

Research Article

On the Hermite-Hadamard Inequality and Other Integral Inequalities Involving Several Functions

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We present some new Hermite-Hadamard-type inequalities and other integral inequalities involving several functions.

1. Introduction and Preliminaries

In 2003, Pachpatte [1] gave some Hermite-Hadamard-type inequalities involving two convex functions, and then Pachpatte [2] also gave, in 2004, some Hermite-Hadamard-type inequalities involving two log-convex functions. In 2007, Kirmaci et al. [3] gave some Hadamard-type inequalities involving s -convex functions. In 2008, Bakula et al. [4] presented some Hadamard-type inequalities involving m -convex functions and (α, m) -convex functions. In 2010, Set et al. [5] gave some new Hermite-Hadamard-type inequalities and other integral inequalities involving two functions. More details about results proved in [5] will be given in Section 2. In this paper, we present more general Hermite-Hadamard-type inequalities and some integral inequalities involving several functions.

Let $f : [a, b] \rightarrow \mathbb{R}$ and $p \geq 1$. The p -norm of the function f on $[a, b]$ is defined by

$$\|f\|_p = \begin{cases} \left(\int_a^b |f(x)|^p dx \right)^{1/p}; & 1 \leq p < \infty, \\ \sup_{x \in [a,b]} |f(x)|; & p = +\infty. \end{cases} \quad (1)$$

Below we recall few well-known inequalities that will be useful in the proofs of our results.

Hermite-Hadamard's Inequality (see [6–8]). If f is a convex function on $[a, b]$, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a) + f(b)}{2}. \quad (2)$$

If f is a concave function on $[a, b]$, then

$$\frac{f(a) + f(b)}{2} \leq \frac{1}{b-a} \int_a^b f(x) dx \leq f\left(\frac{a+b}{2}\right). \quad (3)$$

Barnes-Gugunova-Levin Inequality (see [9–11]). If f and g are nonnegative concave functions on $[a, b]$, and if $p, q > 1$, then

$$\begin{aligned} & \left(\int_a^b f^p(x) dx \right)^{1/p} \left(\int_a^b g^q(x) dx \right)^{1/q} \\ & \leq B(p, q) \int_a^b f(x) g(x) dx, \end{aligned} \quad (4)$$

where

$$B(p, q) = \frac{6(b-a)^{1/p+1/q-1}}{(p+1)^{1/p}(q+1)^{1/q}}. \quad (5)$$

The Power-Mean Inequality (see [12]). Let $x_1, x_2, \dots, x_n, p_1, p_2, \dots, p_n > 0$ and let $r \in \mathbb{R} \cup \{-\infty, +\infty\}$. Then

$$M_n^{[r]} = \begin{cases} \left(\frac{\sum_{i=1}^n p_i x_i^r}{\sum_{i=1}^n p_i} \right)^{1/r}; & r \neq 0, \pm\infty, \\ \left(\prod_{i=1}^n x_i^{p_i} \right)^{1/\sum_{i=1}^n p_i}; & r = 0, \\ \max\{x_1, x_2, \dots, x_n\}; & r = \infty, \\ \min\{x_1, x_2, \dots, x_n\}; & r = -\infty. \end{cases} \quad (6)$$

Notice that if $-\infty \leq r < s \leq +\infty$ then

$$M_n^{[r]} \leq M_n^{[s]}. \quad (7)$$

A Generalization of Hölder Integral Inequality. For any $p_1, p_2, \dots, p_n > 1$, $\sum_{i=1}^n (1/p_i) = 1$, if f_1, f_2, \dots, f_n are nonnegative functions on $[a, b]$ and if $f_1^{p_1}, f_2^{p_2}, \dots, f_n^{p_n}$ are integrable functions on $[a, b]$, then

$$\int_a^b \left(\prod_{i=1}^n f_i(x) \right) dx \leq \prod_{i=1}^n \left(\int_a^b f_i^{p_i}(x) dx \right)^{1/p_i}. \quad (8)$$

A Generalization of Minkowski Integral Inequality. If $p \geq 1$ and if f_1, f_2, \dots, f_n are non-negative functions on $[a, b]$ such that $0 < \int_a^b f_i^p(x) dx < \infty$ for all $i = 1, 2, \dots, n$, then

$$\left(\int_a^b \left(\sum_{i=1}^n f_i(x) \right)^p dx \right)^{1/p} \leq \sum_{i=1}^n \left(\int_a^b f_i^p(x) dx \right)^{1/p}. \quad (9)$$

A Generalization of Young Inequality. If $x_1, x_2, \dots, x_n \geq 0$ and $p_1, p_2, \dots, p_n > 1$, $\sum_{i=1}^n (1/p_i) = 1$, then

$$\prod_{i=1}^n x_i \leq \sum_{i=1}^n \frac{x_i^{p_i}}{p_i}. \quad (10)$$

To prove results, we refer to the following lemma.

Lemma 1 (see [13]). If $p \geq 1$ and if f and g are positive functions on $[a, b]$ such that $0 < m \leq f(x)/g(x) \leq M$ for all $x \in [a, b]$, then

$$\begin{aligned} \left(\int_a^b f^p(x) dx \right)^{1/p} &\leq \frac{M}{M+1} \left(\int_a^b (f(x) + g(x))^p dx \right)^{1/p}, \\ \left(\int_a^b g^p(x) dx \right)^{1/p} &\leq \frac{1}{m+1} \left(\int_a^b (f(x) + g(x))^p dx \right)^{1/p}. \end{aligned} \quad (11)$$

Proof. Let $p \geq 1$. Assume that f and g are positive functions on $[a, b]$ such that $0 < m \leq f(x)/g(x) \leq M$ for all $x \in [a, b]$. Then

$$\begin{aligned} f &\leq Mg = M(f+g) - Mf, \\ g &\leq \frac{1}{m}f = \frac{1}{m}(f+g) - \frac{1}{m}g. \end{aligned} \quad (12)$$

Then

$$\begin{aligned} (M+1)^p f^p &\leq M^p (f+g)^p, \\ \left(\frac{1}{m} + 1 \right)^p g^p &\leq \left(\frac{1}{m} \right)^p (f+g)^p. \end{aligned} \quad (13)$$

Thus,

$$\begin{aligned} \left(\int_a^b f^p(x) dx \right)^{1/p} &\leq \frac{M}{M+1} \left(\int_a^b (f(x) + g(x))^p dx \right)^{1/p}, \\ \left(\int_a^b g^p(x) dx \right)^{1/p} &\leq \frac{1}{m+1} \left(\int_a^b (f(x) + g(x))^p dx \right)^{1/p}. \end{aligned} \quad (14) \quad \square$$

2. Main Results

We start this section with the following.

Theorem 2. Let n be a positive even integer and $p_1, p_2, \dots, p_n > 1$ and let f_1, f_2, \dots, f_n be non-negative functions on $[a, b]$ such that $f_1^{p_1}, f_2^{p_2}, \dots, f_n^{p_n}$ are concave on $[a, b]$. Then

$$\begin{aligned} \prod_{i=1}^n \left(\frac{f_i(a) + f_i(b)}{2} \right) &\leq \frac{1}{(b-a)^{\sum_{i=1}^n (1/p_i)}} \left(\prod_{i=1}^{n/2} B(p_{2i-1}, p_{2i}) \int_a^b f_{2i-1}(x) f_{2i}(x) dx \right), \end{aligned} \quad (15)$$

where

$$B(p_{2i-1}, p_{2i}) = \frac{6(b-a)^{1/p_{2i-1}+1/p_{2i}-1}}{(p_{2i-1}+1)^{1/p_{2i-1}} (p_{2i}+1)^{1/p_{2i}}} \quad (16)$$

for all $i = 1, \dots, n/2$. Moreover, if $\sum_{i=1}^n (1/p_i) = 1$, then

$$\frac{1}{b-a} \int_a^b \left(\prod_{i=1}^n f_i(x) \right) dx \leq \prod_{i=1}^n f_i \left(\frac{a+b}{2} \right). \quad (17)$$

Proof. Applying the inequality (3) with $f_i^{p_i}$, for any $i = 1, \dots, n$, we get

$$\frac{f_i^{p_i}(a) + f_i^{p_i}(b)}{2} \leq \frac{1}{b-a} \int_a^b f_i^{p_i}(x) dx \leq f_i^{p_i} \left(\frac{a+b}{2} \right), \quad (18)$$

and, consequently,

$$\begin{aligned} \left(\frac{f_i^{p_i}(a) + f_i^{p_i}(b)}{2} \right)^{1/p_i} &\leq \frac{1}{(b-a)^{1/p_i}} \left(\int_a^b f_i^{p_i}(x) dx \right)^{1/p_i} \leq f_i \left(\frac{a+b}{2} \right). \end{aligned} \quad (19)$$

By the Barnes-Gudunova-Levin inequality (4), it follows that

$$\begin{aligned}
& \prod_{i=1}^n \left(\frac{f_i^{p_i}(a) + f_i^{p_i}(b)}{2} \right)^{1/p_i} \\
& \leq \frac{1}{(b-a)^{\sum_{i=1}^n (1/p_i)}} \prod_{i=1}^n \left(\int_a^b f_i^{p_i}(x) dx \right)^{1/p_i} \\
& = \frac{1}{(b-a)^{\sum_{i=1}^n (1/p_i)}} \prod_{i=1}^{n/2} \left(\int_a^b f_{2i-1}^{p_{2i-1}}(x) dx \right)^{1/p_{2i-1}} \\
& \quad \times \left(\int_a^b f_{2i}^{p_{2i}}(x) dx \right)^{1/p_{2i}} \\
& \leq \frac{1}{(b-a)^{\sum_{i=1}^n (1/p_i)}} \left(\prod_{i=1}^{n/2} B(p_{2i-1}, p_{2i}) \int_a^b f_{2i-1}(x) f_{2i}(x) dx \right), \tag{20}
\end{aligned}$$

where

$$B(p_{2i-1}, p_{2i}) = \frac{6(b-a)^{1/p_{2i-1}+1/p_{2i}-1}}{(p_{2i-1}+1)^{1/p_{2i-1}} (p_{2i}+1)^{1/p_{2i}}} \tag{21}$$

for all $i = 1, \dots, n/2$.

By the power-mean inequality (7), we have

$$\left(\frac{f_i^{p_i}(a) + f_i^{p_i}(b)}{2} \right)^{1/p_i} \geq \frac{f_i(a) + f_i(b)}{2} \tag{22}$$

for all $i = 1, 2, \dots, n$.

This implies the inequality (15).

Next, we assume that $\sum_{i=1}^n (1/p_i) = 1$. By the inequality (19) and the generalized Hölder inequality, we obtain that

$$\begin{aligned}
\prod_{i=1}^n f_i \left(\frac{a+b}{2} \right) & \geq \frac{1}{(b-a)^{\sum_{i=1}^n (1/p_i)}} \prod_{i=1}^n \left(\int_a^b f_i^{p_i}(x) dx \right)^{1/p_i} \\
& = \frac{1}{b-a} \prod_{i=1}^n \left(\int_a^b f_i^{p_i}(x) dx \right)^{1/p_i} \\
& \geq \frac{1}{b-a} \int_a^b \left(\prod_{i=1}^n f_i(x) \right) dx. \tag{23}
\end{aligned}$$

This proof is completed. \square

It is easy to notice that if we put $n = 2$ in Theorem 2 then we get the following.

Corollary 3 (see [5]). *Let $p, q > 1$ and let f, g be non-negative functions on $[a, b]$ such that f^p, g^q are concave on $[a, b]$. Then*

$$\begin{aligned}
& \left(\frac{f(a) + f(b)}{2} \right) \left(\frac{g(a) + g(b)}{2} \right) \\
& \leq \frac{1}{(b-a)^{1/p+1/q}} \left(B(p, q) \int_a^b f(x) g(x) dx \right), \tag{24}
\end{aligned}$$

where

$$B(p, q) = \frac{6(b-a)^{1/p+1/q-1}}{(p+1)^{1/p} (q+1)^{1/q}}. \tag{25}$$

Moreover, if $1/p + 1/q = 1$, then

$$\frac{1}{b-a} \int_a^b f(x) g(x) dx \leq f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right). \tag{26}$$

Theorem 4. *Let $p \geq 1$ and n be a positive integer such that $n \geq 2$ and let f_1, f_2, \dots, f_n be positive functions on $[a, b]$ such that the functions $f_1^p, f_2^p, \dots, f_n^p$ are integrable functions on $[a, b]$, $0 < \int_a^b f_i^p(x) dx < \infty$ for all $i = 1, 2, \dots, n$, and*

$$0 < m_i \leq \frac{f_i(x)}{f_{i+1}(x)} \leq M_i \tag{27}$$

for all $x \in [a, b]$ and for all $i = 1, \dots, n-1$. Then

$$\frac{\left(\sum_{i=1}^n \|f_i\|_p \right)^n}{\prod_{i=1}^n \|f_i\|_p} \geq \frac{1}{\prod_{i=1}^n s_i}, \tag{28}$$

where

$$\begin{aligned}
s_1 & = \frac{M_1}{M_1 + 1}, & s_n & = \frac{1}{m_{n-1} + 1}, \\
s_i & = \min \left\{ \frac{1}{m_{i-1} + 1}, \frac{M_i}{M_i + 1} \right\} \tag{29}
\end{aligned}$$

for all $1 < i < n$.

Proof. By Lemma 1, we have

$$\begin{aligned}
\left(\int_a^b f_i^p(x) dx \right)^{1/p} & \leq \frac{M_i}{M_i + 1} \left(\int_a^b (f_i(x) + f_{i+1}(x))^p dx \right)^{1/p}, \\
\left(\int_a^b f_{i+1}^p(x) dx \right)^{1/p} & \leq \frac{1}{m_i + 1} \left(\int_a^b (f_i(x) + f_{i+1}(x))^p dx \right)^{1/p} \tag{30}
\end{aligned}$$

for all $i = 1, \dots, n-1$.

Then

$$\begin{aligned}
\left(\int_a^b f_i^p(x) dx \right)^{1/p} & \leq \frac{M_i}{M_i + 1} \left(\int_a^b \left(\sum_{j=1}^n f_j(x) \right)^p dx \right)^{1/p}, \\
\left(\int_a^b f_{i+1}^p(x) dx \right)^{1/p} & \leq \frac{1}{m_i + 1} \left(\int_a^b \left(\sum_{j=1}^n f_j(x) \right)^p dx \right)^{1/p} \tag{31}
\end{aligned}$$

for all $i = 1, \dots, n-1$.

Let $s_1 = M_1/(M_1 + 1)$, $s_n = 1/(m_{n-1} + 1)$, and $s_i = \min\{1/(m_{i-1} + 1), M_i/(M_i + 1)\}$ for all $1 < i < n$.

It follows that

$$\begin{aligned} \left(\int_a^b f_1^p(x) dx \right)^{1/p} &\leq s_1 \left(\int_a^b \left(\sum_{j=1}^n f_j(x) \right)^p dx \right)^{1/p}, \\ \left(\int_a^b f_n^p(x) dx \right)^{1/p} &\leq s_n \left(\int_a^b \left(\sum_{j=1}^n f_j(x) \right)^p dx \right)^{1/p}, \\ \left(\int_a^b f_i^p(x) dx \right)^{1/p} &\leq s_i \left(\int_a^b \left(\sum_{j=1}^n f_j(x) \right)^p dx \right)^{1/p} \end{aligned} \quad (32)$$

for all $1 < i < n$.

By multiplying the above inequalities and the generalized Minkowski inequality, we obtain that

$$\begin{aligned} \prod_{i=1}^n \left(\int_a^b f_i^p(x) dx \right)^{1/p} &\leq \left(\prod_{i=1}^n s_i \right) \left(\left(\int_a^b \left(\sum_{j=1}^n f_j(x) \right)^p dx \right)^{1/p} \right)^n \\ &\leq \left(\prod_{i=1}^n s_i \right) \left(\sum_{j=1}^n \left(\int_a^b f_j^p(x) dx \right)^{1/p} \right)^n. \end{aligned} \quad (33)$$

Then

$$\prod_{i=1}^n \|f_i\|_p \leq \left(\prod_{i=1}^n s_i \right) \left(\sum_{j=1}^n \|f_j\|_p \right)^n. \quad (34)$$

This implies the inequality (28). \square

Notice that from above theorem one can easily get the following.

Corollary 5 (see [5]). *Let $p \geq 1$ and let f, g be positive functions on $[a, b]$ such that $0 < \int_a^b f^p(x) dx < \infty$, $0 < \int_a^b g^p(x) dx < \infty$, and*

$$0 < m \leq \frac{f(x)}{g(x)} \leq M \quad (35)$$

for all $x \in [a, b]$. Then

$$\frac{\|f\|_p^2 + \|g\|_p^2}{\|f\|_p \|g\|_p} \geq \frac{1}{s} - 2, \quad (36)$$

where $s = M/(M+1)(m+1)$.

Proof. By Theorem 4 where $n = 2$, we have

$$\frac{(\|f\|_p + \|g\|_p)^2}{\|f\|_p \|g\|_p} \geq \frac{1}{s_1 s_2}, \quad (37)$$

where

$$s_1 = \frac{M}{M+1}, \quad s_2 = \frac{1}{m+1}. \quad (38)$$

Let $s = s_1 s_2$. Then

$$\begin{aligned} \frac{1}{s} &\leq \frac{(\|f\|_p + \|g\|_p)^2}{\|f\|_p \|g\|_p} \\ &= \frac{\|f\|_p^2 + 2\|f\|_p \|g\|_p + \|g\|_p^2}{\|f\|_p \|g\|_p} \\ &= \frac{\|f\|_p^2 + \|g\|_p^2}{\|f\|_p \|g\|_p} + 2. \end{aligned} \quad (39)$$

This implies the inequality (36). \square

Theorem 6. *Let n be a positive integer such that $n \geq 2$ and $p_1, p_2, \dots, p_n > 1$ and let f_1, f_2, \dots, f_n be non-negative functions on $[a, b]$ such that $f_1^{p_1}, f_2^{p_2}, \dots, f_n^{p_n}$ are concave on $[a, b]$. Then*

$$\prod_{i=1}^n (f_i(a) + f_i(b))^{p_i} \leq \frac{2^{\sum_{i=1}^n p_i}}{(b-a)^n} \prod_{i=1}^n \|f_i\|_{p_i}^{p_i}. \quad (40)$$

Proof. Using the inequality (3) with $f_i^{p_i}$, for any $i = 1, \dots, n$, we obtain

$$\frac{f_i^{p_i}(a) + f_i^{p_i}(b)}{2} \leq \frac{1}{b-a} \int_a^b f_i^{p_i}(x) dx. \quad (41)$$

Then

$$\prod_{i=1}^n \left(\frac{f_i^{p_i}(a) + f_i^{p_i}(b)}{2} \right) \leq \frac{1}{(b-a)^n} \prod_{i=1}^n \left(\int_a^b f_i^{p_i}(x) dx \right). \quad (42)$$

By the power-mean inequality (7), we have

$$\left(\frac{f_i^{p_i}(a) + f_i^{p_i}(b)}{2} \right)^{1/p_i} \geq \frac{f_i(a) + f_i(b)}{2}, \quad (43)$$

so

$$\frac{f_i^{p_i}(a) + f_i^{p_i}(b)}{2} \geq \frac{(f_i(a) + f_i(b))^{p_i}}{2^{p_i}}, \quad (44)$$

for all $i = 1, 2, \dots, n$.

Then

$$\begin{aligned} \frac{1}{(b-a)^n} \prod_{i=1}^n \left(\int_a^b f_i^{p_i}(x) dx \right) &\geq \prod_{i=1}^n \left(\frac{(f_i(a) + f_i(b))^{p_i}}{2^{p_i}} \right) \\ &= \frac{\prod_{i=1}^n (f_i(a) + f_i(b))^{p_i}}{2^{\sum_{i=1}^n p_i}}. \end{aligned} \quad (45)$$

This implies the inequality (40). \square

One can easily check that if we put $n = 2$ in Theorem 6 then we get the following.

Corollary 7 (see [5]). *Let $p, q > 1$ and let f, g be non-negative functions on $[a, b]$ such that f^p, g^q are concave on $[a, b]$. Then*

$$\frac{(f(a) + f(b))^p (g(a) + g(b))^q}{2^{p+q}} \leq \frac{1}{(b-a)^2} \|f\|_p^p \|g\|_q^q. \quad (46)$$

Theorem 8. *Let n be a positive integer such that $n \geq 2$ and $p_1, p_2, \dots, p_n > 1$, $\prod_{i=1}^n (1/p_i) = 1$, and let f_1, f_2, \dots, f_n be positive functions on $[a, b]$ such that the function $f_i^{p_i}$ is integrable on $[a, b]$, $0 < \int_a^b f_i^{p_i}(x) dx < \infty$ for all $i = 1, 2, \dots, n$, and*

$$0 < m_i \leq \frac{f_i(x)}{f_{i+1}(x)} \leq M_i \quad (47)$$

for all $x \in [a, b]$ and $i = 1, \dots, n-1$. Then

$$\int_a^b \left(\prod_{i=1}^n f_i(x) \right) dx \leq \sum_{i=1}^n \left(\frac{s_i}{p_i} \left(\sum_{j=1}^n \|f_j\|_{p_i}^{p_i} \right) \right), \quad (48)$$

where

$$\begin{aligned} s_1 &= 2^{p_1-1} \left(\frac{M_1}{M_1+1} \right)^{p_1}, & s_n &= 2^{p_n-1} \left(\frac{1}{m_{n-1}+1} \right)^{p_n}, \\ s_i &= \min \left\{ 2^{p_i-1} \left(\frac{1}{m_{i-1}+1} \right)^{p_i}, 2^{p_i-1} \left(\frac{M_i}{M_i+1} \right)^{p_i} \right\} \end{aligned} \quad (49)$$

for all $1 < i < n$.

Proof. By Lemma 1, we have

$$\begin{aligned} &\left(\int_a^b f_i^{p_i}(x) dx \right)^{1/p_i} \\ &\leq \frac{M_i}{M_i+1} \left(\int_a^b (f_i(x) + f_{i+1}(x))^{p_i} dx \right)^{1/p_i}, \\ &\left(\int_a^b f_{i+1}^{p_{i+1}}(x) dx \right)^{1/p_{i+1}} \\ &\leq \frac{1}{m_i+1} \left(\int_a^b (f_i(x) + f_{i+1}(x))^{p_{i+1}} dx \right)^{1/p_{i+1}} \end{aligned} \quad (50)$$

for all $i = 1, \dots, n-1$.

Using the elementary inequality $(\alpha + \beta)^p \leq 2^{p-1}(\alpha^p + \beta^p)$ where $p > 1$ and $\alpha, \beta > 0$, we get

$$\begin{aligned} \int_a^b f_i^{p_i}(x) dx &\leq \left(\frac{M_i}{M_i+1} \right)^{p_i} \int_a^b (f_i(x) + f_{i+1}(x))^{p_i} dx \\ &\leq \left(\frac{M_i}{M_i+1} \right)^{p_i} \int_a^b 2^{p_i-1} (f_i^{p_i}(x) + f_{i+1}^{p_i}(x)) dx, \\ \int_a^b f_{i+1}^{p_{i+1}}(x) dx &\leq \left(\frac{1}{m_i+1} \right)^{p_{i+1}} \int_a^b (f_i(x) + f_{i+1}(x))^{p_{i+1}} dx \\ &\leq \left(\frac{1}{m_i+1} \right)^{p_{i+1}} \int_a^b 2^{p_{i+1}-1} (f_i^{p_{i+1}}(x) + f_{i+1}^{p_{i+1}}(x)) dx \end{aligned} \quad (51)$$

for all $i = 1, \dots, n-1$.

Then

$$\begin{aligned} \int_a^b f_i^{p_i}(x) dx &\leq 2^{p_i-1} \left(\frac{M_i}{M_i+1} \right)^{p_i} \int_a^b \left(\sum_{j=1}^n f_j^{p_i}(x) \right) dx \\ &= 2^{p_i-1} \left(\frac{M_i}{M_i+1} \right)^{p_i} \sum_{j=1}^n \left(\int_a^b f_j^{p_i}(x) dx \right), \\ \int_a^b f_{i+1}^{p_{i+1}}(x) dx &\leq 2^{p_{i+1}-1} \left(\frac{1}{m_i+1} \right)^{p_{i+1}} \int_a^b \left(\sum_{j=1}^n f_j^{p_{i+1}}(x) \right) dx \\ &= 2^{p_{i+1}-1} \left(\frac{1}{m_i+1} \right)^{p_{i+1}} \sum_{j=1}^n \left(\int_a^b f_j^{p_{i+1}}(x) dx \right) \end{aligned} \quad (52)$$

for all $i = 1, \dots, n-1$.

Let $s_1 = 2^{p_1-1} (M_1/(M_1+1))^{p_1}$, $s_n = 2^{p_n-1} (1/(m_{n-1}+1))^{p_n}$, and

$$s_i = \min \left\{ 2^{p_i-1} \left(\frac{1}{m_{i-1}+1} \right)^{p_i}, 2^{p_i-1} \left(\frac{M_i}{M_i+1} \right)^{p_i} \right\} \quad (53)$$

for all $1 < i < n$.

It follows that

$$\begin{aligned} \int_a^b f_1^{p_1}(x) dx &\leq s_1 \sum_{j=1}^n \left(\int_a^b f_j^{p_1}(x) dx \right), \\ \int_a^b f_n^{p_n}(x) dx &\leq s_n \sum_{j=1}^n \left(\int_a^b f_j^{p_n}(x) dx \right), \\ \int_a^b f_i^{p_i}(x) dx &\leq s_i \sum_{j=1}^n \left(\int_a^b f_j^{p_i}(x) dx \right) \end{aligned} \quad (54)$$

for all $1 < i < n$.

By the generalized Young inequality, we obtain that

$$\begin{aligned}
 \int_a^b \left(\prod_{i=1}^n f_i(x) \right) dx &\leq \int_a^b \left(\sum_{i=1}^n \frac{1}{p_i} f_i^{p_i}(x) \right) dx \\
 &= \sum_{i=1}^n \frac{1}{p_i} \left(\int_a^b f_i^{p_i}(x) dx \right) \\
 &\leq \sum_{i=1}^n \frac{1}{p_i} \left(s_i \sum_{j=1}^n \left(\int_a^b f_j^{p_i}(x) dx \right) \right) \\
 &\leq \sum_{i=1}^n \left(\frac{s_i}{p_i} \left(\sum_{j=1}^n \|f_j\|_{p_i}^{p_i} \right) \right).
 \end{aligned} \tag{55}$$

This proof is completed. \square

Applying Theorem 8 with $n = 2$ and putting there $p_1 = p$, $p_2 = q$, $2s_1/p_1 = c_1$, and $2s_2/p_2 = c_2$, we get the following.

Corollary 9 (see [5]). *Let $p, q > 1$, $1/p + 1/q = 1$, and let f and g be positive functions on $[a, b]$ such that $0 < \int_a^b f^p(x) dx < \infty$, $0 < \int_a^b g^q(x) dx < \infty$, and*

$$0 < m \leq \frac{f(x)}{g(x)} \leq M \tag{56}$$

for all $x \in [a, b]$. Then

$$\int_a^b f(x) g(x) dx \leq c_1 \left(\frac{\|f\|_p^p + \|g\|_p^p}{2} \right) + c_2 \left(\frac{\|f\|_q^q + \|g\|_q^q}{2} \right), \tag{57}$$

where

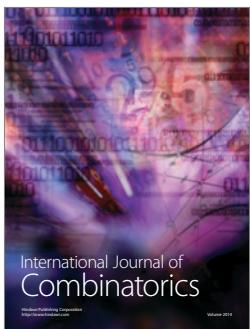
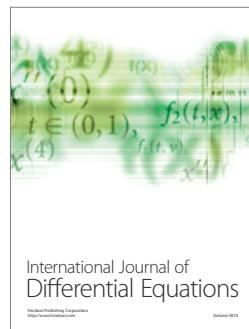
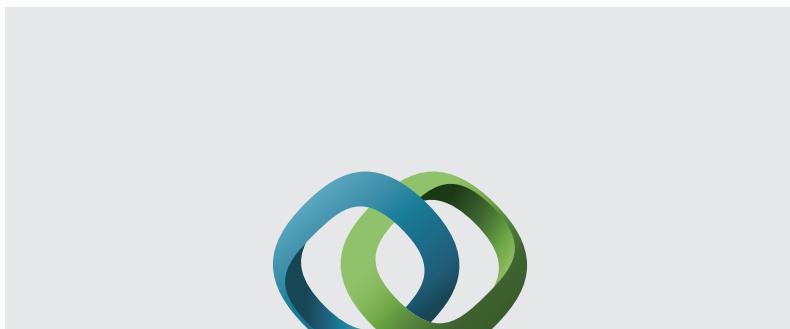
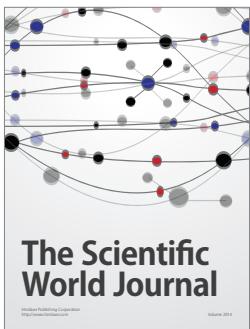
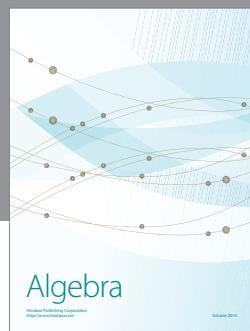
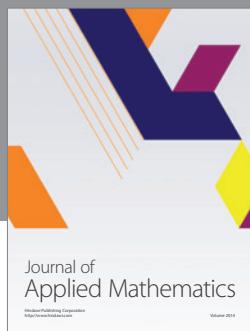
$$c_1 = \frac{2^p}{p} \left(\frac{M}{M+1} \right)^p, \quad c_2 = \frac{2^q}{q} \left(\frac{1}{m+1} \right)^q. \tag{58}$$

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