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Research Article

Endpoint Estimates for Fractional Hardy Operators and Their Commutators on Hardy Spaces

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$(H^p(\mathbb{R}^n), L^q(\mathbb{R}^n))$ bounds of fractional Hardy operators are obtained. Moreover, the estimates for commutators of fractional Hardy operators on Hardy spaces are worked out. It is also proved that the commutators of fractional Hardy operators are mapped from the Herz-type Hardy spaces into the Herz spaces. The estimates for multilinear commutators of fractional Hardy operators are also discussed.

1. Introduction

The most fundamental averaging operator is the Hardy operators defined by

$$H(f)(x) = \frac{1}{x}\int_0^x f(t)\,dt, \tag{1}$$

where the function f is nonnegative integrable on $\mathbb{R}^+ = (0,\infty)$ and $x > 0$. A classical inequality, due to Hardy et al. [1], states that

$$\|H(f)\|_{L^p(\mathbb{R}^+)} \leq \frac{p}{p-1}\|f\|_{L^p(\mathbb{R}^+)}, \tag{2}$$

holds for $1 < p < \infty$ and the constant $p/(p-1)$ is the best possible.

The Hardy integral inequality has received considerable attention. A number of papers involved its alternative proofs, generalizations, variants, and applications. Among numerous papers dealing with such inequalities, we choose to refer to the papers [2–4].

Let f be a locally integrable function on $\mathbb{R}^n$, $0 \leq \beta < n$. In [5], Fu et al. defined n-dimensional fractional Hardy operators $\mathscr{H}_\beta$:

$$\mathscr{H}_\beta f(x) = \frac{1}{|x|^{n-\beta}}\int_{|y|<|x|} f(y)\,dy. \tag{3}$$

When $\beta = 0$, $\mathscr{H}_\beta$ is just n-dimensional Hardy operators $\mathscr{H}$; see [3] for more details.

It is well known that averaging operators play an important role in harmonic analysis. For example, the Hardy-Littlewood maximal operators control many kinds of operators in analysis. Therefore, the study of Hardy operators is meaningful and has been fully discussed (see [4, 6–11]). In 2012, Zhao et al. [11] have shown the following results: when $b \in BMO(\mathbb{R}^n)$, (1) $\mathscr{H}$ maps $H^1(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$; (2) $[b,\mathscr{H}]$ maps $H^1(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$ was false; (3) $[b,\mathscr{H}]$ maps $H^1(\mathbb{R}^n)L^{1,\infty}(\mathbb{R}^n)$. Furthermore, (3) is also true for $b \in C\dot{M}O^q(\mathbb{R}^n)$, $1 \leq q < \infty$. Recently, fractional Hardy operators were studied by many authors (see [5, 12, 13]). In 2007, Fu et al. [5] have obtained that the commutator $[b,\mathscr{H}_\beta]$ is bounded from $\dot{K}_{q_1}^{\gamma,p_1}(\mathbb{R}^n)$ to $\dot{K}_{q_2}^{\gamma,p_2}(\mathbb{R}^n)$, where $\gamma < n(1-1/q_1)$. In 2013, Lu et al. [13] have proved that fractional Hardy operators H_β map $L^p(\mathbb{R}^n)$ into $L^q(\mathbb{R}^n)$, where $1 < p \leq n/\beta$, $1/q = (1/p) - (\beta/n)$, and $\|\mathscr{H}_\beta\|_{L^1 \to L^{(n/(n-\beta)),\infty}} = 1$. Inspired by the above, we consider the endpoint estimates for fractional Hardy operators and their commutators on Hardy-type spaces.

This paper is organized as follows. In the second section, we give the $(H^p(\mathbb{R}^n), L^q(\mathbb{R}^n))$ bounds of fractional Hardy operators. In the third section, we obtain the estimates for commutators of fractional Hardy operators on Hardy spaces. In Section 4, we consider the case on Herz-type Hardy spaces. In Section 5, we obtain the estimate for multilinear commutators of fractional Hardy operators.

Throughout this paper, $C_1 \sim C_2$ denotes that C_1 is equivalent to C_2, which means there exist two positive constants c_1 and c_2 such that $c_1 C_1 \le C_2 \le c_2 C_1$. For $1 < p$, and $p' < \infty$, p and p' are conjugate indices; that is, $(1/p) + (1/p') = 1$.

Let us introduce some definitions below.

Definition 1. Let $1 \le q < \infty$ and $b \in L_{\mathrm{loc}}(\mathbb{R}^n)$. One says that $b \in C\dot{M}O^q(\mathbb{R}^n)$ if and only if

$$\sup_{r>0}\left(\frac{1}{|B(0,r)|}\int_{B(0,r)}\left|b(x)-b_{B(0,r)}\right|^q dx\right)^{1/q} < \infty, \quad (4)$$

where $b_{B(0,r)} = (1/|B(0,r)|)\int_{B(0,r)} b(x)dx$. The $C\dot{M}O^q$ norm of b is defined by

$$\|b\|_{C\dot{M}O^q(\mathbb{R}^n)} = \sup_{r>0}\left(\frac{1}{|B(0,r)|}\int_{B(0,r)}\left|b(x)-b_{B(0,r)}\right|^q dx\right)^{1/q} < \infty. \quad (5)$$

Remark 2. When $1 \le q < \infty$, $BMO(\mathbb{R}^n) \subseteq C\dot{M}O^q(\mathbb{R}^n)$; when $1 \le p < q < \infty$, $C\dot{M}O^q(\mathbb{R}^n) \subseteq C\dot{M}O^p(\mathbb{R}^n)$. We choose to refer to papers [5, 9].

Definition 3. The Lipschitz space $\mathrm{Lip}_\alpha(\mathbb{R}^n)$ is the space of functions f satisfying

$$\|f\|_{\mathrm{Lip}_\alpha(\mathbb{R}^n)} := \sup_{x,h\in\mathbb{R}^n, h\neq 0}\frac{|f(x+h)-f(x)|}{|h|^\alpha} < \infty, \quad (6)$$

where $0 < \alpha \le 1$.

Remark 4. When $0 < \alpha < 1$, $\mathrm{Lip}_\alpha(\mathbb{R}^n) = \dot{\Lambda}_\alpha(\mathbb{R}^n)$, where $\dot{\Lambda}_\alpha(\mathbb{R}^n)$ is the homogeneous Besov-Lipschitz space.

Definition 5 (see [14]). Let $0 < p \le 1$; a function is called $a(p,\infty,s)$-atom, where $s \ge [n(1/p-1)]$, if it satisfies the following conditions:

$$\begin{aligned} &(1)\ \ \mathrm{supp}(a) \subset B(x_0,r); \\ &(2)\ \ \|a\|_{L^\infty} \le |B(x_0,r)|^{-1/p}; \\ &(3)\ \ \int_{\mathbb{R}^n} a(x)x^\gamma dx = 0, \quad \text{where } 0 \le |\gamma| \le s. \end{aligned} \quad (7)$$

As a proper subspace of $L^p(\mathbb{R}^n)$, the atomic Hardy space $H^p(\mathbb{R}^n)$ is defined by

$$H^p(\mathbb{R}^n) = \left\{f \in S'(\mathbb{R}^n) : f(x) \overset{S'}{=} \sum_k \lambda_k a_k(x), \sum_k |\lambda_k|^p < \infty\right\}, \quad (8)$$

where each a_k is a (p,∞,s)-atom and f is a tempered distribution. Set $H^p(\mathbb{R}^n)$ norm of f by

$$\|f\|_{H^p(\mathbb{R}^n)} := \inf\left\{\left(\sum_k^\infty |\lambda_k|^p\right)^{1/p}\right\}, \quad (9)$$

where the infimum has taken over all the decompositions of $f = \sum_k \lambda_k a_k$ as above.

Given a positive integer m and $1 \le i \le m$, we denote by C_i^m the family of all finite subsets $\sigma = \{\sigma(1),\sigma(2),\ldots,\sigma(i)\}$ of $\{1,2,\ldots,m\}$ of j different elements. For $\sigma \in C_i^m$, set $\sigma^c = \{1,\ldots,m\}\backslash\sigma$. For $\vec{b} = (b_1,b_2,\ldots,b_m)$ and $\sigma = \{\sigma_1,\sigma_2,\ldots,\sigma_i\} \in C_j^m$, set $\vec{b}_\sigma = (b_{\sigma_1},\ldots,b_{\sigma_j})$, $b_\sigma = b_{\sigma_1}\cdots b_{\sigma_j}$, and $\|\vec{b}_\sigma\|_{\mathrm{Lip}_\alpha(\mathbb{R}^n)} = \|b_{\sigma_1}\|_{\mathrm{Lip}_\alpha(\mathbb{R}^n)}\cdots\|b_{\sigma_i}\|_{\mathrm{Lip}_\alpha(\mathbb{R}^n)}$.

Definition 6. Let b_i $(i = 1,2,\ldots,m)$ be a locally integrable functions, and $0 < p \le 1$. A bounded measurable function a on $\mathbb{R}^n$ is called $a(p,\vec{b})$-atom, if

(i) $\mathrm{supp}\, a \subset B = B(x_0,r)$,

(ii) $\|a\|_{L^\infty} \le |B(x_0,r)|^{-1/p}$,

(iii) $\int_B a(y)dy = \int_B a(y)\prod_{j\in\sigma} b_j dy = 0$ for any $\sigma \in C_i^m$, $1 \le i \le m$.

A temperate distribution (see [15, 16]) f is said to belong to $H_{\vec{b}}^p(\mathbb{R}^n)$, if, in the Schwartz distribution sense, it can be written as

$$f(x) = \sum_i \lambda_i a_i(x), \quad (10)$$

where a_i is $(p,\vec{b})$-atom, $\lambda_i \in C$, and $\sum_i |\lambda_i|^p < \infty$. Moreover, $\|f\|_{H_{\vec{b}}^p} = \inf(\sum_i |\lambda_i|^p)^{1/p}$, where the infimum has taken over all the decompositions of f as above.

Remark 7. When $m = 1$, $H_{\vec{b}}^1(\mathbb{R}^n) = H_b^1(\mathbb{R}^n)$.

Let $B_k = \{x \in \mathbb{R} : |x| < 2^k\}$ and $E_k = B_k \setminus B_{k-1}$ for $k \in \mathbb{Z}$. Denote $\chi_k = \chi_{E_k}$.

Definition 8. Let $\gamma \in \mathbb{R}$, $0 < p$ and $q \le \infty$.

(i) The homogeneous Herz space $\dot{K}_q^{\gamma,p}(\mathbb{R}^n)$ is defined by

$$\dot{K}_q^{\gamma,p}(\mathbb{R}^n) = \left\{f : f \in L_{\mathrm{loc}}^q(\mathbb{R}^n \setminus \{0\}), \|f\|_{\dot{K}_q^{\gamma,p}} < \infty\right\}, \quad (11)$$

where

$$\|f\|_{\dot{K}_q^{\gamma,p}(\mathbb{R}^n)} = \left\{\sum_{k\in\mathbb{Z}} 2^{k\gamma p}\|f\chi_k\|_{L^q(\mathbb{R}^n)}^p\right\}^{1/p}. \quad (12)$$

(ii) The nonhomogeneous Herz space $K_q^{\gamma,p}(\mathbb{R}^n)$ is defined by

$$K_q^{\gamma,p}(\mathbb{R}^n) = \left\{f : f \in L_{\mathrm{loc}}^q(\mathbb{R}^n \setminus \{0\}), \|f\|_{\dot{K}_q^{\gamma,p}} < \infty\right\}, \quad (13)$$

where

$$\|f\|_{K_q^{\gamma,p}(\mathbb{R}^n)} = \left\{\sum_{k=0}^\infty 2^{k\gamma p}\|f\chi_k\|_{L^q(\mathbb{R}^n)}^p\right\}^{1/p} \quad (14)$$

(the usual modifications are made when $p = \infty$).

Remark 9. When $0 < q \le \infty$, $\dot{K}_q^{0,q}(\mathbb{R}^n) = K_q^{0,q}(\mathbb{R}^n) = L^q(\mathbb{R}^n)$, and $\dot{K}_q^{\gamma/q}(\mathbb{R}^n) = L^q_{|x|^\gamma}(\mathbb{R}^n)$.

Definition 10 (see [15]). Let $\gamma \in \mathbb{R}$, and $0 < p < q < \infty$.

(i) The homogeneous Herz-type Hardy space $H\dot{K}_q^{\gamma,p}(\mathbb{R}^n)$ is defined by

$$H\dot{K}_q^{\gamma,p}(\mathbb{R}^n) = \left\{ f \in S'(\mathbb{R}^n) : G(f) \in \dot{K}_q^{\gamma,p}(\mathbb{R}^n) \right\}, \quad (15)$$

and we define $\|f\|_{H\dot{K}_q^{\gamma,p}(\mathbb{R}^n)} = \|G(f)\|_{\dot{K}_q^{\gamma,p}(\mathbb{R}^n)}$.

(ii) The nonhomogeneous Herz-type Hardy space $HK_q^{\gamma,p}(\mathbb{R}^n)$ is defined by

$$HK_q^{\gamma,p}(\mathbb{R}^n) = \left\{ f \in S'(\mathbb{R}^n) : G(f) \in K_q^{\gamma,p}(\mathbb{R}^n) \right\}, \quad (16)$$

and we define $\|f\|_{HK_q^{\gamma,p}(\mathbb{R}^n)} = \|G(f)\|_{K_q^{\gamma,p}(\mathbb{R}^n)}$.

Remark 11. When $0 < p < \infty$, $H\dot{K}_p^{0,p}(\mathbb{R}^n) = K_p^{0,p}(\mathbb{R}^n) = H^p(\mathbb{R}^n)$ and $H\dot{K}_p^{\gamma/p}(\mathbb{R}^n) = L^p_{|x|^\gamma}(\mathbb{R}^n)$. And when $1 < q < \infty$, we know that $H\dot{K}_q^{\gamma,p}(\mathbb{R}^n) = \dot{K}_q^{\gamma,p}(\mathbb{R}^n)$ and $HK_q^{\gamma,p}(\mathbb{R}^n) = K_q^{\gamma,p}(\mathbb{R}^n)$, where $-n/q < \gamma < n(1-1/q)$. However, when $\gamma \ge n(1-1/q)$, $H\dot{K}_q^{\gamma,p}(\mathbb{R}^n) \ne \dot{K}_q^{\gamma,p}(\mathbb{R}^n)$ and $HK_q^{\gamma,p}(\mathbb{R}^n) \ne K_q^{\gamma,p}(\mathbb{R}^n)$ (see [17, 18]).

2. $(H^p(\mathbb{R}^n), L^q(\mathbb{R}^n))$ Bounds of Fractional Hardy Operators

Theorem 12. *Let $0 < p \le 1$ and $1/q = 1/p - \beta/n$. $\mathscr{H}_\beta$ maps $H^p(\mathbb{R}^n)$ into $L^q(\mathbb{R}^n)$.*

Proof. Assume that a is an atom of $H^p(\mathbb{R}^n)$ and satisfies the following conditions: (i) $\operatorname{supp}(a) \subset B(x_0, r)$, (ii) $\|a\|_{L^\infty} \le |B(x_0,r)|^{-1/p}$, and (iii) $\int a(x)x^\gamma dx = 0$, where $0 \le |\gamma| \le s$, $s \ge [n(1/p-1)]$. We now take $\tilde{a}(x) = a(x+x_0)$; then $\tilde{a}$ satisfies: (i) $\operatorname{supp}(\tilde{a}) \subset B(0,r)$, (ii) $\|\tilde{a}\|_{L^\infty} \le |B(0,r)|^{-1/p}$, and (iii) $\int \tilde{a}(x)dx = 0$.

Suppose that $\operatorname{supp}\tilde{a} \subset B(0,r)$ for $r > 0$. Consider

$$\begin{aligned}
&\int_{\mathbb{R}^n} \left|\mathscr{H}_\beta(\tilde{a})(x)\right|^q dx \\
&= \int_{\mathbb{R}^n} \left| \frac{1}{|x|^{n-\beta}} \int_{|y|<|x|} \tilde{a}(y)\,dy \right|^q dx \\
&= \int_{B(0,r)} \left| \frac{1}{|x|^{n-\beta}} \int_{|y|<|x|} \tilde{a}(y)\,dy \right|^q dx \\
&\quad + \int_{\mathbb{R}^n\setminus B(0,r)} \left| \frac{1}{|x|^{n-\beta}} \int_{|y|<|x|} \tilde{a}(y)\,dy \right|^q dx \\
&= \int_{B(0,r)} \left| \frac{1}{|x|^{n-\beta}} \int_{\{|y|<|x|\}\cap B(0,r)} \tilde{a}(y)\,dy \right|^q dx \\
&\quad + \int_{\mathbb{R}^n\setminus B(0,r)} \left| \frac{1}{|x|^{n-\beta}} \int_{\{|y|<|x|\}\cap B(0,r)} \tilde{a}(y)\,dy \right|^q dx \\
&:= I_1 + I_2,
\end{aligned} \quad (17)$$

where we used the condition $\operatorname{supp}(a) \subset B(0,r)$. For I_1, we have the following estimate

$$\begin{aligned}
I_1 &\le \int_{B(0,r)} \frac{1}{|x|^{(n-\beta)q}} \left(\int_{\{|y|<|x|\}\cap B(0,r)} |\tilde{a}(y)|\,dy \right)^q dx \\
&\le \int_{B(0,r)} \frac{1}{|x|^{(n-\beta)q}} \frac{1}{|B(0,r)|^{q/p}} \left(\int_{|y|<|x|} dy \right)^q dx \\
&\le C \frac{1}{|B(0,r)|^{q/p}} \int_{B(0,r)} \frac{1}{|x|^{(n-\beta)q}} |x|^{nq} dx \\
&\le C \frac{1}{|B(0,r)|^{q/p}} \int_{B(0,r)} |x|^{\beta q} dx \\
&\le C \frac{1}{r^{nq/p}} \cdot r^{n+\beta q} \\
&\le C < \infty,
\end{aligned} \quad (18)$$

where we used the condition $\|\tilde{a}\|_{L^\infty} \le |B(0,r)|^{-1/p}$ and $1/q = (1/p) - (\beta/n)$. For I_2, since $x \in \mathbb{R}^n \setminus B(0,r)$, we have $\{y \in \mathbb{R}^n : |y| < |x|\} \supset \{y \in \mathbb{R}^n : y \in B(0,r)\}$. Then

$$\begin{aligned}
I_2 &= \int_{\mathbb{R}^n\setminus B(0,r)} \left| \frac{1}{|x|^{n-\beta}} \int_{\{|y|<|x|\}\cap B(0,r)} \tilde{a}(y)\,dy \right|^q dx \\
&= \int_{\mathbb{R}^n\setminus B(0,r)} \left| \frac{1}{|x|^{n-\beta}} \int_{B(0,r)} \tilde{a}(y)\,dy \right|^q dx = 0.
\end{aligned} \quad (19)$$

The proof is completed. □

3. Estimates for Commutators of Fractional Hardy Operators on Hardy Spaces

Definition 13 (see [5, 19]). Let b be a locally integrable function on $\mathbb{R}^n$. The commutator of n-dimensional Hardy operators is defined by

$$[b, \mathscr{H}] f = b\mathscr{H} f - \mathscr{H}(bf). \quad (20)$$

Meanwhile, the commutators of n-dimensional fractional Hardy operators are defined by

$$[b, \mathscr{H}_\beta] f = b\mathscr{H}_\beta f - \mathscr{H}_\beta(bf). \quad (21)$$

In general, the properties of commutator are worse than those of the operators themselves (e.g., the Hardy operators [11] and the singular integral operators [20]). Therefore, when b is in $\operatorname{Lip}_\alpha(\mathbb{R}^n)$, we prove that $[b, \mathscr{H}_\beta]$ is not bounded from $H^1(\mathbb{R}^n)$ to $L^{n/(n-\beta-\alpha)}(\mathbb{R}^n)$. Furthermore, we conclude that the commutator maps from $H^1(\mathbb{R}^n)$ to $L^{n/(n-\beta-\alpha),\infty}(\mathbb{R}^n)$ and the commutator maps $H^p_b(\mathbb{R}^n)$ into $L^q(\mathbb{R}^n)$, where $0 < p \le 1$ and $1/q = 1/p - (\beta+\alpha)/n$.

Proposition 14. *If $b \in \operatorname{Lip}_\alpha(\mathbb{R}^n)$, $q = n/(n-\beta-\alpha)$ and $0 < q < \infty$, then $[b, \mathscr{H}_\beta]$ is not bounded from $H^1(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$.*

Proof. We give the proof only for the case $n = 1$, then $q = 1/(1-\beta-\alpha)$. Taking $b(x) = (|x|^\alpha - 2^\alpha)\chi_{(2,\infty)}(x)$ and $f_0(x) = \chi_{(0,2)}(x) - \chi_{(-2,0)}(x)$, it is easy to see that $b \in \mathrm{Lip}_\alpha(\mathbb{R}^n)$. Then for $x > 3$, we have the following estimate:

$$\begin{aligned} &\left|\left[b, \mathscr{H}_\beta\right] f(x)\right| \\ &= \left|\frac{1}{|x|^{1-\beta}} \int_0^x (b(x) - b(y)) f_0(y)\, dy\right| \\ &= \frac{1}{x^{1-\beta}} \int_0^2 (x^\alpha - 2^\alpha - 0) \times 1 dy \geq \frac{C_\alpha}{x^{1-\beta-\alpha}}. \end{aligned} \tag{22}$$

Here C_α is the constant dependent on α. So we get

$$\int_{\mathbb{R}^1} \left|\left[b, \mathscr{H}_\beta\right] f(x)\right|^q dx \geq \int_3^\infty \left(\frac{2}{x^{1-\beta-\alpha}}\right)^q dx = \infty. \tag{23}$$

□

Theorem 15. *Let $b \in \mathrm{Lip}_\alpha(\mathbb{R}^n)$, $q = n/(n-\beta-\alpha)$, and $0 < q < \infty$ then $[b, \mathscr{H}_\beta]$ maps $H^1(\mathbb{R}^n)$ into $L^{q,\infty}(\mathbb{R}^n)$.*

Proof. It is enough to prove that

$$\begin{aligned} &\lambda \left|\left\{x \in \mathbb{R}^n : \left|\left[b, \mathscr{H}_\beta\right](\widetilde{a})(x)\right| > \lambda\right\}\right|^{1/q} \\ &\quad \leq C\|b\|_{\mathrm{Lip}_\alpha(\mathbb{R}^n)} \|\widetilde{a}\|_{H^1(\mathbb{R}^n)} \end{aligned} \tag{24}$$

holds for any $\widetilde{a}$ defined in the proof of Theorem 12 and the constant C is independent of $\widetilde{a}$. Suppose that $\operatorname{supp} \widetilde{a} \subset B(0,r)$ for $r > 0$. Then

$$\begin{aligned} &\left|\left\{x \in \mathbb{R}^n : \left|\left[b, \mathscr{H}_\beta\right](\widetilde{a})(x)\right| > \lambda\right\}\right|^{1/q} \\ &\quad \leq \max\left\{1, n^{(1/q)-1}\right\} \\ &\quad\quad \times \left\{\left|\left\{x \in B(0,r) : \left|\left[b, \mathscr{H}_\beta\right](\widetilde{a})(x)\right| > \frac{\lambda}{2}\right\}\right|^{1/q}\right. \\ &\quad\quad\quad \left. + \left|\left\{x \in \mathbb{R}^n \setminus B(0,r) : \left|\left[b, \mathscr{H}_\beta\right](\widetilde{a})(x)\right| > \frac{\lambda}{2}\right\}\right|^{1/q}\right\}. \end{aligned} \tag{25}$$

For the first term, $|y| < |x|$ implies $|x - y| \leq |x| + |y| \leq 2|x|$; then we have

$$\begin{aligned} &\lambda^q \left|\left\{x \in B(0,r) : \left|\left[b, \mathscr{H}_\beta\right](\widetilde{a})(x)\right| > \frac{\lambda}{2}\right\}\right| \\ &\quad \leq C \int_{B(0,r)} \left|\left[b, \mathscr{H}_\beta\right](\widetilde{a})(x)\right|^q dx \\ &\quad \leq C \int_{B(0,r)} \left|\frac{1}{|x|^{n-\beta}} \int_{|y|<|x|} \widetilde{a}(y)(b(x) - b(y))\, dy\right|^q dx \\ &\quad \leq C\|b\|^q_{\mathrm{Lip}_\alpha(\mathbb{R}^n)} \\ &\quad\quad \times \int_{B(0,r)} \left(\frac{1}{|x|^{n-\beta}} \int_{|y|<|x|} |\widetilde{a}(y)|\, |x-y|^\alpha dy\right)^q dx \\ &\quad \leq C\|b\|^q_{\mathrm{Lip}_\alpha(\mathbb{R}^n)} \int_{B(0,r)} \frac{1}{|B(0,r)|^q} \\ &\quad\quad \times \left(\frac{1}{|x|^{n-\beta}} \int_{|y|<|x|} |x-y|^\alpha dy\right)^q dx \\ &\quad \leq C\|b\|^q_{\mathrm{Lip}_\alpha(\mathbb{R}^n)} \frac{1}{|B(0,r)|^q} \int_{B(0,r)} |x|^{\beta q + \alpha q} dx \\ &\quad \leq C\|b\|^q_{\mathrm{Lip}_\alpha(\mathbb{R}^n)} r^{-nq} \cdot r^{n+\alpha q+\beta q} \\ &\quad = C\|b\|^q_{\mathrm{Lip}_\alpha(\mathbb{R}^n)}, \end{aligned} \tag{26}$$

where $q = n/(n-\beta-\alpha)$. For the last term, by $x \in \mathbb{R}^n \setminus B(0,r)$, we have

$$\begin{aligned} &\left|\frac{1}{|x|^{n-\beta}} \int_{|y|<r} (b(x) - b(y))\, \widetilde{a}(y)\, dy\right| \\ &\quad \leq \frac{1}{|x|^{n-\beta}} \int_{|y|<r} |b(x) - b(y)|\, |\widetilde{a}(y)|\, dy \\ &\quad \leq C\|b\|_{\mathrm{Lip}_\beta(\mathbb{R}^n)} \frac{1}{|x|^{n-\beta}} \int_{|y|<r} |x-y|^\alpha \frac{1}{|B(0,r)|} dy \\ &\quad \leq C\|b\|_{\mathrm{Lip}_\beta(\mathbb{R}^n)} \frac{1}{|x|^{n-\beta}} \int_{|y|<r} |x|^\alpha \frac{1}{|B(0,r)|} dy \\ &\quad \leq C\|b\|_{\mathrm{Lip}_\beta(\mathbb{R}^n)} \frac{1}{|x|^{n-\beta-\alpha}}. \end{aligned} \tag{27}$$

So we obtain that

$$\begin{aligned} &\left|\left\{x \in \mathbb{R}^n \setminus B(0,r) : \left|\left[b, \mathscr{H}_\beta\right](\widetilde{a})(x) > \frac{\lambda}{2}\right|\right\}\right| \\ &\quad \leq \left|\left\{x \in \mathbb{R}^n \setminus B(0,r) : C\|b\|_{\mathrm{Lip}_\beta(\mathbb{R}^n)} \frac{1}{|x|^{n-\beta-\alpha}} > \lambda\right\}\right| \\ &\quad = C \int_r^{(C\|b\|_{\mathrm{Lip}_\beta(\mathbb{R}^n)}/\lambda)^{1/(n-\alpha-\beta)}} t^{n-1} dt \\ &\quad \leq C\left(\frac{\|b\|_{\mathrm{Lip}_\beta(\mathbb{R}^n)}}{\lambda}\right)^{n/(n-\alpha-\beta)} \\ &\quad = C\left(\frac{\|b\|_{\mathrm{Lip}_\beta(\mathbb{R}^n)}}{\lambda}\right)^q. \end{aligned} \tag{28}$$

Combining all the above estimates, we complete the proof of Theorem 15. □

Theorem 16. *Let $b \in \mathrm{Lip}_\alpha(\mathbb{R}^n)$, $0 < p \leq 1$ and $1/q = 1/p - (\beta+\alpha)/n$ and $0 < q < \infty$; then $[b, \mathscr{H}_\beta]$ maps $H^p_b(\mathbb{R}^n)$ into $L^q(\mathbb{R}^n)$.*

Proof. Similar to the proof above, suppose that $\operatorname{supp} \widetilde{a} \subset B(0,r)$ for $r > 0$. Consider

$$\begin{aligned} &\int_{\mathbb{R}^n} \left|\left[b, \mathscr{H}_\beta\right](\widetilde{a})(x)\right|^q dx \\ &\quad = \int_{\mathbb{R}^n} \left|\frac{1}{|x|^{n-\beta}} \int_{|y|<|x|} \widetilde{a}(y)(b(x) - b(y))\, dy\right|^q dx \\ &\quad = \int_{B(0,r)} \left|\frac{1}{|x|^{n-\beta}}\right. \\ &\quad\quad \times \left.\int_{\{|y|<|x|\}\cap B(0,r)} \widetilde{a}(y)(b(x) - b(y))\, dy\right|^q dx \end{aligned}$$

$$
\begin{aligned}
&+\int_{\mathbb{R}^n\setminus B(0,r)}\left|\frac{1}{|x|^{n-\beta}}\int_{\{|y|<|x|\}\cap B(0,r)}\widetilde{a}(y)\right.\\
&\qquad\left.\times(b(x)-b(y))\,dy\right|^q dx\\
&\leq\int_{B(0,r)}\left|\frac{1}{|x|^{n-\beta}}\right.\\
&\qquad\left.\times\int_{\{|y|<|x|\}\cap B(0,r)}\widetilde{a}(y)(b(x)-b(y))\,dy\right|^q dx\\
&\quad+C\int_{\mathbb{R}^n\setminus B(0,r)}\left|\frac{1}{|x|^{n-\beta}}\int_{\{|y|<|x|\}\cap B(0,r)}\widetilde{a}(y)b(x)\,dy\right|^q dx\\
&\quad+C\int_{\mathbb{R}^n\setminus B(0,r)}\left|\frac{1}{|x|^{n-\beta}}\int_{\{|y|<|x|\}\cap B(0,r)}\widetilde{a}(y)b(y)\,dy\right|^q dx\\
&:=J_1+J_2+J_3.
\end{aligned}
\tag{29}
$$

For J_1, by the fact that $|y|<|x|$ implies $|x-y|\leq|x|+|y|\leq 2|x|$, we have that

$$
\begin{aligned}
J_1&=\int_{B(0,r)}\left|\frac{1}{|x|^{n-\beta}}\int_{\{|y|<|x|\}\cap B(0,r)}\widetilde{a}(y)(b(x)-b(y))\,dy\right|^q dx\\
&\leq C\|b\|^q_{\mathrm{Lip}_\alpha(\mathbb{R}^n)}\\
&\quad\times\int_{B(0,r)}\left|\frac{1}{|x|^{n-\beta}}\frac{1}{|B(0,r)|^{1/p}}\int_{|y|<|x|}|x|^\alpha dy\right|^q dx\\
&\leq C\|b\|^q_{\mathrm{Lip}_\alpha(\mathbb{R}^n)}\int_{B(0,r)}\left||x|^{\beta+\alpha}\frac{1}{|B(0,r)|^{1/p}}\right|^q dx\\
&\leq C\|b\|^q_{\mathrm{Lip}_\alpha(\mathbb{R}^n)}.
\end{aligned}
\tag{30}
$$

By $\int\widetilde{a}(y)dy=\int\widetilde{a}(y)b(y)dy=0$, we obtain that $J_2=J_3=0$. Then $[b,\mathscr{H}_\beta]$ maps $H^p_b(\mathbb{R}^n)$ into $L^q(\mathbb{R}^n)$. □

Theorem 17. *Let $b\in\mathrm{Lip}_\alpha(\mathbb{R}^n)$, $1/q=1/p-(\beta+\alpha)/n$, and $0<p\leq n/(n+\alpha)$; $0<q<\infty$, then the following two conditions are equivalent:*

(i) *$[b,\mathscr{H}_\beta]$ is bounded from $H^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$;*

(ii) *for all $a(p,\infty,0)$-atom, $\int_{\mathbb{R}^n}a(y)b(y)dy=0$.*

Proof. By Theorem 16, it is clear that (ii) ⇒ (i) is obvious. We only need to prove (i) ⇒ (ii). We assume that $\operatorname{supp}a\subset B=B(x_0,r)$ for $r>0$. Let

$$
\begin{aligned}
v_1(x)&=\chi_B\left[b,\mathscr{H}_\beta\right]a(x),\\
v_2(x)&=\chi_{\mathbb{R}^n\setminus B}(x)\,b(x)\frac{1}{|x|^{n-\beta}}\int_B a(y)\,dy,\\
v_3(x)&=\chi_{\mathbb{R}^n\setminus B}(x)\frac{1}{|x|^{n-\beta}}\int_B a(y)\,b(y)\,dy.
\end{aligned}
\tag{31}
$$

Then

$$\left[b,\mathscr{H}_\beta\right]a(x)=v_1(x)+v_2(x)-v_3(x). \tag{32}$$

Obviously, $\|v_1\|_q\leq C$, and $\|v_2\|_q=0$. For $N>2$,

$$
\begin{aligned}
C&\geq\|v_3\|_q\\
&\geq\left|\int_B a(y)\,b(y)\,dy\right|\left\{\int_{r<|x-x_0|\leq Nr}\frac{1}{|x|^{(n-\beta)q}}dx\right\}^{1/q}.
\end{aligned}
\tag{33}
$$

By $0<p\leq n/(n+\alpha)$, we have $(n-\beta)q\leq n$.

When $N\to\infty$, $\int_{r<|x-x_0|\leq Nr}(1/|x|^{(n-\beta)q})dx\to\infty$; then

$$\int_B a(y)\,b(y)\,dy=0. \tag{34}$$

□

When b is in $C\dot{M}O(\mathbb{R}^n)$, the commutator $[b,\mathscr{H}_\beta]$ has the similar properties.

Proposition 18. *If $b\in BMO(\mathbb{R}^n)$ and $q=n/(n-\beta)$, then $[b,\mathscr{H}_\beta]$ is not bounded from $H^1(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$.*

Theorem 19. *Let $b\in C\dot{M}O^{\max\{sq,s\}}(\mathbb{R}^n)$ and $q=n/(n-\beta)$, $1<s<\infty$, then $[b,\mathscr{H}_\beta]$ maps $H^1(\mathbb{R}^n)$ into $L^{q,\infty}(\mathbb{R}^n)$.*

Remark 20. In 2013, Yu and Lu [21] have given the analogous results of Proposition 18 and Theorem 19.

Theorem 21. *Let $b\in C\dot{M}O^{\max\{sq,s\}}(\mathbb{R}^n)$, $0<p\leq 1$ and $1<s<\infty$, $1/q=1/p-\beta/n$; then $[b,\mathscr{H}_\beta]$ is bounded from $H^p_b(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$.*

Theorem 22. *Let $b\in C\dot{M}O^{\max\{sq,s\}}(\mathbb{R}^n)$ and $1/q=1/p-\beta/n$, $1<s<\infty$, and $0<p\leq 1$; then the following two conditions are equivalent:*

(i) *$[b,\mathscr{H}_\beta]$ maps $H^p(\mathbb{R}^n)$ into $L^q(\mathbb{R}^n)$;*

(ii) *for all $a(p,\infty,0)$-atom, $\int_{\mathbb{R}^n}a(y)b(y)dy=0$.*

4. Estimates for Commutators of Fractional Hardy Operators on Herz-Type Hardy Spaces

The boundedness of $[b,\mathscr{H}_\beta]$ on the Herz spaces $\dot{K}^{\gamma,p}_q(\mathbb{R}^n)$ has been obtained as the following, where $\gamma<n(1-1/q_1)$.

Proposition 23 (see [22]). *Let $\beta\geq 0$, $0<p_1\leq p_2<\infty$, $1/q_2=1/q_1-(\beta+\alpha)/n$, $1<q_1,q_2<\infty$, $1/q_1+1/q_1'=1$ and $b\in\mathrm{Lip}_\alpha(\mathbb{R}^n)$; then $\gamma<n/q_1'$ implies that $[b,\mathscr{H}_\beta]$ is bounded from $\dot{K}^{\gamma,p_1}_{q_1}(\mathbb{R}^n)$ to $\dot{K}^{\gamma,p_2}_{q_2}(\mathbb{R}^n)$.*

Proposition 24 (see [5]). *Let $\beta\geq 0$, $0<p_1\leq p_2<\infty$, $1/q_2=1/q_1-\beta/n$, $1<q_1<\infty$, $1/q_1+1/q_1'=1$ and $b\in C\dot{M}O^{\max(q_2,q_1')}(\mathbb{R}^n)$; then $\gamma<n/q_1'$ implies that $[b,\mathscr{H}_\beta]$ is bounded from $\dot{K}^{\gamma,p_1}_{q_1}(\mathbb{R}^n)$ to $\dot{K}^{\gamma,p_2}_{q_2}(\mathbb{R}^n)$.*

In this section, we discuss the case $\gamma\geq n(1-1/q_1)$.

Definition 25. Let $\gamma \in \mathbb{R}$ and $1 < q < \infty$. A function is called central (γ, q)-atom, if it satisfies the following conditions:

(i) $\operatorname{supp} a \subset B(0, r)$;

(ii) $\|a\|_q \leq |B(0, r)|^{-\gamma/n}$;

(iii) $\int_{\mathbb{R}^n} a(x)x^s dx = 0$, where $|s| \leq [\gamma - n(1 - 1/q)]$.

Lemma 26 (see [18]). *Let $0 < p < \infty$, $1 < q < \infty$ and $\gamma \geq n(1 - 1/q)$. Consider that $f \in H\dot{K}_q^{\gamma,p}(\mathbb{R}^n)$ if and only if*

$$f = \sum_{k \in \mathbb{Z}} \lambda_k a_k, \quad \text{in the sense of } S'(\mathbb{R}^n), \tag{35}$$

where each a_k is a central (γ, q)-atom with the support B_k and $\sum_{k \in \mathbb{Z}} |\lambda_k|^p < \infty$. Moreover,

$$\|f\|_{H\dot{K}_q^{\gamma,p}(\mathbb{R}^n)} \sim \inf \left\{ \left(\sum_{k \in \mathbb{Z}} |\lambda_k|^p \right)^{1/p} \right\}, \tag{36}$$

where the infimum has taken over all above decompositions of f.

Theorem 27. *Let $b \in \operatorname{Lip}_\alpha(\mathbb{R}^n)$, $\gamma \geq n(1 - 1/q_1) + \alpha$, $0 < p < \infty$, $1 < q_1, q_2 < \infty$, $1/q_2 = 1/q_1 - (\beta + \alpha)/n$, then the following two conditions are equivalent:*

(i) *$[b, \mathscr{H}_\beta]$ maps $H\dot{K}_{q_1}^{\gamma,p}(\mathbb{R}^n)$ into $\dot{K}_{q_2}^{\gamma,p}(\mathbb{R}^n)$;*

(ii) *for all central $a(\gamma, q_1)$-atom, $\int_{\mathbb{R}^n} b(y)a(y)dy = 0$.*

Proof. (i) $\Rightarrow$ (ii). Let a be a central (γ, q_1)-atom, with the support B_k. In fact, if $\operatorname{supp} a(x) \subset B(0, r)$, let $k \in \mathbb{Z}$ such that $2^k - 1 < r \leq 2^k$, and for $C(n, q_1, \alpha) = 2^{-\gamma}$, set $a' = C(n, q_1, \alpha)a$; then a' is a central (γ, q_1)-atom with the support B_k.

Similar to the proof of Theorem 16, we have

$$\begin{aligned} v_1(x) &= \chi_{B_k} \left[b, \mathscr{H}_\beta\right] a(x), \\ v_2(x) &= \chi_{\mathbb{R}^n \setminus B_k}(x)\, b(x) \frac{1}{|x|^{n-\beta}} \int_B a(y)\, dy, \\ v_3(x) &= \chi_{\mathbb{R}^n \setminus B_k}(x) \frac{1}{|x|^{n-\beta}} \int_B a(y)\, b(y)\, dy. \end{aligned} \tag{37}$$

Then

$$\left[b, \mathscr{H}_\beta\right] a(x) = v_1(x) + v_2(x) - v_3(x). \tag{38}$$

Using the fact that $[b, \mathscr{H}_\beta]$ maps $L^{q_1}(\mathbb{R}^n)$ into $L^{q_2}(\mathbb{R}^n)$

$$\begin{aligned} \|v_1\|_{\dot{K}_q^{\gamma,p}(\mathbb{R}^n)} &\leq \left(\sum_{j=-\infty}^{k+1} 2^{j\gamma p} \left\| \chi_j \left(\left[b, \mathscr{H}_\beta\right] a \right) \right\|_{q_2}^p \right)^{1/p} \\ &\leq C \left(\sum_{j=-\infty}^{k+1} 2^{j\gamma p} \|a\|_{q_1}^p \right)^{1/p} \\ &\leq C. \end{aligned} \tag{39}$$

When $|x| \geq 2^{k+1}$, we have $\|v_2\|_{\dot{K}_{q_2}^{\gamma,p}(\mathbb{R}^n)} = 0$. On the other hand, $\|a\|_{H\dot{K}_{q_1}^{\gamma,p}(\mathbb{R}^n)} \leq 1$, so $\|[b, \mathscr{H}_\beta]a\|_{\dot{K}_{q_2}^{\gamma,p}} \leq C$. Combining all the above estimates, we have $\|v_3\|_{\dot{K}_{q_2}^{\gamma,p}(\mathbb{R}^n)} \leq C$. For $N > 3$, by $\gamma \geq n(1 - 1/q_1 + \alpha)$, we have that

$$\begin{aligned} C^p &\geq \sum_{j=k+2}^{\infty} 2^{j\gamma p} \left(\int_{E_j} \left| \frac{1}{|x|^{n-\beta}} \int_{|y|<|x|} b(y)\, a(y)\, dy \right|^{q_2} dx \right)^{p/q_2} \\ &\geq \sum_{j=k+3}^{k+N} 2^{j\gamma p} \left(\left| \int_{\mathbb{R}^n} b(y)\, a(y)\, dy \right| \int_{E_j} |x|^{(\beta-n)q_2} dx \right)^{p/q_2} \\ &\geq C \sum_{j=k+3}^{k+N} 2^{j(\gamma - n(1-1/q_1) + \alpha)p} \left| \int_{\mathbb{R}^n} b(y)\, a(y)\, dy \right|^p. \end{aligned} \tag{40}$$

Set $N \to \infty$; then

$$\int_{\mathbb{R}^n} b(y)\, a(y)\, dy = 0. \tag{41}$$

(ii) $\Rightarrow$ (i). Similar to the previous proof, by $\int_{\mathbb{R}^n} b(y)a(y)dy = 0$, we obtain $\|v_3\|_{\dot{K}_{q_2}^{\gamma,p}(\mathbb{R}^n)} = 0$. Combining $\|v_1\|_{\dot{K}_{q_2}^{\gamma,p}(\mathbb{R}^n)} \leq C$ and $\|v_2\|_{\dot{K}_{q_2}^{\gamma,p}(\mathbb{R}^n)} = 0$, $[b, \mathscr{H}_\beta]$ maps $H\dot{K}_{q_1}^{\gamma,p}(\mathbb{R}^n)$ into $\dot{K}_{q_2}^{n(1-1/q_1)+\beta+\alpha,p}(\mathbb{R}^n)$. □

When b is in $C\dot{M}O(\mathbb{R}^n)$, the commutator $[b, \mathscr{H}_\beta]$ also has the similar properties.

Theorem 28. *Let $b \in C\dot{M}O^{\max\{sq,s\}}(\mathbb{R}^n)$, $\gamma \geq n(1 - 1/q_1)$, $1 < s < \infty$, $0 < p < \infty$, $1 < q_1, q_2 < \infty$ and $1/q_2 = 1/q_1 - \beta/n$; then the following two conditions are equivalent:*

(i) *$[b, \mathscr{H}_\beta]$ maps $H\dot{K}_{q_1}^{\gamma,p}(\mathbb{R}^n)$ into $\dot{K}_{q_2}^{\gamma,p}(\mathbb{R}^n)$;*

(ii) *for all central $a(\gamma, q_1)$-atom, $\int_{\mathbb{R}^n} b(y)a(y)dy = 0$.*

5. Estimates for Multilinear Commutators of Fractional Hardy Operators

Definition 29. The multilinear commutator of fractional Hardy operators is defined by

$$\left[\vec{b}, \mathscr{H}_\beta\right] = \frac{1}{|x|^{n-\beta}} \int_{|y|<|x|} \prod_{i=1}^{m} \left(b_i(x) - b_i(y)\right) f(y)\, dy, \tag{42}$$

where $\vec{b} = (b_1, b_2, \ldots, b_m)$ is a m-dimensional vector. When $m = 0$, $[\vec{b}, \mathscr{H}_\beta] = \mathscr{H}_\beta$. When $m = 1$, $[\vec{b}, \mathscr{H}_\beta] = [b, \mathscr{H}_\beta]$.

The study of multilinear operators is motivated by a mere quest to generalize the theory of linear operators and by their natural appearance in analysis (see [23–25]). In this section, we consider the multilinear commutators of fractional Hardy operators on Hardy spaces.

Theorem 30. *Let $\vec{b} = (b_1, b_2, \ldots, b_m)$, $b_i \in \operatorname{Lip}_\alpha(\mathbb{R}^n)$, $1/q = 1/p - (\beta + m\alpha)/n$, $0 < p \leq 1$ and $0 < q < \infty$; then $[\vec{b}, \mathscr{H}_\beta]$ maps $H_{\vec{b}}^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$.*

Proof. Similar to the proof above, when $x \in \mathbb{R}^n \setminus B(0,r)$, we have

$$\int_{\mathbb{R}^n \setminus B(0,r)} \left|\left[\vec{b}, \mathscr{H}_\beta\right](\tilde{a})(x)\right|^q dx = 0, \tag{43}$$

so it is enough to prove that

$$\left(\int_{B(0,r)} \left|\left[\vec{b}, \mathscr{H}_\beta\right](\tilde{a})(x)\right|^q dx\right)^{1/q} \le C\|b\|_{\mathrm{Lip}_\alpha(\mathbb{R}^n)}\|\tilde{a}\|_{H^1_{\vec{b}}}. \tag{44}$$

By $b_i \in \mathrm{Lip}_\alpha(\mathbb{R}^n)$, we obtain $|b_i(x) - b_i(y)| \le \|b_i\|_{\mathrm{Lip}_\alpha(\mathbb{R}^n)}|x - y|^\alpha$, $i = 1, 2, \ldots, m$, and

$$\begin{aligned}
&\left(\int_{B(0,r)} \left|\left[\vec{b}, \mathscr{H}_\beta\right](\tilde{a})(x)\right|^q dx\right)^{1/q} \\
&\le \left(\int_{B(0,r)} \left|\frac{1}{|x|^{n-\beta}} \int_{|y|<|x|} \tilde{a}(y) \times \prod_{i=1}^{m} (b_i(x) - b_i(y))\, dy\right|^q dx\right)^{1/q} \\
&\le \prod_{i=1}^{m} \|b_i\|_{\mathrm{Lip}_\alpha(\mathbb{R}^n)} \times \left(\int_{B(0,r)} \left|\frac{1}{|x|^{n-\beta}} \times \int_{|y|<|x|} |\tilde{a}(y)|\,|x-y|^{m\alpha} dy\right|^q dx\right)^{1/q} \\
&\le C\prod_{i=1}^{m} \|b_i\|_{\mathrm{Lip}_\alpha(\mathbb{R}^n)} \times \left(\int_{B(0,r)} \left|\frac{1}{|x|^{n-\beta}} \times \int_{|y|<|x|} r^{-n/p} \cdot 2|x|^{m\alpha} dy\right|^q dx\right)^{1/q} \\
&\le C\prod_{i=1}^{m} \|b_i\|_{\mathrm{Lip}_\alpha(\mathbb{R}^n)} \times \left(\int_{B(0,r)} \left|\frac{1}{|x|^{n-\beta}} \int_{|y|<|x|} r^{-n/p+m\alpha} dy\right|^q dx\right)^{1/q} \\
&\le C\prod_{i=1}^{m} \|b_i\|_{\mathrm{Lip}_\alpha(\mathbb{R}^n)} \left(\int_{B(0,r)} \left|r^{\beta} \cdot r^{-n/p+m\alpha}\right|^q dx\right)^{1/q} \\
&\le C\prod_{i=1}^{m} \|b_i\|_{\mathrm{Lip}_\alpha(\mathbb{R}^n)},
\end{aligned} \tag{45}$$

where $|x| < r$ and $1/q = (1/p) - ((\beta + m\alpha)/n)$; we complete the proof of Theorem 30. □

Theorem 31. *Let $\vec{b} = (b_1, b_2, \ldots, b_m)$, $b_i \in \mathrm{Lip}_{\alpha}(\mathbb{R}^n)$, $q = n/(n - \beta - m\alpha)$, and $0 < q < \infty$; then $[\vec{b}, \mathscr{H}_\beta]$ maps $H^p(\mathbb{R}^n)$ into $L^{q,\infty}(\mathbb{R}^n)$.*

Proof. It is similar to the proof of Theorem 15. □

When b is in $C\dot{M}O(\mathbb{R}^n)$, we suppose the multilinear commutator of fractional Hardy operators maps $H^1(\mathbb{R}^n)$ into $L^{n/(n-\beta),\infty}$, but we cannot prove it. However, we get the following result.

Theorem 32. *Let $\vec{b} = (b_1, b_2, \ldots, b_m)$ and $b_i \in C\dot{M}O^{\max\{s_i q, s_i\}}(\mathbb{R}^n)$, $1/q = 1/p - \beta/n$, $0 < p \le 1$, $1 < s_i < \infty$, then $[\vec{b}, \mathscr{H}_\beta]$ maps $H^p_{\vec{b}}(\mathbb{R}^n)$ into $L^q(\mathbb{R}^n)$.*

Proof. Similar to the proof of Theorem 12, it is enough to prove that

$$\left\|[\vec{b}, \mathscr{H}_\beta](\tilde{a})(x)\right\|_{L^q} \le C\prod_{i=1}^{m} \|b_i\|_{C\dot{M}O(\mathbb{R}^n)}\|\tilde{a}\|_{H^1_{\vec{b}}}, \tag{46}$$

where $\tilde{a}$ is a center $(p, \vec{b})$-atom supported on a ball $B = B(0,r)$. We write

$$\begin{aligned}
&\left[\vec{b}, \mathscr{H}_\beta\right](\tilde{a})(x) \\
&= \frac{1}{|x|^{n-\beta}} \int_{|y|<|x|} \tilde{a}(y) \prod_{i=1}^{m} (b_i(x) - b_i(y))\, dy \\
&= \frac{1}{|x|^{n-\beta}} \int_{|y|<|x|} \tilde{a}(y) \sum_{i=0}^{m} \sum_{\sigma \in C_i^m} \prod_{i \in \sigma} (b_i(x) - \lambda_i) \times \prod_{i \in \sigma^c} (\lambda_i - b_i(y))\, dy \\
&= \frac{1}{|x|^{n-\beta}} \prod_{i=1}^{m} (b_i(x) - \lambda_i) \int_{|y|<|x|} \tilde{a}(y)\, dy \\
&\quad + \sum_{i=1}^{m-1} \sum_{\sigma \in C_i^m} \frac{1}{|x|^{n-\beta}} \prod_{i \in \sigma} (b_i(x) - \lambda_i) \times \int_{|y|<|x|} \tilde{a}(y) \prod_{i \in \sigma^c} (\lambda_i - b_i(y))\, dy \\
&\quad + \frac{1}{|x|^{n-\beta}} \int_{|y|<|x|} \tilde{a}(y) \prod_{i=1}^{m} (\lambda_i - b_i(y))\, dy \\
&:= K_1 + K_2 + K_3,
\end{aligned} \tag{47}$$

where $\lambda_i = (b_i)_B$. By Minkowski inequality, we have

$$\begin{aligned}
&\left\|\left[\vec{b}, \mathscr{H}_\beta\right](\tilde{a})(x)\right\|_{L^q} \\
&= \int_{\mathbb{R}^n} \left|\left[\vec{b}, \mathscr{H}_\beta\right](\tilde{a})(x)\right|^q dx \\
&\le \left(\int_{\mathbb{R}^n} |K_1 + K_2 + K_3|^q dx\right)^{1/q} \\
&\le \left(\int_{\mathbb{R}^n} |K_1|^q dx\right)^{1/q} + \left(\int_{\mathbb{R}^n} |K_2|^q dx\right)^{1/q} \\
&\quad + \left(\int_{\mathbb{R}^n} |K_3|^q dx\right)^{1/q}
\end{aligned}$$

$$\begin{aligned}&\leq \left(\int_{B(0,r)} |K_1|^q dx\right)^{1/q} + \left(\int_{\mathbb{R}^n\setminus B(0,r)} |K_1|^q dx\right)^{1/q}\\&+ \left(\int_{B(0,r)} |K_2|^q dx\right)^{1/q} + \left(\int_{\mathbb{R}^n\setminus B(0,r)} |K_2|^q dx\right)^{1/q}\\&+ \left(\int_{B(0,r)} |K_3|^q dx\right)^{1/q} + \left(\int_{\mathbb{R}^n\setminus B(0,r)} |K_3|^q dx\right)^{1/q}\\&:= K_{11} + K_{12} + K_{21} + K_{22} + K_{31} + K_{32}.\end{aligned} \tag{48}$$

For $x \in B(0,r)$, we have $|x| < r$; then

$$\begin{aligned}K_{11} &= \left(\int_{B(0,r)} \left|\frac{1}{|x|^{n-\beta}}\prod_{i=1}^m \left(b_j(x) - \lambda_i\right)\right.\right.\\&\qquad \left.\left.\times \int_{|y|<|x|} \widetilde{a}(y)\,dy\right|^q dx\right)^{1/q}\\&\leq C\left(\int_{B(0,r)} \left|\frac{1}{|x|^{n-\beta}} \cdot \prod_{i=1}^m \left|b_i(x) - \lambda_i\right| \cdot |x|^n\right.\right.\\&\qquad \left.\left.\cdot |B(0,r)|^{-1/p}\right|^q dx\right)^{1/q}\\&\leq Cr^{(\beta-n/p)}\left(\int_{B(0,r)} \prod_{i=1}^m \left|b_i(x) - \lambda_i\right|^q dx\right)^{1/q}\\&\leq C\left(\frac{1}{B(0,r)}\int_{B(0,r)} \prod_{i=1}^m \left|b_i(x) - \lambda_i\right|^q dx\right)^{1/q}\\&\leq C\prod_{i=1}^m \|b_i\|_{C\dot{M}O^{s_i q}(\mathbb{R}^n)},\end{aligned}$$

$$\begin{aligned}K_{31} &= \left(\int_{B(0,r)} \left|\frac{1}{|x|^{n-\beta}}\right.\right.\\&\qquad \left.\left.\times \int_{|y|<|x|} \widetilde{a}(y)\prod_{i=1}^m \left(\lambda_i - b_i(y)\right) dy\right|^q dx\right)^{1/q}\\&\leq C\left(\int_{B(0,r)} \left|\frac{1}{|x|^{n-\beta}} \cdot |B(0,r)|^{-1}\right.\right.\\&\qquad \left.\left.\times \int_{|y|<|x|} \prod_{i=1}^m \left(\lambda_i - b_i(y)\right) dy\right|^q dx\right)^{1/q}\\&\leq C\left(\int_{B(0,r)} \left||B(0,r)|^{-1/p} \cdot |x|^\beta \cdot \frac{1}{B(0,|x|)}\right.\right.\\&\qquad \left.\left.\times \int_{|y|<|x|} \prod_{i=1}^m \left(\lambda_i - b_i(y)\right) dy\right|^q dx\right)^{1/q}\\&\leq C\left(\int_{B(0,r)} \left||B(0,r)|^{-1/p} \cdot |x|^\beta\right.\right.\\&\qquad \left.\left.\cdot \prod_{i=1}^m \|b\|_{C\dot{M}O^{s_i}(\mathbb{R}^n)}\right|^q dx\right)^{1/q}\\&\leq C\prod_{i=1}^m \|b\|_{C\dot{M}O^{s_i}(\mathbb{R}^n)}\left(r^{(\beta-n/p)q}\int_{B(0,r)} dx\right)^{1/q}\\&\leq C\prod_{i=1}^m \|b\|_{C\dot{M}O^{s_i}(\mathbb{R}^n)}.\end{aligned} \tag{49}$$

For K_{21},

$$\begin{aligned}&\left(\int_{B(0,r)} \left|\frac{1}{|x|^{n-\beta}}\prod_{i\in\sigma} \left(b_i(x) - \lambda_i\right)\right.\right.\\&\qquad \left.\left.\times \int_{|y|<|x|} \widetilde{a}(y)\prod_{i\in\sigma^c} \left(\lambda_i - b_i(y)\right) dy\right|^q dx\right)^{1/q}\\&\leq C\left(\int_{B(0,r)} \left|\frac{1}{r^{n/p-\beta}}\right.\right.\\&\qquad \left.\left.\times \prod_{i\in\sigma} \left(b_i(x) - \lambda_i\right)\prod_{i\in\sigma^c} \|b\|_{C\dot{M}O^{s_i}(\mathbb{R}^n)}\right|^q dx\right)^{1/q}\\&\leq C\prod_{i\in\sigma^c} \|b\|_{C\dot{M}O^{s_i}(\mathbb{R}^n)}\left(r^{-n}\int_{B(0,r)} \left|\prod_{i\in\sigma} \left(b_i(x) - \lambda_i\right)\right|^q dx\right)^{1/q}\\&\leq C\prod_{i\in\sigma^c} \|b\|_{C\dot{M}O^{s_i}(\mathbb{R}^n)} \cdot \prod_{i\in\sigma} \|b\|_{C\dot{M}O^{s_i q}(\mathbb{R}^n)}\\&\leq C\prod_{i=1}^m \|b\|_{C\dot{M}O^{\max\{s_i q, s_i\}}(\mathbb{R}^n)}.\end{aligned} \tag{50}$$

For $x \in \mathbb{R}^n \setminus B(0,r)$, we have $|x| \geq r$; then $\{y : |y| < |x|\} \bigcap \{y : |y| < r\} = \{y : |y| < r\}$. By the condition of $\int_B a(y)dy = \int_B a(y)\prod_{j\in\sigma} b_j dy = 0$ for any $\sigma \in C_i^m$, $1 \leq j \leq m$, we have that

$$\begin{aligned}&\int_{|y|<r} \widetilde{a}(y)\prod_{i\in\sigma^c} \left(\lambda_i - b_i(y)\right) dy\\&\quad = \int_{|y|<r} \widetilde{a}(y)\prod_{i=1}^m \left(\lambda_i - b_i(y)\right) dy = 0,\end{aligned} \tag{51}$$

so,

$$K_{12} = K_{22} = K_{32} = 0. \tag{52}$$

Combining all the above estimates, we complete the proof of Theorem 32. □

Conflict of Interests

The authors declare that they do not have any commercial or associative interests that represents a conflict of interests in connection with the work submitted.

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