_n}} \left| p_{n_{sj}} - p_{n_{s,j+1}} \right| + \sum_{\substack{j \in \mathbb{N} \\ n_{sj} \leq m_n < n_{s,j+1}}} p_{n_{sj}} \right) \\ &\leq \frac{1}{P_{m_n}} \times \left(2 \sum_{k=1}^{m_{k_{i_0}}} p_k + \sum_{\tau=i_0}^{i} \sum_{s=1}^{j_{\tau+1}-1} \sum_{k=1}^{r_{\tau s}} \left(p_{\xi_{i s k 1}} + \sum_{l=1}^{e_{\tau s k}-1} \left| p_{\xi_{\tau s k l}} - p_{\xi_{\tau,s,k,l+1}} \right| + p_{\xi_{i s k e_{i s k}}} \right) \right) \\ &\leq \frac{1}{P_{m_n}} \left(2 P_{m_{k_{i_0}}} + \sum_{\tau=i_0}^{i} \left(16 \varepsilon_{\tau+1}^{1/8} + 4 \varepsilon_{\tau+1}^{1/2} \right) P_{m_{\tau+1}} \right). \end{aligned} \tag{49}$$

If $k_{i+1} = k_i + 1$ then $n = k_{i+1}$, so $P_n = P_{k_{i+1}}$. If $k_{i+1} > k_i + 1$ then $P_n \geq P_{k_i} \geq 8^{-1} P_{k_{i+1}}$. Hence

$$\begin{aligned} B_n &\leq \frac{8}{P_{k_{i+1}}} \left(2 P_{m_{i_0}} + \sum_{\tau=i_0}^{i} \left(16 \varepsilon_{\tau+1}^{1/8} + 4 \varepsilon_{\tau+1}^{1/2} \right) P_{m_{\tau+1}} \right) \\ &\leq \frac{16}{2^{[(i-i_0+1)/2]}} + 32 \sum_{\tau=i_0}^{i} \frac{1}{2^{[(i-\tau)/2]}} \left(4 \varepsilon_{\tau+1}^{1/8} + \varepsilon_{\tau+1}^{1/2} \right). \end{aligned} \tag{50}$$

Consider the matrix $A = (a_{i\tau})$ with $a_{i\tau} := 2^{[(\tau-i)/2]}$ for $\tau \leq i$ and $a_{i\tau} := 0$ otherwise. Evidently, A is regular for null sequences, so it sums the null sequence $(4\varepsilon_{\tau+i_0}^{1/8} + \varepsilon_{\tau+i_0}^{1/2})_\tau$ to zero. Therefore $B_n \to 0$ $(n \to \infty)$. So in view of Lemma 13 the inclusion $bs(N) \subset c_{R_{p,m}}$ holds. □

Now having in mind Proposition 2.8 in [7] we can prove the following theorem.

Theorem 15. *Let $p = (p_i)$ be a sequence of positive reals with $p \notin \ell^1$. Then one considers the following conditions:*

(a) *$\ell^\infty \cap c_{R_{p,m}}$ has the matrix Hahn property,*

(b) *$R_{p,m}$ has spreading rows; that is, $\lim_n (1/P_{m_n}) \sup_{1 \leq k \leq m_n} p_k = 0$,*

(c) *$d_{R_{p,m}}$ is strongly nonatomic,*

(d) *$R_{p,m} \in KG$,*

(e) *$\ell^\infty \cap c_{R_{p,m}}$ has the separable Hahn property,*

(f) *$\ell^\infty \cap c_{R_{p,m}}$ has the Hahn property,*

(g) *$\chi(c_{R_{p,m}} \cap \ell^\infty)^\beta = \ell^1$.*

Then the implications (c)⇔(b)⇒(f)⇒(e)⇒(a)⇒(d)⇔(g) hold.

Proof. The implication chain (f)⇒(e)⇒(a)⇒(d) is obvious. (c)⇔(b) is Theorem 10, (b)⇒(f) is Theorem 14, and (d)⇔(g) follows from Corollary 5. □

The author did not succeed to show that the condition

$$\lim_n \frac{1}{P_{m_n}} \sup_{1 \leq k \leq m_n} p_k = 0 \tag{51}$$

is necessary for $R_{p,m} \in$ KG. The following example gives a generalized Riesz matrix $R_{p,m}$ not having spreading rows and failing to have $R_{p,m} \in$ KG.

Example 16. Suppose $p \notin \ell^1$ and

$$\liminf_n \left(\frac{1}{P_{m_n}} \right) \sup_{1 \leq k \leq m_n} p_k > \frac{1}{2}. \tag{52}$$

Then there exists $\delta > 1/2$ and $n_0 \in \mathbb{N}$ such that $(1/P_{m_n}) \sup_{1 \leq k \leq m_n} p_k \geq \delta$ for $n \geq n_0$.

Let $k_n \leq m_n$ be an integer such that $p_{k_n} = \sup_{1 \leq k \leq m_n} p_k$. Then

$$\frac{1}{P_{m_n}} \sum_{\substack{k=1 \\ k \neq k_n}}^{m_n} p_k \leq 1 - \delta < \frac{1}{2}. \tag{53}$$

Since $P_n \to \infty$ we can choose an index sequence (n_i) such that (k_{n_i}) is increasing. Now if $x \in \chi \cap c_{R_{p,m}}$ then there exists $i_0 = i_0(x)$ such that $x_{k_{n_i}} = 0$ for all $i \geq i_0$ or $x_{k_{n_i}} = 1$ for all $i \geq i_0$. We set $y_{k_{n_i}} := (-1)^i/i$ for $i \in \mathbb{N}$ and $y_k := 0$ for $k \notin \{k_{n_i} \mid i \in \mathbb{N}\}$. Then the series

$$\sum_{k=k_{n_{i_0}}}^{\infty} x_k y_k = \sum_{i=i_0}^{\infty} \frac{(-1)^i}{i} x_{k_{n_i}} \tag{54}$$

converges, so $y \in \chi(c_{R_{p,m}} \cap \ell^\infty)^\beta \setminus \ell^1$.

Problem 4. In Theorem 15 we have shown that the fact that $R_{p,m}$ has spreading rows implies $R_{p,m} \in \mathrm{KG}$. Does the inverse implication hold?

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