

Research Article

Higher Order Sobolev-Type Spaces on the Real Line

Bogdan Bojarski,¹ Juha Kinnunen,² and Thomas Zürcher^{3,4}

¹ Institute of Mathematics, Polish Academy of Sciences, 00-956 Warsaw, Poland

² Department of Mathematics, School of Science and Technology, Aalto University, P.O. Box 11100, 00076 Aalto, Finland

³ Department of Mathematics and Statistics, University of Jyväskylä, P.O. Box 35 (MaD), 40014 Jyväskylä, Finland

⁴ Mathematics Institute, University of Warwick, Coventry CV4 7AL, UK

Correspondence should be addressed to Juha Kinnunen; juha.kinnunen@aalto.fi

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This paper gives a characterization of Sobolev functions on the real line by means of pointwise inequalities involving finite differences. This is also shown to apply to more general Orlicz-Sobolev, Lorentz-Sobolev, and Lorentz-Karamata-Sobolev spaces.

1. Introduction

The general opinion in the literature on Sobolev spaces $W^{m,p}(\mathbb{R}^n)$ and their generalizations is that the one-dimensional case $n = 1$ may not be interesting. In the first order case $m = 1$, the theory is essentially contained in the fundamental theorem of calculus

$$f(x) - f(y) = \int_x^y f'(t) dt, \quad x, y \in \mathbb{R}, \quad (1)$$

and the higher order case $m > 1$ reduces recursively to the first order case according to the following definition.

Definition 1 (Sobolev spaces). Assume that $\Omega \subset \mathbb{R}$ is an open interval with finite Lebesgue measure in \mathbb{R} and let X be a vector space of functions. One says that $f \in W_X^m$ if there exists $F \in X$ that agrees with f almost everywhere and has weak derivatives of order m in X and the derivatives of order r , $0 \leq r \leq m - 1$, are absolutely continuous.

The main goal in this paper is to point out that there are interesting phenomena related to the Sobolev spaces already in the one-dimensional case. More precisely, we give a direct and relatively elementary proof of the following result.

Theorem 2. Suppose that $\Omega \subset \mathbb{R}$ is an open interval with finite measure. A real valued (continuous) function $f : \Omega \rightarrow \mathbb{R}$ is in the Sobolev space $W^{m,p}(\Omega) := W_{L^p}^m(\Omega)$ ($m \geq 1$, $p > 1$) if and

only if for each x and $y \in \mathbb{R}$ the m th difference of the function f at the point x with step $h = (y - x)/m$

$$\Delta^m f(x; y) := \Delta_h^m f(x) := \sum_{j=0}^m (-1)^{m-j} \binom{m}{j} f(x + jh) \quad (2)$$

satisfies the inequality

$$|\Delta^m f(x; y)| \leq |x - y|^m [a_f(x) + a_f(y)] \quad (3)$$

for some function $a_f \in L^p(\Omega)$.

Note that this result is false for domains with infinite measure as nonzero constant functions satisfy (3) with $a_f = 0$ but are not in $L^p(\Omega)$ (we leave it to the reader to formulate an appropriate version of Theorem 2 for homogeneous Sobolev spaces, which avoids this deficiency). We will also consider this result in the more general framework of Banach function spaces. However, the material is presented in such a manner that it is accessible to readers who are only interested in the classical case. Inequality (3) in Theorem 2 is one of the examples of pointwise inequalities characterising Sobolev functions, proved and propagated by Bojarski and his students in a series of papers over the last almost 30 years; see, for example, [1–6]. The case $m = 1$ lies at the initial concepts and cornerstones of analysis on general measure metric spaces [7].

Let us give an overview of the paper. The proof of Theorem 2 is presented in a series of results. We first give a detailed proof of the fact that Sobolev functions satisfy the pointwise inequality (3). The idea is to consider the Taylor expansion of the function. Using the linearity of the finite differences, we only need to compute and bound the finite differences of polynomials $(x - y)^{m-1}$ and of terms like $(x - y)_+^{m-1}$. The so-called *Hermite-Genocchi formula* is an integral representation of the finite differences (actually divided differences), which makes the computation rather easy. It turns out that the functions a_f are multiples of the Hardy-Littlewood maximal function of the corresponding derivatives. Reference [8] studies the possibility to find a_f whose definition does not involve derivatives at all. The other direction is easily shown to hold for smooth functions and the general result follows from this. Thereafter, we recall the theorems from literature necessary to be able to deduce that our main results also apply to Orlicz, Lorentz, and Lorentz-Karamata spaces.

2. Sobolev Functions Satisfy Pointwise Inequality

We refer the reader to Chapter 7 in [9] for the definition and properties of absolutely continuous functions. We write $f \in L_1^m[a, b]$ to express that the function $f : [a, b] \rightarrow \mathbb{R}$ is $(m-1)$ -times differentiable and that the derivative of order $m-1$ is absolutely continuous. The aim of this section is to show that membership in a Sobolev space implies (3), where the finite difference is replaced by *divided differences*.

Definition 3 (divided difference). Let y_1, \dots, y_{r+1} be points in a domain $\Omega \subset \mathbb{R}$. One assumes that if y_i has multiplicity n , then $f : \Omega \rightarrow \mathbb{R}$ is $(n-1)$ -times differentiable in y_i . First one considers the case where $y_1 \leq y_2 \leq \dots \leq y_{r+1}$ are ordered. One defines for $r \geq 1$

$$[y_1] f := f(y_1),$$

$$[y_1, \dots, y_{r+1}] f := \begin{cases} \frac{[y_2, \dots, y_{r+1}] f - [y_1, \dots, y_r] f}{y_{r+1} - y_1}, & \text{if } y_1 \neq y_{r+1}, \\ \frac{f^{(r)}(y_1)}{r!} & \text{otherwise.} \end{cases} \quad (4)$$

We extend the definition to unordered tuples in the only possible way that makes the divided difference independent of the ordering.

Remark 4. In the recursion formula, we omitted y_1 in the nominator of the minuend and y_{r+1} in the nominator of the subtrahend and then divided by $y_{r+1} - y_r$. We claim that we obtain the same result if we replace y_1 by y_k and y_{r+1} by y_m as long as $y_k \neq y_m$. This can be seen by induction. We look here only at one of the two more difficult cases when one of the omitted points is either the largest or the smallest (and there

are at least three points). The omitted points are denoted by a hat. We also omit the function. Consider

$$\begin{aligned} & \frac{[\hat{y}_1] - [\hat{y}_m]}{y_m - y_1} \\ &= \left(\frac{[\hat{y}_1, \hat{y}_m] - [\hat{y}_1, \hat{y}_{r+1}]}{y_{r+1} - y_m} - \frac{[\hat{y}_1, \hat{y}_m] - [\hat{y}_m, \hat{y}_{r+1}]}{y_{r+1} - y_1} \right) \\ & \quad \times (y_m - y_1)^{-1} \\ &= ((y_{r+1} - y_1) [\hat{y}_1, \hat{y}_m] - (y_{r+1} - y_1) [\hat{y}_1, \hat{y}_{r+1}] \\ & \quad - (y_{r+1} - y_m) [\hat{y}_1, \hat{y}_m] + (y_{r+1} - y_m) [\hat{y}_m, \hat{y}_{r+1}]) \\ & \quad \times ((y_{r+1} - y_m)(y_m - y_1)(y_{r+1} - y_1))^{-1}. \end{aligned} \quad (5)$$

We collect the first and third term and expand the second one:

$$\begin{aligned} & \frac{[\hat{y}_1] - [\hat{y}_m]}{y_m - y_1} \\ &= ((y_m - y_1) [\hat{y}_1, \hat{y}_m] - (y_m - y_1) [\hat{y}_1, \hat{y}_{r+1}] \\ & \quad - (y_{r+1} - y_m) [\hat{y}_1, \hat{y}_{r+1}] + (y_{r+1} - y_m) [\hat{y}_m, \hat{y}_{r+1}]) \\ & \quad \times ((y_{r+1} - y_m)(y_m - y_1)(y_{r+1} - y_1))^{-1}. \end{aligned} \quad (6)$$

Collecting the first two terms and the second two, cancelling, and applying the induction hypothesis give

$$\frac{[\hat{y}_1] - [\hat{y}_m]}{y_m - y_1} = \frac{[\hat{y}_1]}{y_{r+1} - y_1} - \frac{[\hat{y}_{r+1}]}{y_{r+1} - y_1} = [y_1, \dots, y_{r+1}]. \quad (7)$$

The following formula is a useful tool to compute divided differences.

Lemma 5 (Hermite-Genocchi formula). Let y_1, \dots, y_{r+1} be points in a domain $\Omega \subset \mathbb{R}$. One supposes that $f : \Omega \rightarrow \mathbb{R}$ is such that $f^{(n-1)}$ is absolutely continuous if not all y_i coincide and that additionally $f^{(n)}(y_1)$ exists if all points are the same. Then

$$\begin{aligned} & [y_0, \dots, y_n] f \\ &= \int_0^1 dt_1 \int_0^{t_1} dt_2 \\ & \quad \dots \int_0^{t_{n-1}} f^{(n)}(y_0 + (y_1 - y_0)t_1 \\ & \quad \quad \quad + \dots + (y_n - y_{n-1})t_n) dt_n. \end{aligned} \quad (8)$$

Proof (see page 122 in [10] as well). If $n = 1$, we have that

$$\begin{aligned} & \int_0^1 f'(y_0 + (y_1 - y_0)t_1) dt_1 \\ &= \begin{cases} \frac{f(y_1) - f(y_0)}{y_1 - y_0}, & y_0 \neq y_1, \\ f'(y_0), & y_0 = y_1, \end{cases} \\ &= \begin{cases} [y_0, y_1] f = [y_1, y_0] f, & y_0 \neq y_1, \\ [y_0, y_0] f, & y_0 = y_1. \end{cases} \end{aligned} \quad (9)$$

Assume now that the claim is true for some n . We assume here that $y_{n+1} \neq y_n$ and leave the case when y_n and y_{n+1} coincide to the reader (computation (14) might be helpful). Then, evaluating the right most integral,

$$\begin{aligned}
 & \int_0^1 dt_1 \int_0^{t_1} dt_2 \\
 & \quad \cdots \int_0^{t_n} f^{(n+1)} \left(y_0 + \sum_{k=1}^n (y_k - y_{k-1}) t_k \right. \\
 & \quad \quad \left. + (y_{n+1} - y_n) t_{n+1} \right) dt_{n+1} \\
 &= \int_0^1 dt_1 \int_0^{t_1} dt_2 \\
 & \quad \cdots \int_0^{t_{n-1}} \left(f^{(n)} \left(y_0 + \sum_{k=1}^{n-1} (y_k - y_{k-1}) t_k \right. \right. \\
 & \quad \quad \left. \left. + (y_{n+1} - y_{n-1}) t_n \right) \right. \\
 & \quad \quad \left. - f^{(n)} \left(y_0 + \sum_{k=1}^n (y_k - y_{k-1}) t_k \right) \right) dt_{n+1} \\
 & \quad \cdot \frac{1}{y_{n+1} - y_n} \\
 &= \frac{1}{y_{n+1} - y_n} ([y_0, \dots, y_{n-1}, y_{n+1}] f - [y_0, \dots, y_n] f) \\
 &= [y_0, \dots, y_{n+1}] f.
 \end{aligned} \tag{10}$$

□

We use the Hermite-Genocchi formula to compute finite differences of polynomials.

Lemma 6. Let a be a point in \mathbb{R} and $m \in \mathbb{N}_0$. We set

$$p_{m,a}(y) := (y - a)^m. \tag{11}$$

Then

$$[y_0, \dots, y_n] p_{m,a} = \begin{cases} 0, & 0 \leq m < n, \\ 1, & m = n. \end{cases} \tag{12}$$

Proof. As $p_{m,a}$ is smooth, we can use the Hermite-Genocchi formula, Lemma 5. If $0 \leq m < n$, then $p_{m,a}^{(n)} = 0$; hence the Hermite-Genocchi formula gives the claim. If $m = n$, then $p_{m,a}^{(m)} = m!$; we compute the integral in the Hermite-Genocchi formula via induction over m .

We let first for fixed $t_0 \geq 0$

$$\langle y_0, \dots, y_m \rangle_{t_0} := \int_0^{t_0} dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{m-1}} 1 dt_m. \tag{13}$$

We claim that

$$\langle y_0, \dots, y_m \rangle_{t_0} = \frac{t_0^m}{m!}, \quad m \in \mathbb{N}. \tag{14}$$

For $m = 1$, this can easily be seen. Assume that the claim holds for some m . Then

$$\begin{aligned}
 \langle y_0, \dots, y_{m+1} \rangle_{t_0} &= \int_0^{t_0} dt_1 \langle y_0, \dots, y_m \rangle_{t_1} \\
 &= \int_0^{t_0} \frac{t_1^m}{m!} dt_1 = \frac{t_0^{m+1}}{(m+1)!}.
 \end{aligned} \tag{15}$$

The Hermite-Genocchi formula tells us $[y_0, \dots, y_m]_{p_{m,a}} = \langle y_0, \dots, y_m \rangle_1 \cdot m!$, giving the proof. □

The following notation is on page 15 in [11].

Definition 7. Set

$$(x - y)_+^0 = \begin{cases} 1, & x \geq y, \\ 0, & x < y, \end{cases} \tag{16}$$

and for $m > 1$

$$(x - y)_+^{m-1} = \begin{cases} (x - y)^{m-1}, & x \geq y, \\ 0, & x < y. \end{cases} \tag{17}$$

The definition below is a variant of Definition 4.12 in [11]. However, as our starting point is different than Schumaker's, there is a slight regularity issue when x does not differ from at least two y_i . However, we are only interested in cases where Q_i^m lives under an integral, and we agree here to set $Q_i^m(x) = 0$ in points where we do not have enough regularity.

Definition 8 (B-spline). Let $\cdots \leq y_{-1} \leq y_0 \leq y_1 \leq y_2 \leq \cdots$ be a sequence of real numbers. Given integers i and $m > 0$, one defines (note that x is fixed so that the finite difference is with respect to y)

$$Q_i^m(x) = \begin{cases} (-1)^m [y_i, \dots, y_{i+m}] (x - y)_+^{m-1}, & \text{if } y_i < y_{i+m}, \\ 0, & \text{otherwise.} \end{cases} \tag{18}$$

We call Q_i^m the m th order B-spline associated with the knots y_i, \dots, y_{i+m} .

This is Theorem 2.2 in [11]. Schumaker does not give the proof but writes that it is similar to the one of Theorem 2.1 in his book.

Theorem 9 (Dual Taylor Expansion). Let $f \in L_1^m[a, b]$. Then for all $a \leq y \leq b$

$$\begin{aligned}
 f(y) &= \sum_{j=0}^{m-1} \frac{(-1)^j D^j f(b) (b - y)^j}{j!} \\
 &\quad + \int_a^b \frac{(x - y)_+^{m-1} (-1)^m D^m f(x)}{(m-1)!} dx.
 \end{aligned} \tag{19}$$

Proof. The proof uses induction and integration by parts. Note that, by replacing a by y in the integral, we may delete the $+$ in $(x - y)_+$. The case $m = 1$ reduces to the fundamental theorem of calculus for absolutely continuous functions; see, for example, Theorem 7.1.15 in [9]. Let us assume that the statement holds for some m and fix $f \in L_1^{m+1}[a, b]$. Then

$$\begin{aligned} & \int_y^b \frac{(x - y)^m (-1)^{m+1} D^{m+1} f(x)}{m!} dx \\ &= \frac{(b - y)^m (-1)^{m+1} D^m f(b)}{m!} \\ & \quad - \int_y^b \frac{(x - y)^{m-1} (-1)^{m+1} D^m f(x)}{(m-1)!} dx \quad (20) \\ &= -\frac{(b - y)^m (-1)^m D^m f(b)}{m!} \\ & \quad - \sum_{j=0}^{m-1} \frac{(-1)^j D^j f(b) (b - y)^j}{j!} + f(y). \end{aligned}$$

□

We now cite a part of Theorem 4.23 in [11]. Note that m is a positive integer and D_+^j means that we take the right hand derivative of order j . We only need and prove the equality for $j = 0$. We also assume that not all points coincide.

Theorem 10 (Peano representation). *Fix $0 \leq j \leq m - n$, where n is the maximum multiplicity of y_i, \dots, y_{i+m} . Then*

$$[y_i, \dots, y_{i+m}] f = \int_{y_i}^{y_{i+m}} \frac{(-1)^j D_+^j Q_i^m(x) D^{m-j} f(x)}{(m-1)!} dx \quad (21)$$

for all $f \in L_1^{m-j}[\min_i\{y_i\}, \max_i\{y_i\}]$.

Proof of the Peano Representation for $j = 0$. We take the dual Taylor expansion from Theorem 9. Then, we apply $[y_i, \dots, y_{i+m}]$ on both sides and remember that we computed the divided differences of polynomials in Lemma 6. □

Our goal is to use the Peano representation to obtain an upper bound for the divided differences. We start with establishing an upper bound of the factor $D_+ Q_i^m$ in the integrand.

Lemma 11. *Suppose that $y_i \leq \dots \leq y_{i+m}$ and $y_i < y_{i+m}$. Then there is a constant C such that*

$$|D_+ Q_i^m(x)| = |[y_i, \dots, y_{i+m}](x - y)_+^{m-1}| \leq \frac{C}{y_{i+m} - y_i}. \quad (22)$$

Proof. We want to estimate

$$\begin{aligned} & |[y_i, \dots, y_{i+m}](x - y)_+^{m-1}| \\ &= \frac{[y_{i+1}, \dots, y_{i+m}](x - y)_+^{m-1} - [y_i, \dots, y_{i+m-1}](x - y)_+^{m-1}}{y_{i+m} - y_i}. \end{aligned} \quad (23)$$

Assume $k < m - 1$. We claim that

$$\frac{d^k}{dy^k} (x - y)_+^{m-1} = (-1)^k (m-1) \cdots (m-k) (x - y)_+^{m-1-k}. \quad (24)$$

This claim is easily seen to be true for $y \neq x$. That it is true also for $y = x$ follows by induction. We also note that

$$\frac{d^{m-2}}{dy^{m-2}} (x - y)_+^{m-1} \quad (25)$$

is absolutely continuous on bounded intervals and that

$$(-1)^{(m-1)} (m-1)! (x - y)_+^0 \quad (26)$$

is a weak derivative that equals the derivative whenever $y \neq x$. We first assume that not all y_j, \dots, y_{j+m-1} coincide. An application of the Hermite-Genocchi formula, Lemma 5, gives

$$\begin{aligned} & [y_j, \dots, y_{j+m-1}](x - y)_+^{m-1} \\ &= \int_0^1 dt_1 \int_0^{t_1} dt_2 \\ & \quad \cdots \int_0^{t_{m-2}} (-1)^{(m-1)} (m-1)! \\ & \quad \cdot \left(x - \left(y_j + \sum_{k=1}^{m-1} (y_{j+k} - y_{j+k-1}) t_k \right) \right)_+^0 dt_{m-1}. \end{aligned} \quad (27)$$

We use now the fact that the integrand is bounded from above by $(m-1)!$ and the computation (14) to obtain

$$|[y_j, \dots, y_{j+m-1}](x - y)_+^{m-1}| \leq 1. \quad (28)$$

Suppose now that $y_j = \dots = y_{j+m-1} \neq x$. Then $||[x_j, \dots, x_{j+m-1}](x - y)_+^{m-1}|$ is bounded from above by 1 if $x > y_i$ and vanishes if $y_i < x$. In the end, we find a constant C such that

$$|D_+ [y_i, \dots, y_{i+m}](x - y)_+^{m-1}| \leq \frac{C}{y_{i+m} - y_i}. \quad (29)$$

□

In the next result, we use the notation

$$Mg(x) = \sup_z \int_x^z |g| \quad (30)$$

for the famous Hardy-Littlewood maximal function of g introduced in the seminal paper [12] of Hardy-Littlewood in Acta Math. (1930).

Theorem 12. *Suppose $y_0 \leq \dots \leq y_m$, $y_0 < y_m$ and f is in $L_1^m[y_0, y_m]$ for some $m \in \mathbb{N}$. Then there is some constant C such that*

$$|[y_0, \dots, y_m] f| \leq C (M(|D^m f|)(y_0) + M(|D^m f|)(y_m)). \quad (31)$$

Proof. We use the Peano representation, Theorem 10, with $i = j = 0$. Then, using the bound of $|D_+ Q_i^m(x)|$ found in Lemma 11,

$$\begin{aligned} |[y_0, \dots, y_m] f| &\leq \frac{1}{(m-1)!} \int_{y_0}^{y_m} |D_+ Q_i^m(x)| |D^m f|(x) dx \\ &\leq \frac{C}{y_m - y_0} \int_{y_0}^{y_m} |D^m f|(x) dx. \end{aligned} \quad (32)$$

For $y \in [y_0, y_m]$, we have

$$\begin{aligned} |[y_0, \dots, y_m] f| &\leq \frac{C}{y_m - y_0} \left(\int_{y_0}^y |D^m f|(x) dx + \int_y^{y_m} |D^m f|(x) dx \right) \end{aligned} \quad (33)$$

and further

$$\begin{aligned} |[y_0, \dots, y_m] f| &\leq C \left(\int_{y_0}^y \frac{y - y_0}{y_m - y_0} |D^m f|(x) dx \right. \\ &\quad \left. + \int_y^{y_m} \frac{y_m - y}{y_m - y_0} |D^m f|(x) dx \right). \end{aligned} \quad (34)$$

The first factors in both integrals are bounded from above by 1. We see that the integrals are bounded from above by the corresponding maximal operators. This gives the claim. \square

The finite differences defined in (2) are, up to a factor, divided differences for equispaced nodes.

Lemma 13. Let x, y be two points in a domain $\Omega \subset \mathbb{R}$. Let $m \in \mathbb{N}$ and $x_k = x_0 + hk$, where $h = (y - x)/m$. Then each $f : \Omega \rightarrow \mathbb{R}$ satisfies

$$[x_0, \dots, x_m] f = \frac{1}{h^m m!} \Delta^m f(x; y) \quad (x = x_0, x_m = y). \quad (35)$$

Proof. We use induction. For $m = 1$, we have

$$[x_0, x_1] f = \frac{f(x_1) - f(x_0)}{x_1 - x_0}. \quad (36)$$

As $h = x_1 - x_0$, $x = x_0$, $y = x_1$, and $m = 1$, we obtain the claim. Suppose that the claim is true for some $m \in \mathbb{N}$. Then

$$\begin{aligned} \Delta_h^{m+1} f(x) &= \sum_{j=0}^{m+1} (-1)^{m+1-j} \binom{m+1}{j} f(x + jh) \\ &= \sum_{j=0}^{m+1} (-1)^{m+1-j} \left(\binom{m}{j-1} + \binom{m}{j} \right) f(x + jh) \end{aligned}$$

$$\begin{aligned} &= \sum_{j=0}^{m+1} (-1)^{m-j} \binom{m}{j} f(x + (j+1)h) \\ &\quad - \sum_{j=0}^{m+1} (-1)^{m-j} \binom{m}{j} f(x + jh) \\ &= \sum_{j=0}^m (-1)^{m-j} \binom{m}{j} f(x_1 + jh) \\ &\quad - \sum_{j=0}^m (-1)^{m-j} \binom{m}{j} f(x + jh) \\ &= h^m m! ([x_1, \dots, x_{m+1}] f - [x_0, \dots, x_m] f) \\ &= h^m m! (h(m+1)) [x_0, \dots, x_{m+1}] f. \end{aligned} \quad (37)$$

\square

We can now formulate Theorem 12 in terms of finite differences.

Corollary 14 (Corollary of Theorem 12). Let $\Omega \subset \mathbb{R}$ be a domain. Suppose $f \in L_1^m(\Omega)$ for some $m \in \mathbb{N}$; then there is a constant $C = C(m)$ such that

$$\begin{aligned} |\Delta^m f(x; y)| &\leq C |x - y|^m (M(|D^m f|)(x) + M(|D^m f|)(y)), \end{aligned} \quad (38)$$

$$x, y \in \Omega.$$

In particular, we have the following special case. Suppose X and Y are normed spaces of functions such that the maximal operator $M : X \rightarrow Y$ is bounded. If $f \in W_X^m(\Omega)$ for some $m \geq 1$, then

$$|\Delta^m f(x; y)| \leq |x - y|^m (a_f(x) + a_f(y)) \quad (39)$$

for some function $a_f \in Y$ depending on f .

Proof. We first use Lemma 13 and then Theorem 12 with step $h = (y - x)/m$. There is some constant $C = C(m)$ such that

$$\begin{aligned} |\Delta^m f(x; y)| &= h^m m! |[x_0, \dots, x_m] f| \\ &\leq Ch^m (M(|D^m f|)(x) + M(|D^m f|)(y)) \\ &\leq |x - y|^m (CM(|D^m f|)(x) + CM(|D^m f|)(y)). \end{aligned} \quad (40)$$

\square

Proof of Theorem 2 (Sobolev functions satisfy pointwise inequality). We combine Corollary 14 with the boundedness of the maximal operator, which is by now classical and can be found, for example, in Theorem 3.3.10 in [13]. \square

3. Pointwise Inequality Implies Membership in Sobolev Space

In this section, our aim is to show that the pointwise inequality (3) implies the existence of the highest order derivative in the correct space. We will take care of the intermediate derivatives in the next section. In this section, the properties of the Lebesgue spaces play an important role. Our point of view is however to extract the properties of the Lebesgue spaces and base our proofs solely on these properties. We hope that the disadvantage of being slightly more abstract is later on compensated when we generalize the results. We start with some auxiliary results.

Lemma 15. *Let $\Omega \subset \mathbb{R}$ be a domain and $\varphi \in \mathcal{C}_0^n(\Omega)$; that is, φ has compact support in Ω and is n -times continuously differentiable. Then*

$$\left| [x_0, \dots, x_n] \varphi - \frac{\varphi^{(n)}(y)}{n!} \right| \rightarrow 0 \quad (41)$$

uniformly in Ω as $x_0, \dots, x_n \rightarrow y$.

Proof. By Taylor's formula, for $x \in \text{conv}\{x_0, \dots, x_n\}$, there is $\xi_x \in \text{conv}\{x_0, \dots, x_n\}$ such that

$$\begin{aligned} \varphi(x) &= \varphi(y) + \varphi'(y)(x-y) + \dots + \varphi^{(n-1)}(y) \frac{(x-y)^{n-1}}{(n-1)!} \\ &\quad + \frac{1}{n!} \varphi^{(n)}(\xi_x) (x-y)^n. \end{aligned} \quad (42)$$

We apply $[x_0, \dots, x_n]$ on both sides (with variable x) and use the computation of the divided differences of polynomials carried out in Lemma 6 to obtain

$$[x_0, \dots, x_n] \varphi(x) = \frac{1}{n!} \varphi^{(n)}(\xi_x). \quad (43)$$

As $\varphi^{(n)}$ is uniformly continuous on compact intervals, the claim follows. \square

Lemma 16. *Assume $\Omega \subset \mathbb{R}$ is open and $K \subset \Omega$ is compact. Then there is a neighborhood of K that is a subset of Ω .*

Proof. For each $x \in K$, we find a ball $B(x, r_x) \subset \Omega$. Suppose the statement of the lemma is false. Then for every $n \in \mathbb{N}$ there is $z_n \in \Omega \setminus K$ with $\text{dist}(z_n, K) < 1/n$. Thus, for each $n \in \mathbb{N}$, there is $y_n \in K$ with $d(z_n, y_n) < 1/n$. Since K is compact, there is a subsequence $(y_{n_k})_k$ of $(y_n)_n$ that converges to some $y \in K$. Now, y lies in one of the balls $B(x, r_x) \subset \Omega$. For k large enough, we have that y_{n_k} and z_{n_k} are in $B(x, r_x) \subset \Omega$. This is a contradiction. \square

Lemma 17. *Suppose $\Omega \subset \mathbb{R}$ is open and $K \subset \Omega$ compact. Then there exists $\varepsilon > 0$ such that*

$$(\Omega - h) \cap K = \Omega \cap K \quad (44)$$

for $|h| < \varepsilon$.

Proof. We first show that any member of the left hand side is contained in the right hand side; this is true for any $h \in \mathbb{R}$. If z is a member of the left hand side, then it is in K , and since K is a subset of Ω , it is also a member of the right hand side. To prove the other inclusion, note that, by Lemma 16, there is $\varepsilon > 0$ such that the ε -neighborhood K_ε of K lies in Ω . Now suppose that z is in the right hand side and $|h| < \varepsilon$. Then for $x := z+h$, we have $x \in K_\varepsilon \subset \Omega$ implying that $z = x-h \in \Omega-h$. That $z \in K$ is clear. This gives the proof. \square

Next, we state basically Definition 3.1.1 in [14]. Readers following [13] might want to have a look at Definitions 1.1.1 and 1.1.3 in their book. To gain familiarity with the concept of Banach function norm, the reader may want to verify that Lebesgue spaces are examples of Banach function spaces. As we are here only interested in Lebesgue measure, we adapt the notion to this case. The set \mathcal{M}^+ denotes the cone of measurable functions whose values lie in $[0, \infty]$.

Definition 18 (Banach function norm; Banach function space). Assume that $R \subset \mathbb{R}$. A Banach function norm on R is a map ρ from $\mathcal{M}^+(R)$ to $[0, \infty]$ such that, for all f, g , and f_n ($n \in \mathbb{N}$) in $\mathcal{M}^+(R)$, all scalars $\lambda \geq 0$, and all measurable subsets E of R , the following are true, where μ is the Lebesgue measure:

- (P1) $\rho(f) = 0$ if, and only if, $f = 0$ μ -almost everywhere, $\rho(\lambda f) = \lambda \rho(f)$, $\rho(f+g) \leq \rho(f) + \rho(g)$;
- (P2) if $0 \leq g \leq f$ μ -almost everywhere, then $\rho(g) \leq \rho(f)$;
- (P3) if $0 \leq f_n \uparrow f$ μ -almost everywhere, we have $\rho(f_n) \uparrow \rho(f)$;
- (P4) if $\mu(E) < \infty$, then $\rho(\chi_E) < \infty$;
- (P5) if $\mu(E) < \infty$, then there is a constant $C = C(E, \rho) < \infty$ such that $\int_E f dx \leq C \rho(f)$.

Given such a function norm ρ , the set $X = X(\rho)$ of all extended measurable functions f (identifying functions equal μ -almost everywhere) such that $\rho(|f|) < \infty$ is called a Banach function space (BFS for short), and we define

$$\|f\|_X = \rho(|f|), \quad f \in X. \quad (45)$$

We leave it to the reader to verify that Lebesgue spaces satisfy the following properties (X' can be chosen as the dual space).

Properties 19. Given a Banach function space X , we require the existence of a Banach function space X' such that

- (B) X contains each essentially bounded function whose support has finite measure,
- (HI) Hölder's inequality: $\int |fg| dx \leq \|f\|_X \|g\|_{X'}$,
- (C) X' is complete,
- (D) $\|g\|_X = \sup\{\int_\Omega |g|h : h \in \mathcal{M}^+, \|h\|_{X'} \leq 1\}$,
- (AC) absolutely continuous norm: if $(\chi_{E_n})_n$ is a sequence of characteristic functions converging pointwise almost everywhere to 0, then $\|f \chi_{E_n}\|_{X'}$ converges to zero for all $f \in X'$,

(A) each function $f \in X'$ can be approximated by simple functions (in X' -norm),

(NC) if $f \in X'$ and h_n are step functions with $0 \leq h_n \uparrow |f|$, then

$$\|f - h_n \operatorname{sgn}(f)\|_{X'} \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (46)$$

Lemma 20. Suppose $\Omega \subset \mathbb{R}$ is open and X is a Banach function space fulfilling the items listed in Properties 19. Assume that $f : \Omega \rightarrow \mathbb{R}$ is in X . We let $\lambda_f : X' \cap \mathcal{C}_0^{r+1}(\Omega) \rightarrow \mathbb{R}$ be defined as

$$\lambda_f(\varphi) = \int_{\Omega} f \varphi^{(r)} dx. \quad (47)$$

If

$$|\Delta^r f(x, y)| \leq |x - y|^r [a_f(x) + a_f(y)], \quad (48)$$

for almost every $x, y \in \Omega$

for some $a_f \in X$, then λ_f has a continuous extension to X' with norm bounded from above by $2r^r \|a_f\|_X$.

Proof. Our goal is to show

$$\left| \int_{\Omega} f(x) \varphi^{(r)}(x) dx \right| \leq 2r^r \|a_f\|_X \|\varphi\|_{X'}, \quad (49)$$

because λ_f extends then by the theorem of Hahn-Banach to a bounded functional on X' with norm bounded from above by $2r^r \|a_f\|_X$. Without loss of generality, we may assume that a_f is nonnegative. If f were be smooth, we could just use integration by parts and use as bound the limiting case of the pointwise inequality (48). As f is not necessarily smooth, we have to operate on the level of finite differences.

Let us choose $\varphi \in \mathcal{C}_0(\Omega)$ and denote its support by K . Lemma 17 ensures the existence of some positive $\varepsilon > 0$ such that $(\Omega - jh) \cap K = \Omega \cap K$ for $0 \leq j \leq r$ as long as $|h| < \varepsilon/r$. We assume in the following that h has been chosen accordingly.

The following integrals exist by Hölder's inequality because φ and thus $\Delta_{-h}^r \varphi(x)$ are bounded with compact support and f is integrable. Consider

$$\begin{aligned} & \int_{\Omega} f(x) \varphi^{(r)}(x) dx \\ &= \int_{\Omega} f(x) (-h)^{-r} \Delta_{-h}^r \varphi(x) dx \\ &+ \int_{\Omega} f(x) (\varphi^{(r)}(x) - (-h)^{-r} \Delta_{-h}^r \varphi(x)) dx. \end{aligned} \quad (50)$$

We compute the following integral:

$$\begin{aligned} & \int_{\Omega} f(x) \Delta_{-h}^r \varphi(x) dx \\ &= \int_{\Omega} f(x) \sum_{j=0}^r (-1)^{r-j} \binom{r}{j} \varphi(x - jh) dx \\ &= \sum_{j=0}^r (-1)^{r-j} \binom{r}{j} \int_{\Omega} f(x) \varphi(x - jh) dx \end{aligned}$$

$$\begin{aligned} &= \sum_{j=0}^r (-1)^{r-j} \binom{r}{j} \int_{\Omega - jh} f(y + jh) \varphi(y) dy \\ &= \sum_{j=0}^r (-1)^{r-j} \binom{r}{j} \int_{(\Omega - jh) \cap K} f(y + jh) \varphi(y) dy. \end{aligned} \quad (51)$$

We have chosen $\varepsilon > 0$ such that $(\Omega - jh) \cap K = \Omega \cap K$. We continue

$$\begin{aligned} & \int_{\Omega} f(x) \Delta_{-h}^r \varphi(x) dx \\ &= \sum_{j=0}^r (-1)^{r-j} \binom{r}{j} \int_{\Omega \cap K} f(y + jh) \varphi(y) dy \\ &= \sum_{j=0}^r (-1)^{r-j} \binom{r}{j} \int_{\Omega} f(y + jh) \varphi(y) dy \\ &= \int_{\Omega} \sum_{j=0}^r (-1)^{r-j} \binom{r}{j} f(y + jh) \varphi(y) dy \\ &= \int_{\Omega} \Delta_h^r f(x) \varphi(x) dx. \end{aligned} \quad (52)$$

With the aid of (52), we can now rewrite (50) as

$$\begin{aligned} & \int_{\Omega} f(x) \varphi^{(r)}(x) dx \\ &= \int_{\Omega} (-h)^{-r} \Delta_h^r f(x) \varphi(x) dx \\ &+ \int_{\Omega} f(x) (\varphi^{(r)}(x) - (-h)^{-r} \Delta_h^r \varphi(x)) dx. \end{aligned} \quad (53)$$

Inequality (48) leads to

$$\begin{aligned} & \left| \int_{\Omega} f(x) \varphi^{(r)}(x) dx \right| \\ &\leq r^r \int_{\Omega} (a_f(x) + a_f(x + rh)) |\varphi(x)| dx \\ &+ \int_{\Omega} |f(x)| |\varphi^{(r)}(x) - (-h)^{-r} \Delta_h^r \varphi(x)| dx. \end{aligned} \quad (54)$$

Using Hölder's inequality (HI) for the first summand, we are left to show that the second summand tends to 0 as h approaches 0. We note first that since the second factor has support in a set of finite measures, we can replace the integration domain by a set of finite measures. Then we apply a (L^1, L^∞) -Hölder inequality and the uniform convergence of the divided difference to $\varphi^{(r)}(x)/r!$ established in Lemmas 13 and 15. \square

After having verified that λ_f has a continuous extension to X' , we want to show that this extension has a representation as integral.

Lemma 21. Assume $\Omega \subset \mathbb{R}$ is open. Suppose the properties listed in Properties 19 are met. Let λ_f be defined as in (47) and bounded on X' . Then there exists $g \in X$ with $\|g\|_X \leq \|\lambda_f\|$ such that

$$\lambda_f(\varphi) = (-1)^r \int_{\Omega} g(x) \varphi(x) dx \quad (55)$$

for all $\varphi \in X'$, especially for all $\varphi \in \mathcal{C}_0^\infty(\Omega)$.

Proof. The proof is heavily influenced by the proof of Theorem 1.4.1 in [13]. We first want to find a signed measure ν such that

$$\lambda_f(\varphi) = \int_{\Omega} \varphi d\nu \quad (56)$$

for each $\varphi \in X'$. Then, we want to apply the Lebesgue-Radon-Nikodym theorem to replace $d\nu$ by $(-1)^r g(x)dx$ for some integrable function g for which we then show that it is contained in X .

We start by assuming that the measure of Ω is finite. For each μ -measurable set A , we set

$$\nu(A) := \lambda_f(\chi_A). \quad (57)$$

We start by proving the σ -additivity of ν (for the μ -measurable sets). In the first step, we just show the additivity. By induction, it is enough to consider only two sets.

Let A_1 and A_2 be two disjoint μ -measurable sets. Then

$$\begin{aligned} \nu(A_1 \cup A_2) &= \lambda_f(\chi_{A_1 \cup A_2}) = \lambda_f(\chi_{A_1} + \chi_{A_2}) \\ &= \lambda_f(\chi_{A_1}) + \lambda_f(\chi_{A_2}) = \nu(A_1) + \nu(A_2). \end{aligned} \quad (58)$$

We move to the verification of the σ -additivity. Let A_1, A_2, \dots be countably many pairwise disjoint μ -measurable sets. We first want to show that $\sum_i \chi_{A_i}$ converges in X' .

We let $E_k := \bigcup_{i=1}^k A_i$ and $E := \bigcup_{i=1}^\infty A_i$. Then, since $(E_k)_k$ is increasing,

$$\mu(E) = \mu\left(\bigcup_{k=1}^\infty E_k\right) = \lim_{k \rightarrow \infty} \mu(E_k). \quad (59)$$

As E has finite measure, given $\varepsilon > 0$, there exists $k_0 \in \mathbb{N}$ such that

$$(1 - \varepsilon) \mu(E) < \mu(E_{k_0}). \quad (60)$$

By the disjointness of the sets A_i , we see that

$$\mu(E) = \mu(E_{k_0}) + \mu\left(\bigcup_{i=k_0+1}^\infty A_i\right). \quad (61)$$

It follows that

$$\begin{aligned} \mu\left(\bigcup_{i=k_0+1}^\infty A_i\right) &= \mu(E) - \mu(E_{k_0}) < \mu(E) - (1 - \varepsilon) \mu(E) \\ &= \varepsilon \mu(E). \end{aligned} \quad (62)$$

We have for $m < n$

$$\left\| \sum_{i=1}^n \chi_{A_i} - \sum_{i=1}^m \chi_{A_i} \right\|_{X'} = \left\| \sum_{i=m+1}^n \chi_{A_i} \right\|_{X'} \leq \|\chi_{\bigcup_{i=m+1}^\infty A_i}\|_{X'}. \quad (63)$$

By the considerations before and the fact that $\|\cdot\|_{X'}$ is absolutely continuous as stipulated in (AC), we see that $\sum_{i=1}^n \chi_{A_i}$ is a Cauchy sequence. Now, as X' is complete by (C), the Cauchy sequence converges in X' . We have, by the continuity of λ_f ,

$$\begin{aligned} \nu(\bigcup_i A_i) &= \lambda_f(\chi_{\bigcup_i A_i}) = \lambda_f\left(\sum_{i=1}^\infty \chi_{A_i}\right) \\ &= \lim_{n \rightarrow \infty} \lambda_f\left(\sum_{i=1}^n \chi_{A_i}\right) = \lim_{n \rightarrow \infty} \sum_{i=1}^n \lambda_f(\chi_{A_i}) \\ &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \nu(A_i) = \sum_{i=1}^\infty \nu(A_i). \end{aligned} \quad (64)$$

Thus ν is σ -additive.

In order to apply the theorem of Lebesgue-Radon-Nikodym, see, for example, Section 6.9 in [15], and concluding the existence of $\hat{g} \in L^1(\Omega)$ such that

$$\nu(A) = \int_A \hat{g} dx, \quad A \mu\text{-measurable}, \quad (65)$$

we need to show that ν is absolutely continuous with respect to μ . To do so, assume that $\mu(N) = 0$. Then $\nu(N) = \lambda_f(\chi_N)$. By the absolute continuity of the norm of X' required in (AC), we see that $\|\chi_N\|_{X'} = 0$, and therefore $\nu(N) = 0$. Replacing \hat{g} by $-\hat{g}$ if necessary, we obtain

$$\lambda_f(\chi_A) = (-1)^r \int_{\Omega} \hat{g}(x) \chi_A(x) dx. \quad (66)$$

Now, we assume that Ω does not necessarily have finite measure. We write $\Omega = \bigcup_n \Omega_n$ as union of open sets with finite measure such that $\Omega_n \subset \Omega_{n+1}$. By the considerations above, we find for each $n \in \mathbb{N}$ a g_n such that

$$\lambda_f(\chi_A) = (-1)^r \int_{\Omega} g_n(x) \chi_A(x) dx \quad (67)$$

for each measurable set $A \subset \Omega_n$. It is easily seen that g_n and g_{n+1} agree almost everywhere on Ω_n . This gives rise to a function g satisfying

$$\begin{aligned} \lambda_f(\chi_A) &= (-1)^r \int_{\Omega} g(x) \chi_A(x) dx, \\ A &\subset \bigcup_{n=1}^N \Omega_n \text{ for some } N. \end{aligned} \quad (68)$$

By linearity, we have for each simple function s :

$$\lambda_f(s) = (-1)^r \int_{\Omega} g(x) s(x) dx. \quad (69)$$

To show that $g \in X$, we want to use property (D).

Let h be a nonnegative simple function with support in some Ω_n . Then $h \operatorname{sgn}(g)$ is also a simple function with support in Ω_n . It is a finite linear combination of characteristic functions of sets in Ω_n . Hence

$$\int |hg| dx = \int h \cdot \operatorname{sgn}(g) g dx = \lambda_f(h \operatorname{sgn}(g)). \quad (70)$$

Using the boundedness of λ_f , we see that

$$\int |hg| dx \leq \|\lambda_f\| \|h\|_{X'}. \quad (71)$$

If h is an arbitrary function in X' , then we may construct a sequence $(h_n)_{n=1}^\infty$ of simple functions, each h_n having support in Ω_n , such that $0 \leq h_n \uparrow |h|$ μ -almost everywhere. By the monotone convergence theorem, we have by (71) that

$$\int |hg| dx = \lim_{n \rightarrow \infty} \int |h_n g| dx \leq \limsup_{n \rightarrow \infty} \|\lambda_f\| \|h_n\|_{X'}. \quad (72)$$

By (P3) in Definition 18, we see that

$$\int |hg| dx \leq \|\lambda_f\| \|h\|_{X'} \quad (73)$$

and Property (D) gives that $g \in X$. In the last step, we want to upgrade the integral representation (69) to hold for arbitrary functions in $\varphi \in X'$. As before, we approximate φ by simple functions h_n with support contained in Ω_n and $0 \leq h_n \uparrow |\varphi|$ in μ -almost everywhere point. Define $s_n := h_n \operatorname{sgn}(\varphi)$. By (NC), s_n converges to φ in X' and as both λ_f and “its integral representation” are continuous and agree on the set of step functions, they agree on X' . \square

Having shown the existence of the weak derivatives of highest order, we are left to show the existence of the intermediate derivatives, which we do in Proposition 22 in the next section. To keep the flow going, we give here the proof that the pointwise inequality implies membership in the corresponding Sobolev space under the assumption that we already know that Proposition 22 holds.

Proof of Theorem 2 (pointwise inequality implies membership in Sobolev space). We combine Lemmas 20 and 21 and apply Proposition 22. \square

4. Equivalent Definitions of Sobolev Spaces

The aim of this section is to prove the following equivalence.

Proposition 22. *Let $\Omega \subset \mathbb{R}$ be a domain with finite Lebesgue measure in \mathbb{R} . Suppose X is a Banach function space satisfying (B) and (D). Assume that $m \in \mathbb{N}$. Then the following sets coincide:*

- (a) $A = \{f \in X : \text{there exists } F \in X, \text{ where } F \text{ agrees almost everywhere with } f \text{ and has weak derivatives up to order } m \text{ in } X\}, \text{ where the derivatives of order } r, 0 \leq r \leq m-1, \text{ are absolutely continuous,}$
- (b) $B = \{f \in X : f^{(k)} \in X, k \in \{0, \dots, m\}\}, \text{ where } f^{(k)} \text{ are the weak derivatives of } f,$

- (c) $C = \{f \in X : f^{(m)} \in X\}, \text{ where } f^{(m)} \text{ is the weak derivative of } f \text{ of order } m.$

We see that $A \subset B \subset C$. We now show that $C \subset A$.

Let us give an outline of the proof. Given $g : \Omega \rightarrow \mathbb{R}$ (we will choose $g = f^{(m)}$), by integrating, we construct in Lemma 23 an absolutely continuous function G_m together with its weak derivatives G_0, \dots, G_{m-1} , which are absolutely continuous, and $G_m^{(m)} = g$. However, G_m and f might not agree almost everywhere. We verify however in the proof of Theorem 26 (with the help of Theorem 24) that we find a polynomial P such that $G_m + P$ agrees with F almost everywhere and shares with G_m the other required properties. We conclude the proof of Theorem 26 by settling the membership questions with the aid of Proposition 25.

Lemma 23. *Assume $\Omega \subset \mathbb{R}$ is a domain. We further assume that $g : \Omega \rightarrow \mathbb{R}$ is in $L^1_{\text{loc}}(\Omega)$. One fixes $x^* \in \Omega$ and defines*

$$G_0(x_0) := g(x_0), \quad (74)$$

$$G_k(x_k) := \int_{x^*}^{x_k} G_{k-1}(x_{k-1}) dx_{k-1}, \quad k \in \mathbb{N},$$

where the integrals are Lebesgue integrals, and we use the notation

$$\begin{aligned} & \int_{x^*}^{x_k} G_{k-1}(x_{k-1}) dx_{k-1} \\ &= \begin{cases} \int_{[x^*, x_k]} G_{k-1}(x_{k-1}) dx_{k-1}, & x^* \leq x_k, \\ -\int_{[x_k, x^*]} G_{k-1}(x_{k-1}) dx_{k-1}, & x_k < x^*. \end{cases} \end{aligned} \quad (75)$$

Then

- (a) G_k is locally absolutely continuous for all $k \in \mathbb{N}$, especially locally integrable,
- (b) if $\varphi \in \mathcal{C}_0^\infty(\Omega)$, and $0 \leq r \leq k$, one has

$$\int_{\Omega} G_k \varphi^{(r)} = (-1)^r \int_{\Omega} G_{k-r} \varphi. \quad (76)$$

Proof. Let us prove (a) by induction. Assume $k = 1$. Note that $G_1(x^*) = 0$. Thus, we have for $x^* \leq x_1$

$$G_1(x_1) - G_1(x^*) = G_1(x_1) = \int_{x^*}^{x_1} g(x_0) dx_0. \quad (77)$$

As $g \in L^1([x^*, x_1])$, we see that G_1 is absolutely continuous. The case $x_1 < x^*$ is treated similarly.

Assume now that the claim is true for some k . Now $G_{k+1}(x_{k+1})$ is the integral of a continuous function and thus locally absolutely continuous.

Let us have a look at (b). Note that the statement is true for $k = 0$; thus we may assume that $k \geq 1$. We make an induction over r . The case $r = 0$ is clear. Let us choose an interval $[a, b] \subset \Omega$ containing a neighborhood of the support of φ . As G_k and $\varphi^{(r)}$ are both absolutely continuous in $[a, b]$, we have

by the integration by part formula for absolutely continuous functions (see, for example, Theorem 7.1.47 in [9]) and the fact that $\varphi^{(r)}(a) = \varphi^{(r)}(b) = 0$

$$\int_{\Omega} G_k \varphi^{(r+1)} = \int_{[a,b]} G_k \varphi^{(r+1)} = (-1) \int_{[a,b]} G'_k \varphi^{(r)}. \quad (78)$$

Noting that $G'_k = G_{k-1}$ almost everywhere, we see that

$$\int_{\Omega} G_k \varphi^{(r+1)} = (-1) \int_{[a,b]} G_{k-1} \varphi^{(r)} = (-1) \int_{\Omega} G_{k-1} \varphi^{(r)}. \quad (79)$$

Using the induction hypothesis, the claim follows. \square

We cite Theorem 2.5.3 in [10] without giving its proof.

Theorem 24. *Let $A = [a, b]$, \mathbb{R} , or \mathbb{R}_+ and $r \geq 0$. If the function $f \in L^1_{\text{loc}}(A)$ satisfies*

$$\int_A f \varphi^{(r)} dx = 0 \quad (80)$$

for all $\varphi \in \mathcal{C}_0^\infty(A)$, then there is a polynomial P of degree $< r$ such that $f = P$ almost everywhere on A .

To show that G_k is in X , we need to verify that (if there is danger of confusion with respect to which variable we take the norm, we use notations as $X(x_k)$)

$$\left\| \int_{x^*}^{x_k} G_{k-1}(x_{k-1}) dx_{k-1} \right\|_{X(x_k)} < \infty \quad (81)$$

under the assumption that G_{k-1} is in X . Actually, bounding G_k from above by its absolute value, this means that we have first to take the L^1 -norm and then the norm in X . We however would like to switch the norms.

We adapt Proposition 2.1 in [16] to our setting.

Proposition 25. *Let ρ be a Banach function norm satisfying property (D). If $f : \Omega \times \Omega \rightarrow \mathbb{R}$ is a measurable function, and one has f_x and f^y functions on Ω such that $f_x(y) = f(x, y) = f^y(x)$, then*

$$\rho(\|f_x\|_1) \leq \|\rho(f^y)\|_1. \quad (82)$$

The cases $X \in \{L^1, L^\infty\}$ are easy to prove. If $X = L^p$ for some $1 < p < \infty$, then the result can be proven by using Fubini's theorem together with property (D). For the general results we refer to Schep's proof.

The next result builds upon Theorem 2.5.4 in [10].

Theorem 26. *Let $\Omega \subset \mathbb{R}$ be a domain, and let $r \geq 1$. Suppose X is a Banach function space satisfying (B) and (D). If f is locally in X and has a generalized r th derivative g locally in X , then there is F agreeing almost everywhere with f , having absolutely continuous intermediate weak derivatives, and the weak derivative of order r agrees almost everywhere with g . Furthermore, the derivatives of F are locally in X . If Ω has finite measure, one obtains the result with "locally" replaced by "globally."*

We thank Pilar Silvestre for pointing out how to prove the membership of the intermediate derivatives in X .

Proof. Choose $x^* \in \Omega$ and let G_k be defined as in Lemma 23. Assume $\varphi \in \mathcal{C}_0^\infty(\Omega)$. Then, using the fact that g is a weak derivative of f of order r and (b) in Lemma 23

$$\begin{aligned} \int_{\Omega} (f - G_r) \varphi^{(r)} dx &= \int_{\Omega} f \varphi^{(r)} dx - \int_{\Omega} G_r \varphi^{(r)} dx \\ &= (-1)^r \int_{\Omega} g \varphi dx - (-1)^r \int_{\Omega} g \varphi dx = 0. \end{aligned} \quad (83)$$

Using Theorem 24, we see that $f - G_r = P$ almost everywhere, where P is a polynomial of degree $< r$. Thus $f = P + G_r$ almost everywhere and for $\varphi \in \mathcal{C}_0^\infty(\Omega)$ and $0 \leq m \leq r$

$$\begin{aligned} \int_{\Omega} f \varphi^{(m)} dx &= \int_{\Omega} (P + G_r) \varphi^{(m)} dx \\ &= (-1)^m \int_{\Omega} (G_{r-m} + P^{(m)}) \varphi dx. \end{aligned} \quad (84)$$

Thus $G_{r-m} + P^{(m)}$ is the m th generalized derivative of f . As it is locally absolutely continuous (if $m < r$) it lies locally in X . In the case $m = r$ the same conclusion holds by the assumption on g .

Moreover, there exists F , for example, $F = P + G_r$, that agrees with f almost everywhere, such that its generalized derivatives up to order $r - 1$ are locally absolutely continuous and the r th generalized derivative equals g almost everywhere.

As g is locally in X and the other derivatives are locally absolutely continuous, they are locally in L^∞ and thus locally in X by (B).

Suppose now that Ω has additionally finite measure. Assume that f and g are in X globally. We want to show that the intermediate derivatives are in X as well.

Note in our setting that, by (P5) in Definition 18, membership in X implies membership in L^1 . We proceed by induction. Suppose that $G_{k-1} \in X$. We will apply Proposition 25. We set, denoting by $\text{conv}\{x^*, x_k\}$ the closed interval with endpoints x^* and x_k ,

$$f(x_{k-1}, x_k) = G_{k-1}(x_{k-1}) \chi_{\text{conv}\{x^*, x_k\}}(x_{k-1}). \quad (85)$$

Then

$$\begin{aligned} \|G_k(x_k)\|_{X(x_k)} &= \left\| \int_{x^*}^{x_k} G_{k-1}(x_{k-1}) dx_{k-1} \right\|_{X(x_k)} \\ &= \left\| \int_{x^*}^{x_k} f(x_{k-1}, x_k) dx_{k-1} \right\|_{X(x_k)} \\ &\leq \|f(x_{k-1}, x_k)\|_{L^1(x_{k-1})} \|1\|_{X(x_k)} \\ &\leq \|f(x_{k-1}, x_k)\|_{X(x_k)} \|1\|_{L^1(x_{k-1})} \\ &\leq \|G_{k-1}(x_{k-1})\| \|\chi_{\Omega}\|_{X(x_k)} \|1\|_{L^1(x_{k-1})} < \infty. \end{aligned} \quad (86)$$

\square

5. Banach Function Space Setting

Here, our main focus is a more advanced readership that likely knows already Banach function spaces. Therefore, we adopt a much briefer style, pointing merely to the literature where the corresponding definitions and results can be found. We start with giving generalized versions of the necessity and sufficiency of Theorem 2.

We list the results and prove them simultaneously by splitting the proofs into parts according to the function spaces.

Theorem 27. *Assume $\Omega \subset \mathbb{R}$ is a domain, X is a normed space, and $m \in \mathbb{N}$. If the maximal function $M : X \rightarrow X$ is bounded, then there is a constant $C = C(X, m)$ such that for each continuous $f \in W_X^m$ there is $a_f \in X$ with $\|a_f\| \leq C\|f\|_{W_X^m}$ and*

$$|\Delta^m f(x; y)| \leq |x - y|^m (a_f(x) + a_f(y)). \quad (87)$$

In particular such a_f exists if

- (i) $1 < p \leq \infty$ and X is the Lebesgue space $L^p(\Omega)$,
- (ii) $1 < p < \infty$, $1 \leq q \leq \infty$, and X is the Lorentz space $L^{p,q}(\Omega)$,
- (iii) $1 < p < \infty$, $1 \leq q \leq \infty$, b is slowly varying, and X is the Lorentz-Karamata space $L_{p,q;b}(\Omega)$,
- (iv) ϕ is an N -function with $\inf_{s>0} (s\phi(s)/\phi(s)) > 1$, where ϕ is the density function of ϕ and X is the Orlicz space $L_\phi^*(\Omega)$,
- (v) $\Omega = \mathbb{R}$ and X is rearrangement-invariant with upper index $\bar{\alpha}_X < 1$.

Theorem 28. *Let $\Omega \subset \mathbb{R}$ be open. Suppose X is a Banach function space such that (AC) holds. Assume further that the function $f : \Omega \rightarrow \mathbb{R}$ is in X . If for some $r \in \mathbb{N}_0$, the following inequality holds*

$$|\Delta^r f(x, y)| \leq |x - y|^r [a_f(x) + a_f(y)] \quad (88)$$

for some $a_f \in X$, then there exists $g \in X$ with $\|g\|_X \leq 2r^r \|a_f\|_X$ such that

$$\int_\Omega f(x) \varphi^{(r)}(x) dx = (-1)^r \int_\Omega g(x) \varphi(x) dx \quad (89)$$

for all $\varphi \in \mathcal{C}_0^\infty(\Omega)$.

In particular the result holds if X is

- (i) the Lebesgue space $L^p(\Omega)$ for some $1 < p < \infty$,
- (ii) the Lorentz space $L^{p,q}(\Omega)$, where $1 < p < \infty$ and $1 \leq q < \infty$,
- (iii) the Lorentz-Karamata space $L_{p,q;b}(\Omega)$, where b is slowly varying, $1 < p < \infty$, and $1 < q \leq \infty$,

(iv) the Orlicz space $L_\phi^*(\Omega)$ for some N -function ϕ satisfying

$$\inf_{s>0} \frac{s\varphi(s)}{\phi(s)} > 1, \quad (90)$$

where φ is the density function of ϕ .

To help with orientation, let us give an overview of the remaining material in this section. With respect to Theorem 27, note that we carried out the proof of Corollary 14 in a high level of abstraction. Hence, we are left to verify that the spaces under question are normed spaces (we will show that they are even Banach function spaces) and the boundedness of the maximal operator.

Concerning the proof Theorem 28, we proved the corresponding part of Theorem 2 under the assumptions that the spaces under question are Banach function spaces and further satisfy the requirements listed in Properties 19. The existence of X' is classical (Proposition 29) as is the fact that (AC) implies (A) and (NC) (Proposition 30). Hence, we only need to point to the literature where (AC) was verified.

Proposition 29. *Given a Banach function space, there is a space X' such that properties (B), (HI), (C), and (D) are satisfied.*

Proof. We refer the reader to Sections 1.1 and 1.2 in [13] or to Section 3.1 in [14]. \square

The definition of absolutely continuous norm can be found as Definition 3.1.11 in [14] and Definition 1.3.1 in [13].

Proposition 30. *A Banach function space that has absolutely continuous norm satisfies (A) and (NC).*

Proof. The first statement follows from Theorem 1.3.13 in [13] and the second from Proposition 1.3.6 in the same source. \square

5.1. General Statements

Proof of Theorem 27 (general statement). We basically have already proven the general statement in Corollary 14. \square

Proof of Theorem 28 (general statement). We gave the proof of the Lebesgue space case in such generality that the general statement follows from the proof of Theorem 2 and Propositions 29 and 30. \square

5.2. Lorentz-Karamata Spaces. Lorentz-Karamata spaces are detailed in [14], mainly Section 3.4.3.

Proof of Theorem 27, (ii) and (iii). The part that is left to prove is the boundedness of the maximal operator, which is guaranteed by Remark 3.5.17 in [14]. \square

Proof of Theorem 28, (ii) and (iii). By Theorem 3.4.41 in [14], we see that X is a rearrangement-invariant Banach space. A combination of Lemma 3.4.39 in [14] together with Lemma 3.4.43 in [14] gives the absolute continuity (AC) of the norm of X' . \square

5.3. *Orlicz Spaces.* The setting is the one of [17].

Lemma 31. *If ϕ is an N -function satisfying the Δ_2 -condition, then the Orlicz space L_ϕ^* has absolutely continuous norm.*

Proof. Let $f \in L_\phi^*(\Omega)$. We verify first that

$$\int_{\Omega} \phi(|f|) dx < \infty. \quad (91)$$

Let $\varepsilon > 0$. By the Δ_2 -condition, there exists a constant $\kappa = \kappa(\|f\|_{(\phi)} + \varepsilon)$ such that

$$\phi((\|f\|_{(\phi)} + \varepsilon)s) \leq \kappa \phi(s) \quad (92)$$

for all $s \geq 0$. Thus

$$\begin{aligned} \int_{\Omega} \phi(|f|) dx &= \int_{\Omega} \phi\left(\frac{|f|}{\|f\|_{(\phi)} + \varepsilon} (\|f\|_{(\phi)} + \varepsilon)\right) dx \\ &\leq \kappa \int_{\Omega} \phi\left(\frac{|f|}{\|f\|_{(\phi)} + \varepsilon}\right) dx \leq \kappa. \end{aligned} \quad (93)$$

Suppose that E is a measurable set. Then

$$\begin{aligned} \|f\chi_E\|_{(\phi)} &= \inf \left\{ \lambda > 0 : \int_{\Omega} \phi(\lambda^{-1} |f\chi_E|) dx \leq 1 \right\} \\ &= \inf \left\{ \lambda > 0 : \int_{\Omega} \phi(\lambda^{-1} |f|) \chi_E dx \leq 1 \right\}. \end{aligned} \quad (94)$$

Let us fix $\lambda > 0$. Again using the Δ_2 -condition, we have $\phi(\lambda^{-1}s) \leq \kappa(\lambda^{-1})\phi(s)$. Hence

$$\int_{\Omega} \phi(\lambda^{-1} |f|) \chi_E dx \leq \kappa \int_{\Omega} \phi(|f|) \chi_E dx. \quad (95)$$

Example 1.3.3 in [13] tells us that L^1 has absolutely continuous norm. Thus if χ_{E_n} converges μ -almost everywhere to 0, then $\int_{\Omega} \phi(|f|) \chi_{E_n}$ converges to zero as well. Thus, we have for n large enough that

$$\kappa \int_{\Omega} \phi(|f|) \chi_{E_n} dx \leq 1 \quad (96)$$

implying that $\|f\chi_{E_n}\|_{(\phi)} \leq \lambda$. Since $\lambda > 0$ was arbitrary, the absolute continuity of f follows. \square

Proof of Theorem 27 (iv). We conclude from Gallardo's result, Theorem 2.1 in [17], that it is sufficient that the complementary N -function satisfies the Δ_2 -condition in $[0, \infty)$. Again resorting to Gallardo, this time to Proposition 1.4 in [17], we obtain the conclusion. \square

Proof of Theorem 28 (iv). That X is a Banach function space follows from Theorem 3.4.16 in [14]. To prove that X' has absolutely continuous norm, we note that inequality (90) together with Proposition 1.4 in [17] ensures that the complementary N -function of ϕ satisfies the Δ_2 -condition. Lemma 31 verifies that X' has absolutely continuous norm. \square

5.4. *Rearrangement-Invariant Spaces.* The necessary definitions can be found in [13].

Proof of Theorem 27, (v). We are only left to show the boundedness of the maximal operator. However, this is the content of a result by Lorentz and Shimogaki, for example, stated as Theorem 3.5.17 in [13]. \square

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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