Hindawi Publishing Corporation
Journal of Function Spaces
Volume 2014, Article ID 326940, 6 pages
http://dx.doi.org/10.1155/2014/326940

Research Article

Molecular Characterization of Hardy Spaces Associated with Twisted Convolution

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Received 15 April 2014; Accepted 10 June 2014; Published 22 July 2014

Academic Editor: Kishin Sadarangani

We give a molecular characterization of the Hardy space associated with twisted convolution. As an application, we prove the boundedness of the local Riesz transform on the Hardy space.

1. Introduction

In this paper, we consider the $2n$ linear differential operators

$$Z_j = \frac{\partial}{\partial z_j} + \frac{1}{4}\overline{z}_j, \qquad \overline{Z}_j = \frac{\partial}{\partial \overline{z}_j} - \frac{1}{4}z_j, \qquad \text{on } \mathbb{C}^n,\ j = 1, 2, \ldots, n. \tag{1}$$

Together with the identity they generate a Lie algebra h^n which is isomorphic to the $2n+1$ dimensional Heisenberg algebra. The only nontrivial commutation relations are

$$\left[Z_j, \overline{Z}_j\right] = -\frac{1}{2}I, \quad j = 1, 2, \ldots, n. \tag{2}$$

The operator L defined by

$$L = -\frac{1}{2}\sum_{j=1}^{n}\left(Z_j\overline{Z}_j + \overline{Z}_jZ_j\right) \tag{3}$$

is nonnegative, self-adjoint, and elliptic. Therefore, it generates a diffusion semigroup $\{T_t^L\}_{t>0} = \{e^{-tL}\}_{t>0}$. The operators in (1) generate a family of "twisted translations" τ_w on $\mathbb{C}^n$ defined on measurable functions by

$$\begin{aligned}(\tau_w f)(z) &= \exp\left(\frac{1}{2}\sum_{j=1}^{n}\left(w_jz_j + \overline{w}_j\overline{z}_j\right)\right) f(z) \\ &= f(z+w)\exp\left(\frac{i}{2}\operatorname{Im}(z\cdot s\overline{w})\right).\end{aligned} \tag{4}$$

The "twisted convolution" of two functions f and g on $\mathbb{C}^n$ can now be defined as

$$\begin{aligned}(f\times g)(z) &= \int_{\mathbb{C}^n} f(w)\,\tau_{-w}g(z)\,dw \\ &= \int_{\mathbb{C}^n} f(z-w)\,g(w)\,\overline{\omega}(z,w)\,dw,\end{aligned} \tag{5}$$

where $\omega(z,w) = \exp((i/2)\operatorname{Im}(z\cdot\overline{w}))$. More about twisted convolution can be found in [1–3].

In [4], the authors defined the Hardy space $H_L^1(\mathbb{C}^n)$ associated with twisted convolution. They gave several characterizations of $H_L^1(\mathbb{C}^n)$ via maximal functions, the atomic decomposition, and the behavior of the local Riesz transform. As applications, the boundedness of Hömander multipliers on Hardy spaces is considered in [5]. The "twisted cancellation" and Weyl multipliers were introduced for the first time in [6]. Recently, Huang and Wang [7] defined the Hardy space $H_L^p(\mathbb{C}^n)$ associated with twisted convolution for $2n/(2n+1) < p < 1$. Huang gave the characterizations of the Hardy space associated with twisted convolution by the Lusin area integral function and Littlewood-Paley function in [8] and established the boundedness of the Weyl multiplier on the Hardy space associated with twisted convolution by these characterizations in [9]. The purpose of this paper is to give a molecular characterization for $H_L^p(\mathbb{C}^n)$. As an application, we prove the boundedness of the local Riesz transform on the Hardy space $H_L^p(\mathbb{C}^n)$.

We first give some basic notations about $H_L^p(\mathbb{C}^n)$. Let $\mathscr{B}$ denote the class of C^∞-functions φ on $\mathbb{C}^n$, supported on the ball $B(0,1)$ such that $\|\varphi\|_\infty \le 1$ and $\|\nabla\varphi\|_\infty \le 2$. For $t > 0$, let $\varphi_t(z) = t^{-2n}\varphi(z/t)$. Given $\sigma > 0$, $0 < \sigma \le +\infty$, and a tempered distribution f, define the grand maximal function

$$M_\sigma f(z) = \sup_{\varphi\in\mathscr{B}} \sup_{0<t<\sigma} \left|\varphi_t \times f(z)\right|. \tag{6}$$

Then, the Hardy space $H_L^p(\mathbb{C}^n)$ can be defined by

$$H_L^p(\mathbb{C}^n) = \left\{f \in \mathcal{S}'(\mathbb{C}^n) : M_\infty f \in L^p(\mathbb{C}^n)\right\}. \tag{7}$$

For any $f \in H_L^p(\mathbb{C}^n)$, define $\|f\|_{H_L^p(\mathbb{C}^n)} = \|M_\infty f\|_{L^p}$.

Definition 1. Let $0 < p \le 1 \le q \le \infty$ and $p \ne q$. A function $a(z)$ is a $H_L^{p,q}$-atom for the Hardy space $H_L^p(\mathbb{C}^n)$ associated to a ball $B(z_0, r)$ if

(1) $\operatorname{supp} a \subset B(z_0, r)$;

(2) $\|a\|_q \le |B(z_0, r)|^{1/q-1/p}$;

(3) $\int_{\mathbb{C}^n} a(w)\overline{\omega}(z_0, w)dw = 0$.

We define the atomic Hardy space $H_L^{p,q}(\mathbb{C}^n)$ to be the set of all tempered distributions of the form $\sum_j \lambda_j a_j$ (the sum converges in the topology of $\mathcal{S}'(\mathbb{C}^n)$), where a_j are $H_L^{p,q}$-atoms and $\sum_j |\lambda_j|^p < +\infty$.

The atomic quasinorm in $H_L^{p,q}(\mathbb{C}^n)$ is defined by

$$\|f\|_{L\text{-atom}} = \inf\left\{\left(\sum_j \left|\lambda_j\right|^p\right)^{1/p}\right\}, \tag{8}$$

where the infimum is taken over all decompositions $f = \sum_j \lambda_j a_j$ and a_j are $H_L^{p,q}$-atoms.

The following result has been proved in [4, 7].

Proposition 2. *Let $2n/(2n+1) < p \le 1$. Then, for a tempered distribution f on $\mathbb{C}^n$, the following are equivalent:*

(i) $M_\infty f \in L^p(\mathbb{C}^n)$;

(ii) *for some σ, $0 < \sigma < +\infty$, $M_\sigma f \in L^p(\mathbb{C}^n)$;*

(iii) *for some radial function $\varphi \in \mathcal{S}$, such that $\int_{\mathbb{C}^n} \varphi(z)dz \ne 0$, we have*

$$\sup_{0<t<1} \left|\varphi_t \times f(z)\right| \in L^p(\mathbb{C}^n); \tag{9}$$

(iv) *f can be decomposed as $f = \sum_j \lambda_j a_j$, where a_j are $H_L^{p,q}$-atoms and $\sum_j |\lambda_j|^p < +\infty$.*

Corollary 3. *Let $2n/(2n+1) < p \le 1$ and $1 < q \le \infty$. Then, $H_L^{p,q}(\mathbb{C}^n) = H_L^p(\mathbb{C}^n)$ with equivalent norms.*

In order to give the main result of this paper, we need the dual space of Hardy space $H_L^p(\mathbb{C}^n)$.

Definition 4. Let $0 \le \alpha < 1/2n$; a locally integrable function f is said to be in the Campanato type space Λ_α^L if there exists a constant $K > 0$ such that, for every ball $B = B(z, r)$,

$$|B|^{-\alpha}\left(\int_B \left|f(v) - \left(\frac{1}{|B|}\int_B f(u)\,\overline{\omega}(z,u)\,du\right)\omega(z,v)\right|^2 \times \frac{dv}{|B|}\right)^{1/2} \le K. \tag{10}$$

The norm $\|f\|_{\Lambda_\alpha^L}$ of f is the least value of K for which the above inequality holds.

The dual space of $H_L^1(\mathbb{C}^n)$ is the BMO type space $\mathrm{BMO}_L(\mathbb{C}^n)$ (cf. [4]). Note that Λ_0^L is identified with BMO_L. Let $\mathscr{H}_L^{p,2,a}$ denote the space of finite linear combinations of $H_L^{p,2}$-atoms, which coincides with $L_c^2(\mathbb{C}^n)$, the space of square integrable functions with compact support. By Proposition 2, $\mathscr{H}_L^{p,2,a}$ is a dense subspace of $H_L^p(\mathbb{C}^n)$. Set

$$\mathscr{L}_g(f) = \int_{\mathbb{C}^n} f(z)\,\overline{g}(z)\,dz, \quad f \in \mathscr{H}_L^{p,2,a},\ g \in L^2_{\mathrm{loc}}(\mathbb{C}^n). \tag{11}$$

Similar to the classical case in [10], we immediately obtain the following theorem which proves that $\Lambda_{1/p-1}^L$ is the dual space of $H_L^p(\mathbb{C}^n)$ for $2n/(2n+1) < p < 1$.

Theorem 5. *Let $2n/(2n+1) < p < 1$. Then*

(a) *suppose $g \in \Lambda_{(1/p)-1}^L$; then $\mathscr{L}_g$ given by* (11) *extends to a bounded linear functional on $H_L^p(\mathbb{C}^n)$ and satisfies*

$$\left\|\mathscr{L}_g\right\| \le C\,\|g\|_{\Lambda_{(1/p)-1}^L}; \tag{12}$$

(b) *conversely, every bounded linear functional $\mathscr{L}$ on $H_L^p(\mathbb{C}^n)$ can be realized as $\mathscr{L} = \mathscr{L}_g$ with $g \in \Lambda_{(1/p)-1}^L$ and*

$$\|g\|_{\Lambda_{(1/p)-1}^L} \le C\,\|\mathscr{L}\|. \tag{13}$$

Remark 6. We may define the space $\Lambda_{(1/p)-1,q'}^L$, $2n/(2n+1) < p < 1$, $1 \le q' \le \infty$, by

$$|B|^{1-(1/p)}\left(\int_B \left|f(v) - \left(\frac{1}{|B|}\int_B f(u)\,\overline{\omega}(z,u)\,du\right)\omega(z,v)\right|^{q'} \times \frac{dv}{|B|}\right)^{1/q'} \le K, \tag{14}$$

where $B = B(z, r)$. The norm $\|f\|_{\Lambda_{(1/p)-1,q'}^L}$ of f is the least value of K for which the above inequality holds. Due to Theorem 5, $\Lambda_{(1/p)-1,q'}^L$ is also identified with the dual space of $H_L^p(\mathbb{C}^n)$. The proof is almost the same as that of Theorem 5. Thus, the space $\Lambda_{(1/p)-1,q'}^L$ coincides with $\Lambda_{(1/p)-1}^L$ and $\|f\|_{\Lambda_{(1/p)-1,q'}^L} \sim \|f\|_{\Lambda_{(1/p)-1}^L}$.

Definition 7. Let $2n/(2n+1) < p \le 1 \le q \le \infty$, $p \ne q$, and $\epsilon > (1/p) - 1$. Set $a = 1 - (1/p) + \epsilon$, $b = 1 - (1/p) + \epsilon$. A function $M \in L^q$ is called a $H_L^{p,q,\epsilon}$-molecule with the center z_0 if

(1) $|z|^{2nb} M(z) \in L^q$,

(2) $\mathcal{N}(M) = \|M\|_{L^q}^{a/b} \, \||\cdot - z_0|^{2nb} M\|_{L^q}^{1-(a/b)} < \infty$,

(3) $\int_{\mathbb{C}^n} M(z)\overline{\omega}(z_0, z)\, dz = 0$.

Then, we can obtain a molecular characterization of $H_L^p(\mathbb{C}^n)$ as follows.

Theorem 8. *Given p, q, ϵ as in Definition 7, then $f \in H_L^p$ if and only if f can be written as $f = \sum_j \lambda_j M_j$, where M_j are $H_L^{p,q,\epsilon}$-molecules and $\sum_j |\lambda_j|^p < \infty$. The sum converges in H_L^p norm and also in $(\Lambda_{(1/p)-1}^L)^*$ when $2n/(2n+1) < p < 1$. Moreover,*

$$\|f\|_{H_L^p} \sim \|f\|_{H_L^{p,q,\epsilon,M}} = \inf\left\{\left(\sum_j |\lambda_j|^p\right)^{1/p}\right\}, \tag{15}$$

where the infimum is taken over all decompositions of f into $H_L^{p,q,\epsilon}$-molecules.

Let ψ be a C^∞-function on $\mathbb{C}^n$ with compact support and such that $\psi \equiv 1$ on a neighborhood of zero. Define

$$R_j(z) = \frac{z_j}{|z|^{2n+1}}\psi(z), \qquad \overline{R}_j(z) = \frac{\overline{z}_j}{|z|^{2n+1}}\psi(z), \tag{16}$$

for $j = 1, 2, \ldots, n$.

We refer to the singular integral operators R_j, $\overline{R}_j$ defined by left twisted convolution with these kernels as the local Riesz transforms. The terminology is motivated by the fact that they are essentially the operators which are formally defined as $Z_j L^{-1/2}$, $\overline{Z}_j L^{-1/2}$, $j = 1, 2, \ldots, n$.

As an application of Theorem 8, we can prove the following.

Theorem 9. *The local Riesz transforms R_j, $\overline{R}_j$, $j = 1, 2, \ldots, n$ are bounded on $H_L^p(\mathbb{C}^n)$, where $2n/(2n+1) < p \le 1$.*

Remark 10. When $p = 1$, Theorem 9 is proved by the connection between $H_L^1(\mathbb{C}^n)$ and Hardy space on the Heisenberg group $H^1(\mathbb{H}_n)$ (cf. Lemma 4.9 in [4]).

Throughout the paper, we will use C to denote a positive constant, which is independent of main parameters and may be different at each occurrence. By $B_1 \sim B_2$, we mean that there exists a constant $C > 1$ such that $1/C \le B_1/B_2 \le C$.

2. Molecule Characterization of $H_L^p(\mathbb{C}^n)$

In this section, we prove the main result of this paper. Firstly, we have the following lemma.

Lemma 11. *If a is a $H_L^{p,q}$-atom for $2n/(2n+1) < p \le 1 \le q \le +\infty$ supported in $B(z_0, r)$, then a is a $H_L^{p,q,\varepsilon}$-molecule centered at z_0 and*

$$\mathcal{N}(a) \le C, \tag{17}$$

where $\epsilon > 0$ and C is a positive constant that is independent of a.

Proof. Since

$$\|a\|_q \le |B|^{1/q-1/p} = |B|^{a-b}, \tag{18}$$

we get

$$\left\||\cdot - z_0|^{2nb} a(\cdot)\right\|_q \le r^{2nb}\|a\|_q \le C|B|^b|B|^{a-b} = C|B|^a. \tag{19}$$

Therefore,

$$\mathcal{N}(a) \le C|B|^{(a/b)(a-b)}|B|^{a(1-(a/b))} = C. \tag{20}$$

This proves that a is a molecule with center at z_0. □

The following lemma is the key step for the proof of Theorem 8.

Lemma 12. *If M is a $H_L^{p,q,\epsilon}$-molecule with center at z_0, then $M \in H_L^{p,q}(\mathbb{C}^n)$ and*

$$\|M\|_{H_L^p} \le C\mathcal{N}(M), \tag{21}$$

where C is independent of M.

Proof. If $q = 2$, let $\sigma = \|M\|_2^{1/(2n(a-b))}$, $E_0 = \{z \in \mathbb{C}^n : |z - z_0| \le \sigma\}$, and $E_k = \{z \in \mathbb{C}^n : 2^{k-1}\sigma < |z - z_0| \le 2^k\sigma\}$. Denote $M_k = M\chi_k$, where χ_k is the characteristic function of E_k.

Let

$$P_k(z) = \frac{1}{|E_k|}\int_{E_k} M(u)\,\overline{\omega}(z_0, u)\, du\,\omega(z_0, z)\,\chi_k(z). \tag{22}$$

Then

$$\begin{aligned}\int_{\mathbb{C}^n} &(M_k(z) - P_k(z))\,\overline{\omega}(z_0, z)\, dz \\ &= \int_{E_k} M(z)\,\overline{\omega}(z_0, z)\, dz - \int_{E_k} M(z)\,\overline{\omega}(z_0, z)\, dz = 0.\end{aligned} \tag{23}$$

Without loss of generality, we can assume that $\mathcal{N}(M) = 1$. Then

$$\left\||\cdot - z_0|^{2nb} M(\cdot)\right\|_2^{1-(a/b)} = \|M\|_2^{-a/b}. \tag{24}$$

Therefore,

$$\left\||\cdot - z_0|^{2n((1/2)+\epsilon)} M(\cdot)\right\|_2 = \|M\|_2^{-a/(b-a)} = \sigma^{2na}. \tag{25}$$

Let $B_k = \{z \in \mathbb{C}^n : |z - z_0| \le 2^k\sigma\}$. Then

$$\operatorname{supp}(M_k - P_k) \subseteq E_k \subseteq B_k. \tag{26}$$

In the following, we will prove

$$\frac{1}{|B_k|}\int_{B_k}|M_k-P_k|^2dz\le\frac{C}{|E_k|}\int_{E_k}|M_k(z)|^2dz. \tag{27}$$

In fact, by

$$\int_{E_k}(M_k(z)-P_k(z))\overline{P}_k(z)\,dz=0, \tag{28}$$

we have

$$\begin{aligned}\frac{1}{|B_k|}&\int_{B_k}|M_k(z)-P_k(z)|^2dz\\&=\frac{1}{|B_k|}\int_{B_k}(M_k(z)-P_k(z))\left(\overline{M}_k(z)-\overline{P}_k(z)\right)dz\\&=\frac{1}{|B_k|}\int_{B_k}(M_k(z)-P_k(z))\overline{M}_k(z)\,dz.\end{aligned} \tag{29}$$

Since

$$\begin{aligned}&\int_{E_k}\overline{M}_k(z)P_k(z)\,dz\\&\quad=\int_{E_k}\overline{M}_k(z)\frac{1}{|E_k|}\int_{E_k}M(u)\overline{\omega}(z_0,u)\,du\omega(z_0,z)\,dz\\&\quad=\frac{1}{|E_k|}\int_{E_k}M(u)\overline{\omega}(z_0,u)\,du\int_{E_k}\overline{M}_k(z)\omega(z_0,z)\,dz\\&\quad=|E_k|\,|P_k(z)|^2,\end{aligned} \tag{30}$$

we get

$$\begin{aligned}\frac{1}{|B_k|}&\int_{B_k}|M_k(z)-P_k(z)|^2dz\\&=\frac{1}{|B_k|}\int_{E_k}|M_k(z)|^2dz-\int_{E_k}\frac{|E_k|}{|B_k|}|P_k(z)|^2dz\\&\le\frac{1}{|B_k|}\int_{E_k}|M_k(z)|^2dz\le\frac{C}{|E_k|}\int_{E_k}|M_k(z)|^2dz.\end{aligned} \tag{31}$$

Therefore, (27) holds true. In particular, we have

$$\begin{aligned}\frac{1}{|B_0|}&\int_{B_0}|M_0(z)-P_0(z)|^2dz\\&\le\frac{C}{|E_0|}\int_{E_0}|M_0(z)|^2dz\\&\le C\sigma^{-2n}\sigma^{4n((1/2)-(1/p))}=C|B_0|^{-2/p}.\end{aligned} \tag{32}$$

For $k\ge1$,

$$\begin{aligned}\frac{1}{|B_k|}&\int_{B_k}|M_k(z)-P_k(z)|^2dz\\&\le\frac{C}{|E_k|}\int_{E_k}|M_k(z)|^2dz\\&=\frac{C}{|E_k|}\int_{E_k}|M_k(z)|^2|z-z_0|^{2n(1+2\epsilon)}\\&\qquad\times|z-z_0|^{-2n(1+2\epsilon)}dz\\&\le C\left(2^k\sigma\right)^{-2n}\left(2^{k-1}\sigma\right)^{-2n(1+2\epsilon)}\\&\quad\times\int_{E_k}|M_k(z)|^2|z-z_0|^{2n(1+2\epsilon)}dz\\&\le C\sigma^{-4n(1+\epsilon)}2^{-4kn-4kn\epsilon}\sigma^{4na}\\&\le C2^{-4kn-4kn\epsilon-4kn/p}\left(2^k\sigma\right)^{-2n/p}\\&=C2^{-4kna}|B_k|^{-2/p},\end{aligned} \tag{33}$$

where C depends on n,ϵ. This proves that $M_k-P_k=\lambda_k a_k$, where a_k is a $H_L^{p,2}$-atom supported on B_k and $|\lambda_k|\le C2^{-2kna}$.

Now, we prove that $\sum_{k=1}^{\infty}P_k(z)$ has atomic decomposition. For $k\ge1$,

$$\begin{aligned}|P_k(z)|&\le\frac{1}{|E_k|}\int_{E_k}|M(u)|\,du\\&=\frac{1}{|E_k|}\int_{E_k}|u-z_0|^{2nb}|M(u)|\,|u-z_0|^{-2nb}du\\&\le C(2^k\sigma)^{-2nb}\frac{1}{|E_k|}\left\||\cdot-z_0|^{2nb}M(\cdot)\right\|_2|E_k|^{1/2}\\&\le C\left(2^k\sigma\right)^{-2nb-n}\sigma^{2na}\\&=C2^{-kn(1+2b)}\sigma^{-2n/p}.\end{aligned} \tag{34}$$

Therefore,

$$P_k(z)=\sum_{l=k+1}^{\infty}\left(P_{l-1}(z)-P_l(z)\right). \tag{35}$$

Let

$$N^k=\sum_{l=k}^{\infty}\int_{E_l}M(u)\overline{\omega}(z_0,u)\,du. \tag{36}$$

Then,

$$\begin{aligned}N^0&=\sum_{l=0}^{\infty}\int_{E_l}M(u)\overline{\omega}(z_0,u)\,du\\&=\int_{\mathbb{C}^n}M(u)\overline{\omega}(z_0,u)\,du=0.\end{aligned} \tag{37}$$

Thus, by Abel transform,

$$\begin{aligned}\sum_{k=0}^{\infty} P_k(z) &= \sum_{k=0}^{\infty}\sum_{l=k+1}^{\infty}\left(P_{l-1}(z)-P_l(z)\right)\\ &= \sum_{k=0}^{\infty} N^{k+1}\left\{\left|E_k\right|^{-1}\omega\left(z_0,z\right)\chi_k(z)\right.\\ &\qquad\left.-\left|E_{k+1}\right|^{-1}\omega\left(z_0,z\right)\chi_{k+1}(z)\right\}.\end{aligned} \tag{38}$$

Following from (34), we obtain

$$\begin{aligned}&\left|N^{k+1}\left\{\left|E_k\right|^{-1}\omega\left(z_0,z\right)\chi_k(z)\right.\right.\\ &\qquad\left.\left.-\left|E_{k+1}\right|^{-1}\omega\left(z_0,z\right)\chi_{k+1}(z)\right\}\right|\\ &\qquad\le C2^{-2n(k+1)\epsilon}\sigma^{2n-(2n/p)}\left|B_{k+1}\right|^{-1}\\ &\qquad= C2^{-2n(k+1)\epsilon}\sigma^{2n-(2n/p)}\left(2^{k+1}\sigma\right)^{-1}\\ &\qquad= C2^{-2na(k+1)}\left|B_{k+1}\right|^{-1/p}.\end{aligned} \tag{39}$$

Let $\mu_k = C2^{-2na(k+1)}$ and

$$\begin{aligned}b_k(z) = C^{-1}2^{2na(k+1)}N^{k+1}&\left\{\left|E_k\right|^{-1}\omega\left(z_0,z\right)\chi_k(z)\right.\\ &\left.-\left|E_{k+1}\right|^{-1}\omega\left(z_0,z\right)\chi_{k+1}(z)\right\}.\end{aligned} \tag{40}$$

Then, b_k are $H_L^{p,\infty}$-atoms, $\sum_{k=0}^{\infty}|\mu_k|^p < \infty$, and

$$\sum_{k=0}^{\infty} P_k(z) = \sum_{k=0}^{\infty}\mu_k b_k(z). \tag{41}$$

Therefore,

$$M(z) = \sum_{k=0}^{\infty}\lambda_k a_k(z) + \sum_{k=0}^{\infty}\mu_k b_k(z) \tag{42}$$

holds pointwise, where a_k are $H_L^{p,2}$-atoms and b_k are $H_L^{p,\infty}$-atoms, and

$$\sum_{k=0}^{\infty}\left\{\left|\lambda_k\right|^p + \left|\mu_k\right|^p\right\} < \infty. \tag{43}$$

When $p = 1$, it is easy to see that the sum in (42) converges in L^1.

To prove $M \in H_L^{p,q}$ for $2n/(2n+1) < p < 1$, we need to show that, for every $g \in \Lambda_{(1/p)-1}^{L}$,

$$\begin{aligned}&\int_{\mathbb{C}^n} M(z)\,g(z)\,dz\\ &\qquad= \lim_{m\to\infty}\sum_{k=0}^{m}\int_{\mathbb{C}^n}\left\{\lambda_k a_k(z) + \mu_k b_k(z)\right\}g(z)\,dz.\end{aligned} \tag{44}$$

In fact, (44) implies that (42) holds in $\mathcal{S}'(\mathbb{C}^n)$.

For any $z \in \mathbb{C}^n$, there exists $k \ge 0$ such that $z \in E_k$. If $k = 0$, then

$$M(z) = \lambda_0 a_0(z) + \mu_0 b_0(z). \tag{45}$$

If $k \ge 1$, then

$$M(z) = \lambda_k(z)\,a_k(z) + \sum_{j=k-1}^{k}\mu_j b_j(z). \tag{46}$$

Therefore, when $|z - z_0| \le 2^m\sigma$,

$$\begin{aligned}&\sum_{k=0}^{\infty}\left(\lambda_k a_k(z) + \mu_k b_k(z)\right)\\ &\qquad= \sum_{k=0}^{m}\left(\lambda_k a_k(z) + \mu_k b_k(z)\right) = M(z).\end{aligned} \tag{47}$$

Thus,

$$\begin{aligned}&\int_{\{z:|z-z_0|\le 2^m\sigma\}}\sum_{k=0}^{m}\left(\lambda_k a_k(z) + \mu_k b_k(z)\right)g(z)\,dz\\ &\qquad= \int_{\{z:|z-z_0|\le 2^m\sigma\}} M(z)\,g(z)\,dz.\end{aligned} \tag{48}$$

Let $m \to \infty$; the right side is $\int_{\mathbb{C}^n} M(z)g(z)dz$. The left side is

$$\begin{aligned}&\lim_{m\to\infty}\sum_{k=0}^{m}\int_{\{z:|z-z_0|\le 2^m\sigma\}}\left(\lambda_k a_k(z) + \mu_k b_k(z)\right)g(z)\,dz\\ &\qquad= \lim_{m\to\infty}\sum_{k=0}^{m}\int_{\mathbb{C}^n}\left(\lambda_k a_k(z) + \mu_k b_k(z)\right)g(z)\,dz.\end{aligned} \tag{49}$$

This proves (42) and the case of $q = 2$ for Lemma 12 is proved. Similarly, the case of $q \ne 2$ can be proved as the case of $q = 2$. Lemma 12 is proved. □

Proof of Theorem 8. Theorem 8 follows from Lemmas 11 and 12. □

3. The Boundedness of Local Riesz Transform on $H_L^p(\mathbb{C}^n)$

In this section, we prove the boundedness of local Riesz transform on $H_L^p(\mathbb{C}^n)$ by using Theorem 8.

Proof of Theorem 9. By Theorem 8, it is sufficient to prove that, for any $H_L^{p,2}$-atom a, $R_j(a)$ is a $H_L^{p,2,\varepsilon}$-molecule and the norm $\mathcal{N}(R_j(a)) \le C$, where C is independent of a.

Assume that $\operatorname{supp} a \subset B(z_0, r)$; then

$$\begin{aligned}&\int_{\mathbb{C}^n} R_j(a)(z)\,\overline{\omega}\left(z_0,z\right)dz\\ &\qquad= \int_{\mathbb{C}^n}\left(\int_{\mathbb{C}^n} a(z-u)\,\frac{u_i}{|u|^{2n+1}}\psi(u)\,\overline{\omega}(z,u)\,du\right)\\ &\qquad\quad\times\overline{\omega}\left(z_0,z\right)dz\end{aligned}$$

$$= \int_{\mathbb{C}^n} \frac{u_i}{|u|^{2n+1}} \psi(u) \left(\int_{\mathbb{C}^n} a(z-u) \overline{\omega}(u+z_0, z)\, dz \right) \times \overline{\omega}(z_0, z)\, du = 0, \tag{50}$$

where the last equality is valid because $a(\cdot - u)$ is an atom supported on $B(u + z_0, r)$. This proves that $R_j(a)$ satisfies moment condition.

Denote $M(z) = R_j(a)(z)$. Then, we have

$$\|M\|_2 = \left\| R_j(a) \right\|_2 \le C\|a\|_2 \le |B|^{1/2-1/p} = C|B|^{a-b}. \tag{51}$$

Let $B^* = \{z \in \mathbb{C}^n : |z - z_0| \le 2r\}$. Then,

$$\begin{aligned} &\int_{\mathbb{C}^n} |z-z_0|^{2n(1+2\epsilon)} |M(z)|^2 dz \\ &\quad = \int_{B^*} |z-z_0|^{2n(1+2\varepsilon)} M(z)^2 dz \\ &\qquad + \int_{(B^*)^c} |z-z_0|^{2n(1+2\epsilon)} |M(z)|^2 dz = I + II. \end{aligned} \tag{52}$$

For I,

$$I \le C|B|^{1+2\epsilon} \int_{\mathbb{C}^n} |M(z)|^2 dz \le C|B|^{2+2\epsilon-(2/p)} = C|B|^{2a}. \tag{53}$$

For II, since

$$\begin{aligned} &\left| R_j(a)(z) \right| \\ &\quad = \left| \int_{\mathbb{C}^n} a(u) \frac{z_j - u_i}{|z-u|^{2n+1}} \psi(z-u) \overline{\omega}(z, u)\, du \right| \\ &\quad = \left| \int_{\mathbb{C}^n} a(u) \overline{\omega}(z_0, u) \left(\frac{z_j - u_i}{|z-u|^{2n+1}} \psi(z-u) \right.\right. \\ &\qquad\qquad \times \omega(z_0 - z, u) \\ &\qquad\qquad - \frac{z_j - z_{0,j}}{|z-z_0|^{2n+1}} \psi(z-z_0) \\ &\qquad\qquad \left.\left. \times \omega(z_0 - z, z_0) \right) du \right| \\ &\quad \le C \int_{\mathbb{C}^n} \frac{|u-z_0|}{|z-z_0|^{2n+1}} |a(u)|\, du \\ &\quad \le Cr|B|^{1-1/p} \frac{1}{|z-z_0|^{2n+1}}, \end{aligned} \tag{54}$$

we get

$$II \le Cr^2 |B|^{2-2/p} \int_{(B^*)^c} \frac{|z-z_0|^{2n(1+2\epsilon)}}{|z-z_0|^{4n+2}} dz. \tag{55}$$

Let $0 < \epsilon < 1/2n$. Then

$$II \le C|B|^{2+2\epsilon-2/p} = C|B|^{2a}. \tag{56}$$

Therefore,

$$\begin{aligned} \mathcal{N}(M) &= \|M\|_2^{a/b} \left\| |\cdot - z_0|^{n(1+2\epsilon)} M(\cdot) \right\|_2^{1-(a/b)} \\ &\le C|B|^{(a/b)(a-b)} |B|^{a(1-(a/b))} = C. \end{aligned} \tag{57}$$

This completes the proof of Theorem 9. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

This research is supported by the National Natural Science Foundation of China (11001002 and 10901018), the Beijing Natural Science Foundation (1142005), and the Program for New Century Excellent Talents in University.

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