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Research Article

Positive Solutions of a Kind of Equations Related to the Laplacian and p-Laplacian

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Positive solutions of a kind of equations related to the Laplacian and p-Laplacian on a bounded domain in $\mathbb{R}^N$ with $N \geqslant 1$ are studied by using variational method. A sufficient condition of the existence of positive solutions is characterized by the eigenvalues of linear and another nonlinear eigenvalue problems.

1. Introduction

In this paper, we study the equation

$$\begin{aligned} -\Delta u - \Delta_p u &= f(x,u), \quad \text{in } \Omega, \\ u &= 0, \quad \text{on } \partial\Omega, \end{aligned} \tag{1}$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, $p > 2$, Ω is a smooth bounded domain in $\mathbb{R}^N$ for $N \geqslant 1$, and f satisfies the following conditions:

(f_1) $f \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$, $f(x,t) \geqslant 0$ for any $x \in \overline{\Omega}, t > 0$ and $f(x,t) = 0$ for any $x \in \overline{\Omega}, t \leqslant 0$;

(f_2) for $f_0, f_\infty < \infty$, the limits

$$\lim_{t \to 0^+} \frac{f(x,t)}{t} = f_0, \qquad \lim_{t \to \infty} \frac{f(x,t)}{t^{p-1}} = f_\infty \tag{2}$$

exist uniformly for $x \in \overline{\Omega}$.

The asymptotic behaviors of f near zero and infinity lead us to define

$$\begin{aligned} \lambda_1 &= \inf\left\{\int_\Omega |\nabla u|^2 : u \in H_0^1(\Omega), \int_\Omega |u|^2 = 1\right\}, \\ \mu_1 &= \inf\left\{\int_\Omega |\nabla u|^p : u \in W_0^{1,p}(\Omega), \int_\Omega |u|^p = 1\right\}, \end{aligned} \tag{3}$$

where $H_0^1(\Omega)$ and $W_0^{1,p}(\Omega)$ are the usual Sobolev spaces defined as the completion of $C_0^\infty(\Omega)$ with respect to the norms $\|u\|_2 = (\int_\Omega |\nabla u|^2)^{1/2}$ and $\|u\|_p = (\int_\Omega |\nabla u|^p)^{1/p}$, respectively. Then it is well known that μ_1 is the first eigenvalue of the nonlinear eigenvalue problem

$$\begin{aligned} -\Delta_p \phi &= \mu |\phi|^{p-2}\phi, \quad \text{in } \Omega, \\ \phi &= 0, \quad \text{on } \partial\Omega. \end{aligned} \tag{4}$$

Moreover μ_1 is a simple eigenvalue of (4), the associated eigenfunction ϕ_1 can be chosen as positive in Ω, and any eigenfunction corresponding to an eigenvalue larger than μ_1 must change sign. The reader is referred to [1–3] for details.

By a solution u of (1), we mean that u solves (1) in the weak sense; that is, u satisfies

$$\begin{aligned} \int_\Omega \nabla u \cdot \nabla v + \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v &= \int_\Omega f(x,u)\, v, \\ v &\in W_0^{1,p}(\Omega). \end{aligned} \tag{5}$$

Moreover, by a positive solution u of (1), we mean that u is a weak solution of (1), $u \neq 0$ and $u(x) \geqslant 0$ for $x \in \Omega$.

Our main result is the following theorem.

Theorem 1. *Suppose that f satisfies (f_1) and (f_2) with $f_0 < \lambda_1$, $f_\infty > \mu_1$. Then (1) has a positive solution.*

Assume that $f = g + h$. Equation (1) can be viewed as combination of the following equations:

$$\begin{aligned} -\Delta u &= g(x,u), \quad \text{in } \Omega, \\ u &= 0, \quad \text{on } \partial\Omega, \\ -\Delta_p u &= h(x,u), \quad \text{in } \Omega, \\ u &= 0, \quad \text{on } \partial\Omega. \end{aligned} \tag{6}$$

In the last decade or so, there was an extensive effort in studying the existence of solutions of (6); see [4–8].

Before concluding this section, we recall a theorem from [9], which will be used to prove our main theorem in this paper.

Theorem 2. *Let $(E, \|\cdot\|)$ be a Banach space and $I \subset \mathbb{R}_+$ an interval. Consider the family of C^1 functionals on E,*

$$J_\nu(u) = S(u) - \nu T(u), \quad \nu \in I, \tag{7}$$

with $J_\nu(0) = 0$, $\nu \in I$, T nonnegative and either $S(u) \to \infty$ or $T(u) \to \infty$ as $\|u\| \to \infty$. For any $\nu \in I$, we set

$$\Gamma_\nu = \{\gamma \in C([0,1], E) : \gamma(0) = 0, J_\nu(\gamma(1)) < 0\}. \tag{8}$$

If for every $\nu \in I$ the set Γ_ν is nonempty and

$$c_\nu = \inf_{\gamma\in\Gamma_\nu} \max_{t\in[0,1]} J_\nu(\gamma(t)) > 0, \tag{9}$$

then for almost every $\nu \in I$ there exists a sequence $\{u_n^\nu\} \subset E$ such that

(i) *$\{u_n^\nu\}$ is bounded;*

(ii) *$J_\nu(u_n^\nu) \to c_\nu$ as $n \to \infty$;*

(iii) *$J_\nu'(u_n^\nu) \to 0$ in the dual E^* as $n \to \infty$.*

Throughout this paper, we denote by X the Sobolev space $W_0^{1,p}(\Omega)$ with the norm $\|u\|_X = \|u\|_p$, by X^* the duality space of X, by $\rightharpoonup$ the weak convergence in X, and by $\langle\cdot,\cdot\rangle$ the duality pairing between X^* and X. The letters $C_1, C_2, \ldots$ will denote various positive constants whose exact values are not essential to the analysis of the problem. Let $P = \{u \in X : u(x) \geqslant 0, \text{a.e. } x \in \Omega\}$ and $p^* = Np/(N-p)$ if $p < N$ or ∞ if $p \geqslant N$.

2. Proof of Theorem 1

In this section, we always assume (f_1) and (f_2) hold with $f_0 < \lambda_1$ and $f_\infty > \mu_1$. Hence, there exist $\varepsilon_1 > 0, C_{\varepsilon_1} > 0$ and $q \in (p, p^*)$ such that

$$F(x,t) \geqslant \frac{1}{p}(\mu_1 + \varepsilon_1) t^p - C_{\varepsilon_1}, \quad x \in \overline{\Omega},\ t \geqslant 0, \tag{10}$$

$$F(x,t) \leqslant \frac{1}{2}(1-\varepsilon_1)\lambda_1 t^2 + C_{\varepsilon_1} t^q, \quad x \in \overline{\Omega},\ t \in \mathbb{R}, \tag{11}$$

where $F(x,t) = \int_0^t f(x,s)ds$. In the following, we utilize Theorem 2 to complete the proof of Theorem 1. In the setting of Theorem 2 we have $E = X, I = [\delta, 1]$ with $\mu_1/(\mu_1 + \varepsilon_1) < \delta < 1$, and

$$\begin{aligned} S(u) &= \frac{1}{2}\|u\|_2^2 + \frac{1}{p}\|u\|_X^p, \qquad T(u) = \int_\Omega F(x,u), \\ J_\nu(u) &= \frac{1}{2}\|u\|_2^2 + \frac{1}{p}\|u\|_X^p - \nu\int_\Omega F(x,u), \quad u \in X,\ \nu \in I. \end{aligned} \tag{12}$$

It is easy to verify that

$$\begin{aligned} &\langle J_\nu'(u), v\rangle \\ &\quad = \int_\Omega \nabla u\cdot\nabla v + \int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla v - \nu\int_\Omega f(x,u)v, \\ &\qquad u, v \in X, \quad \nu \in I. \end{aligned} \tag{13}$$

Firstly, we show that J_ν satisfies the conditions of Theorem 2 by proving several lemmas.

Lemma 3. *$\Gamma_\nu \neq \emptyset$ for any $\nu \in I$.*

Proof. Let $\phi_1 > 0$ be a μ_1-eigenfunction. For $t > 0$, we have by (10) that

$$\begin{aligned} J_\nu(t\phi_1) &= \frac{1}{2}t^2\|\phi_1\|_2^2 + \frac{1}{p}t^p\|\phi_1\|_X^p - \nu\int_\Omega F(x,t\phi_1) \\ &\leqslant \frac{1}{2}t^2\|\phi_1\|_2^2 + \frac{1}{p}\mu_1 t^p\int_\Omega |\phi_1|^p \\ &\quad - \frac{1}{p}(\mu_1+\varepsilon_1)\delta t^p\int_\Omega |\phi_1|^p + C_1 \\ &= \frac{1}{2}t^2\|\phi_1\|_2^2 - \frac{1}{p}C_2 t^p\int_\Omega |\phi_1|^p + C_1, \end{aligned} \tag{14}$$

where $C_2 = (\mu_1 + \varepsilon_1)\delta - \mu_1$. Noting that $C_2 > 0$, we can choose $t_0 > 0$ large enough so that $J_\nu(t_0\phi_1) < 0$, where t_0 is independent of $\nu \in I$. The proof is completed. □

Lemma 4. *There exists a constant $c > 0$ such that $c_\nu \geqslant c$ for any $\nu \in I$.*

Proof. For any $u \in X$, it follows from (11) that

$$\begin{aligned} J_\nu(u) &= \frac{1}{2}\|u\|_2^2 + \frac{1}{p}\|u\|_X^p - \nu\int_\Omega F(x,u) \\ &\geqslant \frac{1}{2}\|u\|_2^2 + \frac{1}{p}\|u\|_X^p - \frac{1}{2}(1-\varepsilon_1)\lambda_1\int_\Omega |u|^2 - C_{\varepsilon_1}\int_\Omega |u|^q \\ &\geqslant \frac{1}{2}\|u\|_2^2 + \frac{1}{p}\|u\|_X^p - \frac{1}{2}(1-\varepsilon_1)\|u\|_2^2 - C_{\varepsilon_1}\int_\Omega |u|^q \\ &\geqslant \frac{1}{p}\|u\|_X^p - C_{\varepsilon_1}\int_\Omega |u|^q. \end{aligned} \tag{15}$$

By Sobolev's embedding theorem, we conclude that there exist $\rho > 0$ and $c > 0$ such that $J_\nu(u) > 0$ for $\|u\| \in (0, \rho]$ and

$$J_\nu(u) \geqslant c, \qquad \|u\|_X = \rho. \tag{16}$$

Fix $\nu \in I$ and $\gamma \in \Gamma_\nu$. By the definition of Γ_ν, we have that $\|\gamma(1)\| > \rho$. Hence, there exists $t_\gamma \in (0,1)$ such that $\|\gamma(t_\gamma)\| = \rho$. So

$$c_\nu = \inf_{\gamma\in\Gamma_\nu} \max_{t\in[0,1]} J_\nu(\gamma(t)) \geqslant \inf_{\gamma\in\Gamma_\nu} J_\nu\left(\gamma\left(t_\gamma\right)\right) \geqslant c. \tag{17}$$

The proof is completed. □

Lemma 5. *For any $\nu \in I$, if $\{u_n\}$ is bounded and $J'_\nu(u_n) \to 0$ in X^* as $n \to \infty$, then $\{u_n\}$ admits a convergent subsequence.*

Proof. Given $\nu \in I$, assume that $\{u_n\}$ is bounded, $J'_\nu(u_n) \to 0$ in X^* as $n \to \infty$. By extracting a subsequence, we may suppose that there exists $u \in X$ such that as $n \to \infty$

$$\begin{gathered} u_n \rightharpoonup u \quad \text{in } X, \\ u_n \longrightarrow u \quad \text{in } L^s(\Omega), \quad s \in [1, p^*). \end{gathered} \tag{18}$$

It follows from (f_1) and (f_2) that there exist $C_1, C_2 > 0$ such that

$$f(x,t) \leqslant C_1|t| + C_2|t|^{p-1}, \quad x \in \overline{\Omega},\ t \in \mathbb{R}. \tag{19}$$

Hence, by Hölder's inequality and Sobolev's embedding theorem, we have

$$\begin{aligned} &\left|\int_\Omega f(x,u_n)(u_n-u)\right| \\ &\quad\leqslant C_1\int_\Omega |u_n||u_n-u| + C_2\int_\Omega |u_n|^{p-1}|u_n-u| \\ &\quad\leqslant C_1\left(\int_\Omega |u_n|^2\right)^{1/2}\left(\int_\Omega |u_n-u|^2\right)^{1/2} \\ &\qquad + C_2\left(\int_\Omega |u_n|^p\right)^{(p-1)/p}\left(\int_\Omega |u_n-u|^p\right)^{1/p} \\ &\quad\leqslant C_3\left(\int_\Omega |u_n-u|^2\right)^{1/2} + C_4\left(\int_\Omega |u_n-u|^p\right)^{1/p} \longrightarrow 0, \\ &\qquad\qquad n \longrightarrow \infty. \end{aligned} \tag{20}$$

Similarly, we have

$$\left|\int_\Omega f(x,u)(u_n-u)\right| \longrightarrow 0, \quad n \longrightarrow \infty. \tag{21}$$

Noting that

$$\begin{aligned} &\left\langle J'_\nu(u_n) - J'_\nu(u), u_n-u\right\rangle \\ &\quad= \left\langle J'_\nu(u_n), u_n-u\right\rangle - \left\langle J'_\nu(u), u_n-u\right\rangle \\ &\quad= \int_\Omega \nabla u_n\cdot\nabla(u_n-u) + \int_\Omega |\nabla u_n|^{p-2}\nabla u_n\cdot\nabla(u_n-u) \\ &\qquad - \nu\int_\Omega f(x,u_n)(u_n-u) - \int_\Omega \nabla u\cdot\nabla(u_n-u) \\ &\qquad - \int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla(u_n-u) + \nu\int_\Omega f(x,u)(u_n-u) \\ &\quad= \int_\Omega |\nabla(u_n-u)|^2 \\ &\qquad + \int_\Omega \left(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u\right)\cdot\nabla(u_n-u) \\ &\qquad - \nu\int_\Omega f(x,u_n)(u_n-u) + \nu\int_\Omega f(x,u)(u_n-u), \end{aligned} \tag{22}$$

and the inequality deduced from an inequality in Appendix of [3],

$$\begin{aligned} &\int_\Omega \left(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u\right)\cdot\nabla(u_n-u) \\ &\qquad\geqslant \frac{2}{p(2^{p-1}-1)}\int_\Omega |\nabla(u_n-u)|^p, \end{aligned} \tag{23}$$

it follows from (20) and (21) that

$$\begin{aligned} \frac{2}{p(2^{p-1}-1)}\int_\Omega |\nabla(u_n-u)|^p &\leqslant \left\langle J'_\nu(u_n) - J'_\nu(u), u_n-u\right\rangle \\ &\quad + \nu\int_\Omega f(x,u_n)(u_n-u) \\ &\quad - \nu\int_\Omega f(x,u)(u_n-u) \\ &\longrightarrow 0, \quad n \longrightarrow \infty, \end{aligned} \tag{24}$$

where we have used the fact that

$$\left\langle J'_\nu(u_n) - J'_\nu(u), u_n-u\right\rangle \longrightarrow 0, \quad n \longrightarrow \infty. \tag{25}$$

Hence $u_n \to u$ in X. The proof is completed. □

Lemma 6. *There exists a sequence $\{\nu_n\} \subset I$ with $\nu_n \to 1^-$ as $n \to \infty$ and $\{u_{\nu_n}\} \subset X$ such that $J_{\nu_n}(u_{\nu_n}) = c_{\nu_n}$, $J'_{\nu_n}(u_{\nu_n}) = 0$.*

Proof. We only need to show that for almost every $\nu \in I$ there exists $u^\nu \in X$ such that $J_\nu(u^\nu) = c_\nu$ and $J'_\nu(u^\nu) = 0$. By Theorem 2, for almost each $\nu \in I$, there exists a bounded sequence $\{u_n^\nu\} \subset X$ such that

$$J_\nu(u_n^\nu) \longrightarrow c_\nu, \quad J'_\nu(u_n^\nu) \longrightarrow 0, \quad n \longrightarrow \infty. \tag{26}$$

By Lemma 5, we may assume that $u_n^\nu \to u^\nu$ in X as $n \to \infty$. Then the continuity of J_ν and J'_ν implies that $J_\nu(u^\nu) = c_\nu$ and $J'_\nu(u^\nu) = 0$. The proof is completed. □

Define $\langle Lu, v\rangle = \int_\Omega f(x,u)v$, $\langle Ku, v\rangle = \int_\Omega |u|^{p-2}uv$, $u, v \in X$. Then we have the following.

Lemma 7. *Supposing* (f_1) *and* (f_2) *hold, then*

$$\lim_{\|u\|_X \to \infty, u\in P} \frac{Lu - f_\infty Ku}{\|u\|_X^{p-1}} = 0. \tag{27}$$

Proof. By (f_1), (f_2), for every $\varepsilon > 0$, there is a constant $C_\varepsilon > 0$ such that

$$\left|f(x,t) - f_\infty t^{p-1}\right| \leqslant C_\varepsilon + \varepsilon t^{p-1}, \quad x \in \overline{\Omega},\ t \geqslant 0. \tag{28}$$

For $u \in P \setminus \{0\}$, letting $w = u/\|u\|_X$, by Hölder's inequality and Sobolev's embedding theorem, we have

$$\begin{aligned}
&\sup_{\|v\|_X \leqslant 1} \left|\left\langle \frac{Lu - f_\infty Ku}{\|u\|_X^{p-1}}, v\right\rangle\right| \\
&\quad \leqslant \sup_{\|v\|_X \leqslant 1} \int_\Omega \frac{\left|f(x,u) - f_\infty u^{p-1}\right|}{\|u\|_X^{p-1}} |v| \\
&\quad \leqslant \sup_{\|v\|_X \leqslant 1} \int_\Omega C_\varepsilon \|u\|_X^{-(p-1)} |v| + \varepsilon w^{p-1} |v| \\
&\quad \leqslant C_6 \|u\|_X^{-(p-1)} + \varepsilon C_5,
\end{aligned} \tag{29}$$

where C_5 is independent of ε. The proof is completed. □

Proof of Theorem 1. By Lemma 6, there exists a sequence $\{\nu_n\} \subset I$ with $\nu_n \to 1^-$ as $n \to \infty$ and $\{u_{\nu_n}\} \subset X$ such that

$$J_{\nu_n}\left(u_{\nu_n}\right) = c_{\nu_n}, \qquad J'_{\nu_n}\left(u_{\nu_n}\right) = 0. \tag{30}$$

By Lemma 4 and (30), we have $c_{\nu_n} \geqslant c > 0$ and $\langle J'_{\nu_n}(u_{\nu_n}), u_{\nu_n}^-\rangle = 0$. Hence $u_{\nu_n} \in P \setminus \{0\}$. In the following, we first claim that $\{u_{\nu_n}\}$ is bounded in X. Suppose by contradiction that $\lim_{n\to\infty}\|u_{\nu_n}\|_X = \infty$. Let $w_n = u_{\nu_n}/\|u_{\nu_n}\|_X$. Hence, we have, for $v \in X$,

$$\begin{aligned}
&\frac{1}{\left\|u_{\nu_n}\right\|_X^{p-2}} \int_\Omega \nabla w_n \nabla v + \int_\Omega \left|\nabla w_n\right|^{p-2} \nabla w_n \nabla v \\
&\quad = \nu_n f_\infty \int_\Omega w_n^{p-1} v + \nu_n \int_\Omega \frac{f\left(x, u_{\nu_n}\right) - f_\infty u_{\nu_n}^{p-1}}{\left\|u_{\nu_n}\right\|_X^{p-1}} v.
\end{aligned} \tag{31}$$

Since $\{w_n\}$ is bounded in X, we may assume that $w_n \rightharpoonup w_0 \in P \subset X$, $w_n \to w_0$ in $L^p(\Omega)$ and $w_n(x) \to w_0(x)$ a.e. on Ω as $n \to \infty$. Letting $v = w_n - w_0$ in (31) and $n \to \infty$, we get

$$\lim_{n\to\infty} \int_\Omega \left|\nabla w_n\right|^{p-2} \nabla w_n \cdot \nabla\left(w_n - w_0\right) = 0. \tag{32}$$

It follows from [5, Theorem 10] that $w_n \to w_0$ in X as $n \to \infty$. Passing to limit $n \to \infty$ in (31), we obtain by Lemma 7 that

$$\int_\Omega \left|\nabla w_0\right|^{p-2} \nabla w_0 \cdot \nabla v = f_\infty \int_\Omega w_0^{p-1} v, \quad v \in X. \tag{33}$$

From (33) and the fact that $\|w_0\|_X = 1$, we know that $f_\infty = \mu_1$, which contradicts the assumption $f_\infty > \mu_1$. Since $\nu_n \to 1^-$, we can show that

$$J'_1\left(u_{\nu_n}\right) \longrightarrow 0 \quad \text{in } X^*, \quad n \longrightarrow \infty. \tag{34}$$

In fact, for any $v \in X$, it follows from (19), Hölder's inequality, and Sobolev's embedding theorem that

$$\begin{aligned}
\left|\int_\Omega f\left(x, u_{\nu_n}\right) v\right| &\leqslant C_1 \int_\Omega \left|u_{\nu_n}\right| |v| + C_2 \int_\Omega \left|u_{\nu_n}\right|^{p-1} |v| \\
&\leqslant C_7 \|v\|_X.
\end{aligned} \tag{35}$$

Furthermore, (30) implies that

$$\left\langle J'_1\left(u_{\nu_n}\right), v\right\rangle + \left(1 - \nu_n\right) \int_\Omega f\left(x, u_{\nu_n}\right) v = \left\langle J'_{\nu_n}\left(u_{\nu_n}\right), v\right\rangle = 0, \quad v \in X. \tag{36}$$

Hence, $J'_1(u_{\nu_n}) \to 0$ in X^* as $n \to \infty$. By Lemma 5, $\{u_{\nu_n}\}$ has a convergent subsequence. Without loss of generality, we may assume that $u_{\nu_n} \to u$ as $n \to \infty$. According to Lemma 4, (30) and

$$\left|\int_\Omega F\left(x, u_{\nu_n}\right)\right| \leqslant C_8, \tag{37}$$

we have

$$\begin{aligned}
J_1(u) &= \lim_{n\to\infty} J_1\left(u_{\nu_n}\right) = \lim_{n\to\infty} J_{\nu_n}\left(u_{\nu_n}\right) \geqslant c > 0, \\
J'_1(u) &= \lim_{n\to\infty} J'_1\left(u_{\nu_n}\right) = 0.
\end{aligned} \tag{38}$$

The standard process shows that u is a positive solution to (1). The proof is completed. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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