Auxiliary Materials

Comparing with the main theorem in paper" Multilinear Commutators of Calderón-Zygmund Operator on Generalized Weighted Morrey Spaces", by Vagif S. Guliyev1, and Farida Ch. Alizadeh in Journal of Function Spaces, Volume 2014, Article ID 710542, our theorems require the conditions that $\varphi_{1i}, i=1, \cdot \cdot\cdot, m$ are all non-decreasing functions. Carefully analyzed their certification process, we found that their proof has a fatal error. Their main result as follows.

Theorem 7. *Let $1 < p < \infty$, $w \in A_p$, and (φ_1, φ_2) satisfies the condition*

$$\int_r^{\infty} \ln^m \left(e + \frac{t}{r}\right) \frac{\operatorname{ess\,inf}_{t<s<\infty} \varphi_1(x,s)\, w(B(x,s))^{1/p}}{w(B(x,t))^{1/p}} \frac{dt}{t} \leq C\varphi_2(x,r), \tag{11}$$

where C does not depend on x and r. Let $\vec{b} = (b_1, \ldots, b_m)$, $b_i \in BMO$, $i = 1, \ldots, m$. Then the operator $T_{\vec{b}}$ is bounded from $M_{p,\varphi_1}(w)$ to $M_{p,\varphi_2}(w)$. Moreover,

$$\|T_{\vec{b}} f\|_{M_{p,\varphi_2}(w)} \lesssim \|\vec{b}\|_* \|f\|_{M_{p,\varphi_1}(w)}. \tag{12}$$

Their proof of Theorem 7 is mainly used the following Lemma:

Theorem 14. *The inequality*

$$\operatorname*{ess\,sup}_{t>0} w(t) H_1 g(t) \leq c \operatorname*{ess\,sup}_{t>0} v(t) g(t) \tag{27}$$

holds for all non-negative and non-increasing g on $(0,\infty)$ if and only if

$$A_1 := \sup_{t>0} \frac{w(t)}{t} \int_0^t \ln^m \left(e + \frac{t}{r}\right) \frac{d\mu(r)}{\operatorname{ess\,sup}_{0<s<r} v(s)} < \infty, \tag{28}$$

and $c \approx A_1$.

Theorem 14 is required that g is non-increasing on $(0,\infty)$.

But, in the process of proof as below, $\|f\|_{L_{p,w}(B(x,t^{-1})) w(B(x,t^{-1}))^{-1/p}$ can not guarantee as a non-increasing.(from second equal sign to next $\leq$). After careful calculation, conditon (11) in Theorem 7 also can not guarantee Theorem 14 $A_1<\infty$

$$
\begin{aligned}
\left\|T_{\vec{b}} f\right\|_{M_{p,\varphi_2}(w)} \lesssim & \left\|\vec{b}\right\|_* \sup_{x\in\mathbb{R}^n, r>0} \varphi_2(x,r)^{-1} \\
& \times \int_r^{\infty} \ln^m\left(e+\frac{t}{r}\right) \|f\|_{L_{p,w}(B(x,t))} \\
& \times w(B(x,t))^{-1/p} \frac{dt}{t} \\
= & \left\|\vec{b}\right\|_* \sup_{x\in\mathbb{R}^n, r>0} \varphi_2(x,r)^{-1} \\
& \times \int_0^{r^{-1}} \ln^m\left(e+\frac{1}{tr}\right) \|f\|_{L_{p,w}(B(x,t^{-1}))} \\
& \times w\left(B\left(x,t^{-1}\right)\right)^{-1/p} \frac{dt}{t} \\
= & \left\|\vec{b}\right\|_* \sup_{x\in\mathbb{R}^n, r>0} \varphi_2\left(x,r^{-1}\right)^{-1} r \frac{1}{r} \\
& \times \int_0^{r} \ln^m\left(e+\frac{r}{t}\right) \|f\|_{L_{p,w}(B(x,t^{-1}))} \\
& \times w\left(B\left(x,t^{-1}\right)\right)^{-1/p} \frac{dt}{t} \\
\lesssim & \left\|\vec{b}\right\|_* \sup_{x\in\mathbb{R}^n, r>0} \varphi_1\left(x,r^{-1}\right)^{-1} \\
& \times w\left(B\left(x,r^{-1}\right)\right)^{-1/p} \|f\|_{L_{p,w}(B(x,r^{-1}))} \\
= & \left\|\vec{b}\right\|_* \sup_{x\in\mathbb{R}^n, r>0} \varphi_1(x,r)^{-1} w(B(x,r))^{-1/p} \\
& \times \|f\|_{L_{p,w}(B(x,r))} \\
= & \left\|\vec{b}\right\|_* \|f\|_{M_{p,\varphi_1}(w)}.
\end{aligned}
$$

Therefore, we added a condition “$\varphi_{1i},i=1,\cdot \cdot\cdot,m$ are all non-decreasing functions” in our mainly results. This avoid the application of Theorem 14.