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Research Article

Dual Algebras and A-Measures

Marek Kosiek[1] and Krzysztof Rudol[2]

[1] *Faculty of Mathematics and Computer Sciences, Jagiellonian University, ulica Prof. St. Lojasiewicza 6, 30-348 Kraków, Poland*
[2] *Faculty of Applied Mathematics, AGH University of Science and Technology, Aleja Mickiewicza 30, 30-059 Kraków, Poland*

Correspondence should be addressed to Krzysztof Rudol; grrudol@cyfronet.pl

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Weak-star closures of Gleason parts in the spectrum of a function algebra are studied. These closures relate to the bidual algebra and turn out both closed and open subsets of a compact hyperstonean space. Moreover, weak-star closures of the corresponding bands of measures are reducing. Among the applications we have a complete solution of an abstract version of the problem, whether the set of nonnegative A-measures (called also Henkin measures) is closed with respect to the absolute continuity. When applied to the classical case of analytic functions on a domain of holomorphy $\Omega \subset \mathbb{C}^n$, our approach avoids the use of integral formulae for analytic functions, strict pseudoconvexity, or some other regularity of Ω. We also investigate the relation between the algebra of bounded holomorphic functions on Ω and its abstract counterpart—the w^* closure of a function algebra A in the dual of the band of measures generated by one of Gleason parts of the spectrum of A.

1. Introduction

The idea of using second duals for algebras of analytic function goes back to the works of Brian Cole and Theodore W. Gamelin. It turned out to be a powerful tool for studying some spectral and approximation properties of algebras of analytic functions over domains Ω or even on complex manifolds. Under certain regularity assumptions on $\Omega \subset \mathbb{C}^N$, the abstract counterpart of $H^\infty(\Omega)$ obtained by duality techniques corresponds isomorphically to the classical algebra of bounded analytic functions, allowing the use of functional analysis tools for dual spaces.

Let us recall that a *uniform algebra* (or *function algebra*) over a compact Hausdorff space X is a closed unital subalgebra A of $C(X)$ separating the points of X. Then we can assume that A is a uniform algebra on its spectrum $\mathtt{Sp}(A)$—the space of nonzero linear and multiplicative homomorphisms of A with Gelfand topology. In this abstract setting the domain Ω corresponds to a nontrivial Gleason part G of $\mathtt{Sp}(A)$ and measures on X absolutely continuous with respect to measures representing certain points of G form a band of measures denoted by $\mathscr{M}_G$. The weak-star density of G in the spectrum of some quotient algebra of A^{**} is an equivalent formulation of the corona problem in $H^\infty(\Omega)$ settled 50 years ago by Lennart Carleson in the unit disc case and still open for the higher dimensional balls or polydiscs. This alone justifies the need for better understanding of the nature of weak-star closures of nontrivial Gleason parts G canonically embedded in the dual space for $\mathscr{M}_G$ (or in some other bidual spaces).

In Section 2 we formulate some preliminary observations. Quite simple distance estimates for quotient norms in dual spaces (modulo the set $A^\perp$ of annihilating measures) are obtained from the corresponding band decompositions.

Section 3 focuses on dualities and the related weak-star topologies (abbreviated as w-*), namely, on the w-* closures. Some natural relations are shown to hold between bands of measures on a compact space X, closed ideals in $C(X)$ and w-* closed ideals in its bidual space $C(X)^{**}$, exploring the fact that the latter is of the form $C(Y)$ for a compact space Y.

Our main result concerns Gleason parts in the spectrum of a uniform algebra $A \subset C(X)$. Due to the canonical embeddings, we consider such parts $G \subset A^*$ as subsets of certain bidual spaces. Theorem 6 describes some unexpected property of their w-*closures, relating them also to the w-* closures of the bands of measures generated by G. The obtained relations imply the compatibility of such

closures with the Lebesgue-type band decompositions. As a corollary, we identify these decompositions as the Arens multiplications by some idempotent $F_0 \in A^{**}$.

One of the consequences of Theorem 6 presented in Section 4 is that the representing measures are supported by w-* closures of the corresponding Gleason parts.

In Section 5 we apply the results of Section 3 relating an abstract algebra of H^∞-type to the algebra of bounded analytic functions on a star-like domain. As a result, we obtain an alternative representation of a predual space to $H^\infty(\Omega)$ and "dual algebra property."

Our results give also an abstract solution to the "*A-measures problem*." The A-measures appearing in Section 6 (an abbreviation for *analytic measures*) were introduced in [1] under the name "L-measures" in the case of $A = A(G)$, the algebra of analytic functions on a domain G, continuous at its Euclidean closure $\overline{G}$. (Some authors call them *Henkin measures*; the notation $A(\overline{G})$ can also be met for the algebra $A(G)$.) These are the measures μ on ∂G, for which the integrals of the pointwise convergent to 0 on G, bounded sequences in A, converge to zero. The problem was to verify whether a measure on ∂G, which is absolutely continuous with respect to some representing measure, is itself an A-measure. In Theorem 19 we obtain a general result on A-measures. Here the domain G is replaced by a Gleason part of A and the measures are supported by X. When applied to concrete algebras $A(G)$, this solves the A-measures problem for a wide range of domains, extending the previously known results.

2. Preliminaries

In what follows A will denote a *uniform algebra* on some compact Hausdorff space X, meaning a closed subalgebra of $C(X)$ containing the constants and separating the points of X. We may assume additionally (see [2], Chapter II) that

$$X \text{ is equal to the spectrum of } A \ \left(\text{denoted by } \mathtt{Sp}(A)\right). \tag{1}$$

A closed linear subspace $\mathscr{M}$ in the space $M(X)$ of regular complex Borel measures on X is called a *band of measures* (cf. [3, 4] V§ 17), if along with any $\mu \in \mathscr{M}$ it contains all measures ν absolutely continuous with respect to μ. For $\mathscr{E} \subset M(X)$ the set $\mathscr{E}^s$ of all measures singular to any $\nu \in \mathscr{E}$ is a band, and $\mathscr{E}^{ss} := (\mathscr{E}^s)^s$ is the smallest band containing $\mathscr{E}$, referred to as *the band generated by* $\mathscr{E}$. In particular, we have $\mathscr{M} = \mathscr{M}^{ss}$. Any $\eta \in M(X)$ has the following Lebesgue-type decomposition:

$$\eta = \eta^{\mathscr{M}} + \eta^s \quad \text{with } \eta^{\mathscr{M}} \in \mathscr{M}, \quad \eta^s \in \mathscr{M}^s. \tag{2}$$

The space $M(X)$ is a direct sum of $\mathscr{M}$ and $\mathscr{M}^s$ and for any pair $\mu_1 \in \mathscr{M}$, $\mu_2 \in \mathscr{M}^s$ their total variation norms satisfy

$$\|\mu_1 + \mu_2\| = \|\mu_1\| + \|\mu_2\|. \tag{3}$$

Let $A^\perp$ be the set of all measures $\mu \in M(X)$ annihilating A, that is, such that $\int f\,d\mu = 0$ for any $f \in A$. We say that a band is called *reducing*, if $\eta^{\mathscr{M}} \in A^\perp$ for any $\eta \in A^\perp$ in the above decomposition (2).

If $\phi : A \to \mathbb{C}$ is a nonzero linear and multiplicative functional (a fact denoted by $\phi \in \mathtt{Sp}(A)$), we say that $\mu \in M(X)$ is a *complex representing measure* for ϕ if

$$\phi(f) = \int_X f\,d\mu \quad \text{for any } f \in A. \tag{4}$$

By *representing measure* we understand a nonnegative representing measure. For $\phi_1, \phi_2 \in \mathtt{Sp}(A)$ belonging to the same *Gleason part* G (i.e., satisfying $\|\phi_1 - \phi_2\| < 2$) the bands generated by all measures representing ϕ_1 (resp., ϕ_2) are equal and we denote this band by $\mathscr{M}_G$. A form of an abstract version of F. and M. Riesz theorem ([4] V 18.2, [2] II 7.6) asserts that $\mathscr{M}_G$ is always a reducing band. It is also well known that Gleason parts are Borel sets in the Gelfand topology, as (with ϕ_1 fixed) the above G is a countable union of compact sets (namely, of $\{\phi \in \mathtt{Sp}(A) : \|\phi_1 - \phi\| \le 2 - (1/n)\}$).

In the next section we need the following properties of three quotient norms of the equivalence classes $[\mu]$ of $\mu \in M(X)$: the first one is the norm in $\mathscr{M}/(A^\perp \cap \mathscr{M})$ (defined as the distance: $\|[\mu]\|_Q = \operatorname{dist}(\mu, A^\perp \cap \mathscr{M}) = \inf\{\|\mu - \eta\| : \eta \in A^\perp \cap \mathscr{M}\}$). The second one, the norm in $M(X)/A^\perp$, is denoted here simply as $\|[\mu]\|$. These both turn out to be equal to the third one, in $\mathscr{M}/(A^\perp + \mathscr{M}^s)$. (Note that each of the equivalence classes, denoted for simplicity as $[\mu]$, is in a different quotient space.)

Lemma 1. *If $\mathscr{M}$ is a reducing band then for any $\mu \in \mathscr{M}$ and $\nu \in \mathscr{M}^s$ one has*

$$\|[\mu]\|_Q = \|[\mu]\|, \quad \textit{that is,} \ \operatorname{dist}(\mu, A^\perp \cap \mathscr{M}) = \operatorname{dist}(\mu, A^\perp), \tag{5}$$

$$\|[\mu + \nu]\| = \|[\mu]\| + \|[\nu]\|. \tag{6}$$

For $\mu \in \mathscr{M}$ the quotient space norm of $[\mu]$ in $M(X)/A^\perp$ satisfies also

$$\begin{aligned} \|[\mu]\| &= \operatorname{dist}(\mu, \mathscr{M}^s + A^\perp) \\ &= \inf\{\|\mu - (\nu + \eta)\| : \nu \in \mathscr{M}^s, \eta \in A^\perp\}. \end{aligned} \tag{7}$$

Proof. Decompose $\eta \in A^\perp$ with respect to $\mathscr{M}$ as in (2). From (3) we obtain

$$\|\mu - \eta\| = \|\mu - \eta^{\mathscr{M}}\| + \|\eta^s\| \ge \|\mu - \eta^{\mathscr{M}}\|. \tag{8}$$

Since $\mathscr{M}$ is a reducing band, $\eta^{\mathscr{M}} \in A^\perp \cap \mathscr{M}$ and the nontrivial inequality in (5)

$$\inf_{\eta \in A^\perp} \|\mu - \eta\| \ge \inf_{\eta' \in (A^\perp \cap \mathscr{M})} \|\mu - \eta'\| \tag{9}$$

follows. The opposite inequality in (5) is obvious, because $A^\perp \cap \mathscr{M} \subset A^\perp$.

Now using (3) for $\mu_1 = \mu - \eta^{\mathscr{M}}$, $\mu_2 = \nu - \eta^s$ we have

$$\|\mu + \nu - \eta\| = \|\mu - \eta^{\mathscr{M}} + (\nu - \eta^s)\| = \|\mu - \eta^{\mathscr{M}}\| + \|\nu - \eta^s\| \tag{10}$$

since $\mu-\eta^{\mathscr{M}} \in \mathscr{M}$, $\nu-\eta^{s} \in \mathscr{M}^{s}$. Hence $\|\mu+\nu-\eta\| \geq \|[\mu]\|+\|[\nu]\|$ for $\eta \in A^{\perp}$. Passing to the infimum over $\eta \in A^{\perp}$ we get finally

$$\|[\mu]\| + \|[\nu]\| \geq \|[\mu+\nu]\| \geq \|[\mu]\| + \|[\nu]\|, \tag{11}$$

showing (6). After replacing ν with $-\nu$ in (10), we obtain the nontrivial inequality "≤" in (7), since $\|\mu-(\nu+\eta)\| \geq \|\mu-\eta^{\mathscr{M}}\| \geq \|[\mu]\|$. The rest follows from the inclusion $A^{\perp} \subset \mathscr{M}^{s}+A^{\perp}$. □

3. Second Duals

If the dual E^* of a Banach space is contained isometrically as a subspace of some Banach space B, one way of representing the second dual, E^{**} (see, e.g., [3]), is to consider the weak-star closure in B^* of E, denoted here by $\overline{E}^{ws}$. In some cases this w-* topology will be denoted more precisely as $\sigma(B^*, B)$, to avoid possible ambiguity.

By the bipolar theorem applied to linear subspaces E, we have $\overline{E}^{ws} = ({}^{\perp}E)^{\perp}$, where ${}^{\perp}E = \{y \in B : \phi(y) = 0 \text{ for any } \phi \in E\}$ is the *preannihilator of* E. The symbol j used for all canonical embeddings in the second duals will (almost always) be suppressed, while taking the w-* closures: instead of $\overline{j(\Delta)}^{ws}$, we write $\overline{\Delta}^{ws}$ for any set Δ in the considered space. Since ${}^{\perp}(j\Delta) = \Delta^{\perp}$, the notation $\Delta^{\perp\perp}$ for this closure $\overline{\Delta}^{ws}$ is justified when Δ is a subspace. Recall that by Goldstine's theorem the unit ball of E is weak-star dense in the unit ball of E^{**}.

The second dual of a uniform algebra A has a multiplication extending that in A, called *the Arens product*. Actually, there are two Arens products (they correspond to different orders of iteration mentioned below), but for subalgebras A of commutative C^*-algebras they both coincide (a fact known as *the Arens regularity* of A). Moreover, $C(X)^{**}$ endowed with the Arens multiplication is a commutative C^*-algebra; hence it is isometrically isomorphic to $C(Y)$ for some extremally disconnected space Y, called *the hyperstonean envelope* of X. In other words,

$$Y = \mathtt{Sp}\left(C(X)^{**}\right). \tag{12}$$

The Arens product of $F, H \in A^{**}$ can also be interpreted as an iterated w-* limit of the product $f_\iota h_\kappa$ of bounded nets f_ι and h_κ in A, weak-star converging to F (resp., to H), and the reverse order of iteration yields the same result, due to the Arens regularity of uniform algebras. We refer to [5] for the details on the Arens product and to [3] for the isometric identification of $A^{\perp\perp}$ with A^{**}. Let us just note that, as $M(X) = C(X)^*$, the isomorphism from $M(X)/A^{\perp}$ onto A^* is obtained through the factorisation of the restriction of $\mu \in M(X)$ to A; that is,

$$\langle [\mu], f\rangle = \int f\, d\mu \quad [\mu] \in M(X)/A^{\perp}, f \in A. \tag{13}$$

Next, denote by $\widetilde{\mu}$ the measure on Y obtained by lifting μ, that is, by representing the functional $C(X) \ni h \to \int h\, d\mu$, extended to $C(Y)$. Given $F \in C(Y) = M(X)^*$ and $[\mu] \in M(X)/A^{\perp}$, we have the natural duality formulae

$$\langle F, [\mu]\rangle = \langle F, \mu\rangle = \int F\, d\widetilde{\mu}, \tag{14}$$

well-defined (independent of the choice of the representative of $[\mu]$) if and only if $F \in A^{\perp\perp}$.

On the other hand, for certain subsets G of the algebra's spectrum $\mathtt{Sp}(A) \subset A^*$, we will take the closure $\overline{G}^{ws}$ in A^{***} endowed with its w^* topology $\sigma(A^{***}, A^{**})$. Then it is natural to ask whether this closure is still a subset of $\mathtt{Sp}(A^{**})$. To see that this is the case, first check that the canonical embedding $j : A^* \to A^{***}$ satisfies $j(\mathtt{Sp}(A)) \subset \mathtt{Sp}(A^{**})$. (Here one can either invoke Lemma 3.6 in [6] or use the iterated limits representation of the Arens product FH and the w^*-continuity on A^{**} of $j(\phi)$ for $\phi \in A^*$, deducing that $\phi(FH) = \phi(F)\phi(H)$ in this case.) Then passing to w^*-limits in A^{***} of the form $\Phi = \lim j(\phi_m)$, where $\phi_m \in G$, we verify the needed multiplicativity of Φ.

Let us look now at the weak-star closure of a band of measures $\mathscr{M} \subset M(X)$ in $M(X)^{**}$. Note that if one identifies $C(X)^{**} = M(X)^*$ with $C(Y)$, then one can write $M(X)^{**} = M(Y)$. If $\mu \in M(X)$ and $h \in L^1(\mu)$, we denote by $h\mu$ the measure on X having the density h with respect to μ.

Lemma 2. *The following formula holds:*

$$\overline{\mathscr{M}}^{ws} + \overline{\mathscr{M}^{s}}^{ws} = M(X)^{**} \ (= M(Y)). \tag{15}$$

Proof. Let $\mu \in M(X)^{**}$. Then there is a net $\{\mu_\alpha\} \subset M(X)$ such that $\|\mu_\alpha\| \leq \|\mu\|$ and $\mu_\alpha \to \mu$ weak-star. For each α we have $\mu_\alpha = \mu_\alpha^{\mathscr{M}} + \mu_\alpha^{s}$, where $\mu_\alpha^{\mathscr{M}} \in \mathscr{M}$, $\mu_\alpha^{s} \in \mathscr{M}^{s}$, and $\|\mu_\alpha\| = \|\mu_\alpha^{\mathscr{M}}\| + \|\mu_\alpha^{s}\|$. We have $\|\mu_\alpha^{\mathscr{M}}\| \leq \|\mu\|$ and $\|\mu_\alpha^{s}\| \leq \|\mu\|$ for each α. Hence, by the Banach-Alaoglu theorem, the net $\{\mu_\alpha^{\mathscr{M}}\}$ has an adherent point μ' and the net $\{\mu_\alpha^{s}\}$ has an adherent point μ''. After passing to a suitable subnet, if necessary, we can write $\mu_\alpha^{\mathscr{M}} \to \mu'$ and $\mu_\alpha^{s} \to \mu''$. Consequently $\mu' \in \overline{\mathscr{M}}^{ws}$, $\mu'' \in \overline{\mathscr{M}^{s}}^{ws}$, and $\mu = \mu' + \mu''$ which completes the proof. □

There is an important, yet easy to verify, relation between bands (which are ideals with respect to an order in $M(X)$) and closed ideals in $C(X)$. The latter are of the form $J_E := \{f \in C(X) : f|_E = 0\}$ for some closed subset $E \subset X$ (see [2]). We also identify $M(E)$ with a band in $M(X)$; namely, $M(E) := \{\mu \in M(X) : \mu = \mu_E\}$, where $\mu_E(L) := \mu(L \cap E)$ for all Borel sets $L \subset X$.

Proposition 3. *If J_1 is an ideal in $C(X)$ and J_2 is an ideal in $C(X)^{**}$, then*

(1) *the annihilator, $J_1^{\perp}$, is a band in $M(X)$;*

(2) *the preannihilator, ${}^{\perp}J_2$, is a weak-star closed band in $M(X)$.*

Conversely, if $\mathscr{M}$ is a band in $M(X)$ then the following holds.

(3) *${}^{\perp}\mathscr{M}$ is a closed ideal in $C(X)$ and $({}^{\perp}\mathscr{M}) \cap A$ is a closed ideal in A.*

(4) *$\mathscr{M}^{\perp}$ is a weak-star closed ideal in $C(X)^{**} = M(X)^*$ and $\mathscr{M}^{\perp} \cap A^{**}$ is a weak-star closed ideal in $A^{**} \approx (A^{\perp})^{\perp}$.*

(5) *For some closed sets $E, K \subset X$ with $E \cup K = X$, one has*

$$\overline{\mathscr{M}}^{ws} = \left({}^{\perp}\mathscr{M}\right)^{\perp} = M(E), \qquad \overline{\mathscr{M}^s}^{ws} = \left({}^{\perp}\mathscr{M}^s\right)^{\perp} = M(K). \tag{16}$$

(6) *For some closed sets $\widetilde{E}, \widetilde{K} \subset Y$ such that $\widetilde{E} \cup \widetilde{K} = Y$, the closures $\overline{\mathscr{M}}^{\sigma}$ (of $j(\mathscr{M})$ in $\sigma(M(Y), C(X)^{**})$, where $j : M(X) \to M(X)^{**}$) and $\overline{\mathscr{M}^s}^{\sigma}$ satisfy*

$$\overline{\mathscr{M}}^{\sigma} = \mathscr{M}^{\perp\perp} = M\left(\widetilde{E}\right), \qquad \overline{\mathscr{M}^s}^{\sigma} = \left(\mathscr{M}^s\right)^{\perp\perp} = M\left(\widetilde{K}\right). \tag{17}$$

(7) *If a band $\mathscr{M} \subset M(X)$ is weak-star closed (namely, $\sigma(M(X), C(X))$-closed), then $\mathscr{M} = M(E)$ for some closed subset $E \subset X$ and in this case, the Lebesgue-type decomposition* (2) *has the simplest form: $\mu^{\mathscr{M}} = \mu_E$, $\mu^s = \mu_{X\setminus E}$.*

Proof. (1) Obviously, $J_1^{\perp}$ is a closed subspace in $M(X)$. It suffices to check that for $\mu \in J_1^{\perp}$ any absolutely continuous measure (which is of the form $h\mu$ with $h \in L^1(\mu)$) is also in $J_1^{\perp}$. Since μ is regular, $C(X)$ is dense in $L^1(\mu)$. For some sequence of $h_n \in C(X)$ we have $h = \lim h_n$ in the norm of $L^1(\mu)$. For $f \in J_1$ we have $fh_n \in J_1$, since J_1 is an ideal. Now $0 = \int fh_n\, d\mu$ converge to $\int fh\, d\mu$. Hence $h\mu \in J_1^{\perp}$.

(2) The only difference from (1) is that given $F \in J_2$, $\mu \in {}^{\perp}J_2$, and h, h_n as above, we have $\langle F, h\mu\rangle = \lim\langle F, h_n\mu\rangle = 0$, where the first equality follows from the convergence $h_n\mu \to h\mu$ in $M(X)$. The second one follows since $j(h_n) \in C(X)^{**}$ and $Fj(h_n) \in J_2$, while $\langle F, h_n\mu\rangle = \langle Fj(h_n), \mu\rangle$, by the definition of the Arens product in $C(X)^{**}$.

(3) and (4) are obvious and follow from the absolute continuity $h\mu \ll \mu$.

(5) The ideals ${}^{\perp}\mathscr{M}$, ${}^{\perp}(\mathscr{M}^s)$ in $C(X)$ are closed. Hence ${}^{\perp}\mathscr{M} = J_E$, ${}^{\perp}(\mathscr{M}^s) = J_K$ for some closed subsets $E, K \subset X$. Obviously, the annihilators of the latter ideals are identified with $M(E)$ and $M(K)$, respectively, showing that $({}^{\perp}\mathscr{M})^{\perp} = M(E)$ and $({}^{\perp}\mathscr{M}^s)^{\perp} = M(K)$.

As $\mathscr{M} \cap (\mathscr{M}^s) = \{0\}$, the equality $E \cup K = X$ follows. Indeed, given an arbitrary $f \in {}^{\perp}\mathscr{M} \cap {}^{\perp}(\mathscr{M}^s)$, decompose any measure $\eta \in M(X)$, like in (2). Then $\int f\, d\eta = \int f\, d\eta^{\mathscr{M}} + \int f\, d\eta^s = 0$, showing that ${}^{\perp}\mathscr{M} \cap {}^{\perp}(\mathscr{M}^s) = \{0\}$. Taking the annihilators we get $E \cup K = X$ by Lemma 2.

(6) Here the proof is quite analogous to that of (5).

The last claim comes from the description of closed ideals in $C(X)$, giving ${}^{\perp}\mathscr{M}$ the form of some J_E and from the equality $\mathscr{M} = ({}^{\perp}\mathscr{M})^{\perp}$, which holds in this case. □

Let us recall from [2], II.12, that a peak set for A is a subset $E \subset X$ on which some $f \in A$ equals 1, while satisfying $|f(y)| < 1$ for any point $y \in X\setminus E$. Sets arising as intersections of a family of peak sets are called p-sets. Let $\mathscr{M}$ be a reducing band for our uniform algebra. By Glicksberg's theorem ([2], 12.7) we get the following.

Corollary 4. *If $\mathscr{M} = M(E)$ is a weak-star closed band in $M(X)$, where $E \subset X$, then E is a p-set if and only if the band $M(E)$ is reducing.*

Example 5. When $A = A(\mathbb{D})$ is the classical disc algebra and $X = \overline{\mathbb{D}}$ is the closed unit disc, take a Cantor-type subset E of the unit circle $\partial\mathbb{D}$ of zero arc length. In the case of bidisc algebra $A(\mathbb{D}^2)$ as E we may take the set $\{z_0\} \times \partial\mathbb{D}$ for some $z_0 \in \mathbb{D}$. In both cases peak functions for E can be constructed directly. Hence we obtain specific examples of reducing bands of the form $M(E)$ for these algebras.

Note that $\overline{\mathscr{M}}^{ws} + \overline{(\mathscr{M}^s)}^{ws}$ is not necessarily a direct sum as the related sets $\widetilde{E}$ and $\widetilde{K}$ may intersect. However, in the most important case, the situation is much better: in what follows, we fix a Gleason part $G \subset \mathrm{Sp}(A)$ and $\mathscr{M}_G$ denotes the band generated by representing measures for the points $\phi \in G$, while $M(\overline{G}^{ws}) = \{\mu_{\overline{G}^{ws}} : \mu \in M(X)\}$ is the set of all regular complex Borel measures on $\overline{G}^{ws}$.

Theorem 6. *If G is a Gleason part of A then $\overline{G}^{ws}$ (the closure of $j(G)$ in the $\sigma(A^{***}, A^{**})$-topology) is a closed-open subset of Y. Moreover*

$$\begin{gathered} Y \setminus \overline{G}^{ws} = \overline{X \setminus G}^{ws}, \qquad \left(\overline{\mathscr{M}_G}^{ws}\right)^s = \overline{(\mathscr{M}_G^s)}^{ws}, \\ \overline{\mathscr{M}_G}^{ws} = M\left(\overline{G}^{ws}\right) \end{gathered} \tag{18}$$

and $\overline{\mathscr{M}_G}^{ws}$ is a reducing band for A^{**}.

Proof. Let $\phi_0 \in G$. Since A^* is isometrically identified with $M(X)/A^{\perp}$, we have $1 = \|\phi_0\| = \|[\nu]\|$ for each representing measure ν of ϕ_0. By (7) of Lemma 1, we therefore get

$$1 = \|[\nu]\| = \operatorname{dist}\left(\nu, \mathscr{M}_G^s + A^{\perp}\right). \tag{19}$$

There exists $F_0 \in A^{**} \approx \overline{A}^{ws} \approx A^{\perp\perp}$ having norm 1 and such that $\langle F_0, \phi_0\rangle = 1$ and $F_0|_{\mathscr{M}_G^s} \equiv 0$. In fact, the norm-attaining (at $[\nu]$) functional on $M(X)/(\mathscr{M}^s + A^{\perp})$ composed with the canonical surjection assigning to μ its equivalence class $[\mu] = \mu + \mathscr{M}^s + A^{\perp}$ does the job.

Let $\phi \in G$. Then $\|\phi_0 - \phi\| < 2$. Since the norm in A^* with respect to A is the same as the norm with respect to A^{**}, ϕ_0 and ϕ (canonically embedded in A^{***}) are also in the same Gleason part in Y. Thus we can find a pair of mutually absolutely continuous Borel regular measures μ_0 and μ on Y such that μ_0 (resp., μ) is a representing measure for ϕ_0 (resp. for ϕ). The equality $1 = \langle F_0, \phi_0\rangle = \int_Y F_0\, d\mu_0$ implies that F_0 as an element of $C(Y)$ equals 1 $[\mu_0]$-almost everywhere. Consequently it is also equal to 1 μ, almost everywhere, and hence $\langle F_0, \phi\rangle = \int_Y F_0\, d\mu = 1$. Since $\phi \in G$ was chosen arbitrarily, we have $\langle F_0, \phi\rangle = 1$ for all $\phi \in G$, and consequently $F_0 \equiv 1$ on $\overline{G}^{ws}$, which is the $\sigma(A^{***}, A^{**})$-closure, since $F_0 \in A^{**}$.

Similarly, as $F_0|_{\mathscr{M}_G^s} \equiv 0$, we have $F_0 \equiv 0$ on $X\setminus G$ and $F_0 \equiv 0$ on $\overline{X \setminus G}^{ws}$, by continuity. But $\overline{M(G)}^{ws} + \overline{M(X \setminus G)}^{ws} = M(Y)$ by Lemma 2 applied to the band $M(G)$. Since $G \subset \overline{G}^{ws}$, we have $M(G) \subset M(\overline{G}^{ws})$. On the other hand, $M(\overline{G}^{ws}) = (J_{\overline{G}^{ws}})^{\perp}$, which implies that it is a weak-star closed band, and hence $\overline{M(G)}^{ws} \subset M(\overline{G}^{ws})$. Similarly $\overline{M(X \setminus G)}^{ws} \subset M(\overline{X \setminus G}^{ws})$, whence

$$M\left(\overline{G}^{ws}\right) + M\left(\overline{X \setminus G}^{ws}\right) = M(Y), \tag{20}$$

which implies

$$\overline{G}^{ws} \cup \overline{X \setminus G}^{ws} = Y. \tag{21}$$

We have had $F_0 \equiv 1$ on $\overline{G}^{ws}$ and $F_0 \equiv 0$ on $\overline{X \setminus G}^{ws}$. So the sets $\overline{G}^{ws}$ and $\overline{X \setminus G}^{ws}$ must be disjoint and closed-open.

We have $\langle F_0, x\rangle = 1$ for any $x \in G$, and hence $F_0 \equiv 1$ almost everywhere $[\nu_x]$ for all its representing measures $\nu_x \in M(Y)$. This equality almost everywhere to 1 occurs also with respect to any linear combination of such ν_x's, generating the band $\mathscr{M}_G$. Hence $F_0 = 1$ almost everywhere with respect to any μ from the band $\mathscr{M}_G$. Passing to w-* limits in the equalities $\langle F_0, \mu\rangle = 1$ valid for probabilistic measures $\mu \in \mathscr{M}_G$, we extend it to any probabilistic $\mu \in \overline{\mathscr{M}_G}^{ws}$.

Consequently, $\langle F_0, |\mu|\rangle = \|\mu\|$ for all measures $\mu \in \overline{\mathscr{M}_G}^{ws}$. Since $F_0|_{\mathscr{M}_G^s} \equiv 0$ (yielding $F_0|_{\overline{\mathscr{M}_G^s}^{ws}} \equiv 0$) and since F_0 as an idempotent element has norm 1, we have

$$\begin{gathered}\|\mu\| = \langle F_0, |\mu|\rangle = \langle F_0, |\mu| - \nu\rangle \le \||\mu| - \nu\| \\ \text{for } \mu \in \overline{\mathscr{M}_G}^{ws}, \quad \nu \in \overline{\mathscr{M}_G^s}^{ws}.\end{gathered} \tag{22}$$

It follows from the last inequality that the bands $\overline{\mathscr{M}_G}^{ws}$ and $\overline{\mathscr{M}_G^s}^{ws}$ are mutually singular. The latter combined with Lemma 2 gives the equality $(\overline{\mathscr{M}_G}^{ws})^s = \overline{(\mathscr{M}_G^s)}^{ws}$. By Proposition 3, there are closed subsets of $\widetilde{E}, \widetilde{K} \subset Y$ such that $\overline{\mathscr{M}_G}^{ws} = M(\widetilde{E})$ and $(\overline{\mathscr{M}_G}^{ws})^s = \overline{(\mathscr{M}_G^s)}^{ws} = M(\widetilde{K})$. Hence $\widetilde{E}, \widetilde{K}$ are disjoint and, by Lemma 2, we have $\widetilde{E} \cup \widetilde{K} = Y$. But it must be $G \subset \widetilde{E}$; hence $\overline{G}^{ws} \subset \widetilde{E}$ because $\widetilde{E}$ is weak-star closed. Since $F_0 \equiv 0$ on $Y \setminus \overline{G}^{ws} = \overline{X \setminus G}^{ws}$ (see (21)) and $\langle F_0, \mu\rangle = 1$ for all probabilistic measures $\mu \in \overline{\mathscr{M}_G}^{ws} = M(\widetilde{E})$, we must have $\widetilde{E} \subset \overline{G}^{ws}$. Hence $\overline{\mathscr{M}_G}^{ws} = M(\overline{G}^{ws})$, and by the properties of F_0 it must be a reducing band. □

From the above proof we also obtain the following.

Corollary 7. *There exists a characteristic function $F_0 \in A^{**}$ vanishing exactly on $Y \setminus \overline{G}^{ws}$. The projection associated with the decomposition into mutually singular components*

$$M(Y) = \overline{\mathscr{M}_G}^{ws} \oplus \overline{\mathscr{M}_G^s}^{ws} \tag{23}$$

is the Arens multiplication by F_0. (Metrically, this sum is of L^1-type.)

Example 8. Let Ω be an open unit ball in $\mathbb{C}^n$. Consider the algebra $A(\Omega)$ of all continuous functions on $\overline{\Omega}$ which are analytic in Ω. Its spectrum consists of one nontrivial Gleason part Ω and the union of singleton Gleason parts filling in its boundary. The fact that the whole Ω is one Gleason part follows from the existence of mutually absolutely continuous representing measures for any pair of its points. Such a pair of representing measures is given by suitable Poisson kernels (see [7], Theorem 3.3.2).

We have the following decomposition:

$$C(\overline{\Omega})^* = M(\overline{\Omega}) = \mathscr{M}_\Omega \oplus \mathscr{S}, \tag{24}$$

where $\mathscr{S} = \mathscr{M}_\Omega^s$. The measures in $\mathscr{S}$ are called *totally singular*; in the $n = 1$ case they are simply the measures singular to the Lebesgue measure on the unit circle. It is known (see [7], Chapter 9) that $\mathscr{S} \cap A(\Omega)^\perp = \{0\}$. Hence $\mathscr{S}/_{A(\Omega)^\perp \cap \mathscr{S}} = \mathscr{S}$, and using Lemma 1 we get

$$A(\Omega)^* = C(\overline{\Omega})^*/_{A(\Omega)^\perp} = \mathscr{M}_\Omega/_{A(\Omega)^\perp \cap \mathscr{M}_\Omega} \oplus \mathscr{S}. \tag{25}$$

Now $(\mathscr{M}_\Omega/_{A(\Omega)^\perp \cap \mathscr{M}_\Omega})^* \sim A(\Omega)^{\perp\perp} \cap \mathscr{M}_\Omega^*$, where the last term is isometrically isomorphic to the w-* closure of $A(\Omega)$ in $\mathscr{M}_\Omega^*$. By Theorem 17 (cf. below) we get

$$A(\Omega)^* = C(\overline{\Omega})^*/_{A(\Omega)^\perp} = H^\infty(\Omega)_* \oplus \mathscr{S}, \tag{26}$$

where $H^\infty(\Omega)_*$ denotes the predual of $H^\infty(\Omega)$. By the equality of Corollary 7 we now obtain

$$C(\overline{\Omega})^{***} = \overline{\mathscr{M}_\Omega}^{ws} \oplus \overline{\mathscr{S}}^{ws}. \tag{27}$$

Hence the measures on the hyperstonian envelope of the ball $\overline{\Omega}$ decompose onto those w^*-approximable by absolutely continuous (resp., totally singular) ones.

4. Closures of Parts and Representing Measures

Using the results of Section 3 we can localise supports of measures ν_x representing the points $x \in \mathtt{Sp}(A)$. These sets supp ν_x are "not too far" from the respective Gleason parts G (such that $x \in G$). In fact, they are contained in the Gelfand-topology closures $\overline{G}$. In the special case of the algebra $R(K)$ for a compact set $K \subset \mathbb{C}$ such a statement appears already in Theorem 3.3 in Chapter VI of [2].

Before stating the result, it is convenient to establish one lemma relevant for any nontrivial Gleason part $G \subset \mathtt{Sp}(A)$. At the beginning of Section 3 we have noted that the second conjugate to $C(X)$ is identified with $C(Y)$. Hence we have the canonical embedding $j : C(X) \to C(Y)$. The canonical surjection

$$\Pi : C(Y)^* \ni \phi \longmapsto \phi \circ j \in C(X)^* \tag{28}$$

restricted to Y (Y meaning here $\{\delta_y : y \in Y\}$, a subset of $C(Y)^*$) yields a canonical map $\Pi_Y : Y \to X$. In fact, the point mass at y is linear and multiplicative on $C(Y)$; hence its restriction to $C(X)$ (which is precisely $\Pi(\delta_y)$) belongs to $\mathtt{Sp}(C(X)) = X$. Similar restriction of the other canonical embedding $j : C(X)^* \to C(Y)^*$ results in $j_X : X \to Y$. From the compactness of $\overline{G}^{ws}$ (which is a simplified notation for the weak-star closure of $j_X(G)$ in Y) and by the w-* continuity of Π_Y, we get the closedness of $\Pi_Y(\overline{G}^{ws})$ resulting in one inclusion in the following lemma. (The other inclusion, namely, $(\subset)$, results just from the continuity of Π_Y.)

Lemma 9. *The canonical projection Π_Y maps this weak-star closure of $j_X(G)$ onto the Gelfand closure of G:*

$$\Pi_Y(\overline{G}^{ws}) = \overline{G}. \tag{29}$$

Theorem 10. *If ν_x is a representing measure for $x \in G$ then* $\operatorname{supp} \nu_x \subset \overline{G}$.

Proof. The functional $j(\nu_x)$ is linear on $C(Y)$ and nonnegative. Hence, according to Riesz theorem, it is represented by a unique measure $\widetilde{\nu_x}$ on Y. For $f \in C(X)$ we have $f \circ \Pi_Y \in C(Y)$; hence $\int f\, d\Pi_Y(\widetilde{\nu_x}) = \int (f \circ \Pi_Y)\, d\widetilde{\nu_x} = \int f\, d\nu_x$, where $\Pi_Y(\widetilde{\nu_x})$ denotes the push-forward measure of $\widetilde{\nu_x}$ by the mapping Π_Y. This means that

$$\Pi_Y\left(\widetilde{\nu_x}\right) = \nu_x. \tag{30}$$

By Theorem 6 the band $\overline{j(\mathscr{M}_G)}^{ws}$ is reducing and we have $M(\overline{G}^{ws}) = \overline{j(\mathscr{M}_G)}^{ws}$. The measure $\widetilde{\nu_x} - \delta_{j(x)}$ is orthogonal to A^{**}. So its singular part $(\widetilde{\nu_x} - \delta_{j(x)})^s$ in the decomposition with respect to the band $\overline{j(\mathscr{M}_G)}^{ws}$ is also orthogonal to A^{**}. But $(\widetilde{\nu_x} - \delta_{j(x)})^s = (\widetilde{\nu_x})^s$, since $\delta_{j(x)}$ is concentrated on $j(G) \subset j(\mathscr{M}_G)$. Taking into account the positivity of $\widetilde{\nu_x}$ and A^{**} possessing the unit, so that $\int_Y 1\, d(\widetilde{\nu_x})^s = 0$, it must be $(\widetilde{\nu_x})^s = 0$ and consequently $\widetilde{\nu_x} \in \overline{j(\mathscr{M}_G)}^{ws} = M(\overline{G}^{ws})$. By the last assertion the measure $\widetilde{\nu_x}$ must be concentrated on $\overline{G}^{ws}$. (Recall that this is $\overline{j(G)}^{ws}$.) Hence, by (30) and Lemma 9, we deduce that ν_x is concentrated on $\overline{G}$. □

Let G be a Gleason part of A. Denote by $\overline{G}^{\sigma}$ its weak-star closure in the $\sigma(C(X)^{**}, C(X)^{*})$-topology.

Proposition 11. *If G_1, G_2 are different Gleason parts of A, then* $\overline{G_1}^{\sigma} \cap \overline{G_2}^{\sigma} = \emptyset$.

Proof. Let $\phi \in G_1$ and $\psi \in G_2$. Since G_1, G_2 are different, we have $\|\phi - \psi\| = 2$. Hence there is a sequence $\{f_n\} \subset A$ such that $\|f_n\| \leq 1$ and $|f_n(\phi) - f_n(\psi)| \to 2$. We may assume that $f_n(\phi) \to 1$, so that $f_n(\psi) \to -1$. By Banach-Alaoglu Theorem, the sequence f_n has an adherent point $F \in A^{**}$. We have $\|F\| = 1$, $F(\phi) = 1$, and $F(\psi) = -1$. For any $\phi' \in G_1$ and any $\psi' \in G_2$ we can find pairs of representing measures μ, μ', ν, ν' on Y for ϕ, ϕ' and ψ, ψ', respectively (treated as elements of Y), such that the measures μ, μ' are mutually absolutely continuous and so are ν, ν'. Hence $F \equiv 1[\mu']$ almost everywhere and $F \equiv -1[\nu']$ almost everywhere. Consequently $F(\phi') = 1, F(\psi') = -1$ and so $F \equiv 1$ on G_1 and $F \equiv -1$ on G_2. By the continuity of F we get $F \equiv 1$ on $\overline{G_1}^{\sigma}$ and $F \equiv -1$ on $\overline{G_2}^{\sigma}$. Let F_1 be constructed for G_1 in the same way as F_0 corresponded to G in Corollary 7. Then $F' := F_1(F+1)/2$ is a characteristic function of $\overline{G_1}^{\sigma}$ and vanishes on $\overline{G_2}^{\sigma}$. □

5. Algebras of H^{∞}-Type

One of the approaches to study properties of $H^{\infty}(\Omega)$, the algebra of bounded analytic functions on a given domain $\Omega \subset \mathbb{C}^N$, is to consider its abstract counterpart, the algebra $H^{\infty}(\mathscr{M}_G)$ corresponding to a nontrivial Gleason part G for a function algebra A. The band of measures $\mathscr{M}_G$ corresponding to G is now considered as a Banach space, so that its dual space $\mathscr{M}_G^*$ carries its weak-star topology.

Denote by

$$H^{\infty}\left(\mathscr{M}_G\right) \tag{31}$$

the weak-star closure of A in $\mathscr{M}_G^*$. Note that there is a unique meaning for the value $f(x)$ of $f \in H^{\infty}(\mathscr{M}_G)$ at any point $x \in G$. We say that $H^{\infty}(\mathscr{M}_G)$ satisfies *the domination condition* if

$$\|f\| = \sup_{x \in G} |f(x)| \quad \text{for any } f \in H^{\infty}\left(\mathscr{M}_G\right). \tag{32}$$

Theorem 12. *If G is a Gleason part of A, then $H^{\infty}(\mathscr{M}_G)$ satisfies the domination condition.*

Proof. By Theorem 6, we have $\mathscr{M}_G \subset \overline{\mathscr{M}_G}^{ws} = M(\overline{G}^{ws})$; hence

$$\|f\| = \sup\{|\langle f, \mu\rangle| : \mu \in \mathscr{M}_G, \|\mu\| = 1\} = \sup_{x \in G} |f(x)| \tag{33}$$

for $f \in H^{\infty}(\mathscr{M}_G)$, which means that $H^{\infty}(\mathscr{M}_G)$ satisfies condition (32). □

Proposition 13. *The algebra $H^{\infty}(\mathscr{M}_G)$ is isometrically and algebraically isomorphic to $A^{**}/_{\mathscr{M}_G^{\perp} \cap A^{**}}$.*

Proof. Let $g \in A^{**}$. Then there is a net $\{g_\alpha\} \subset A$ such that $\langle g_\alpha, \mu\rangle \to \langle g, \mu\rangle$ for $\mu \in M(X)$. If we denote by f the restriction of g to $\mathscr{M}_G$ then f depends only on the equivalence class $[g] \in A^{**}/_{\mathscr{M}_G^{\perp} \cap A^{**}}$ of g and also $\langle g_\alpha, \mu\rangle \to \langle f, \mu\rangle$ for $\mu \in \mathscr{M}_G$; hence $f \in H^{\infty}(\mathscr{M}_G)$. This means that the mapping

$$A^{**}/_{\mathscr{M}_G^{\perp} \cap A^{**}} \ni [g] \longrightarrow g|_{\mathscr{M}_G} \in H^{\infty}\left(\mathscr{M}_G\right) \tag{34}$$

is well defined.

From the obvious relations $\|g'\| \geq \|g'|_{\mathscr{M}_G}\|$ for any $g' \in [g]$, we get $\|[g]\| \geq \|g|_{\mathscr{M}_G}\|$.

On the other hand, let $f \in H^{\infty}(\mathscr{M}_G)$. By Lemma 2.2 of [8], we can find a net $\{f_\alpha\} \subset A$ such that $f_\alpha \to f$ and $\|f_\alpha\| \leq \|f\|$ for each α. Hence, by Banach-Alaoglu Theorem, the net $\{f_\alpha\}$ has an adherent point $g \in A^{**}$. So $\langle f, \mu\rangle = \langle g, \mu\rangle$ for $\mu \in \mathscr{M}_G$ and $\|g\| \leq \|f\|$. Consequently the norm of $[g]$ in $A^{**}/_{\mathscr{M}_G^{\perp} \cap A^{**}}$ is less than or equal to $\|f\|$. It means that the mapping (34) is isometric and surjective. By Nagasawa's theorem (see [9] V.31), it is also isomorphic (preserves both linear and multiplicative structure). □

Corollary 14. *G is a subset of the spectrum of $H^{\infty}(\mathscr{M}_G)$.*

The following proposition (for its proof and details, see Proposition 2.8 of [10]) is a consequence of the Hahn-Banach theorem and Theorem 12.

Proposition 15. *The band $\mathscr{M}_G$ is equal to the norm closed linear span of all representing measures for points in G, taken in the quotient space $M(X)/A^{\perp}$.*

Note that for $f \in H^{\infty}(\mathscr{M}_G)$ and $z \in G$ we can define $f(z)$ as the value of f on a representing measure ν_z for z. By the weak-star density of A in $H^{\infty}(\mathscr{M}_G)$, this value $f(z)$ does not depend on the choice of representing measure. So

the elements of $H^\infty(\mathcal{M}_G)$ can be regarded as functions on G. Similarly as in Proposition 3.6 of [11], we can show the following.

Proposition 16. *If G is a bounded domain in $\mathbb{C}^n$ and $f \in H^\infty(\mathcal{M}_G)$ then the defined above mapping $G \ni z \to f(z)$ is a bounded analytic function of $z \in G$.*

Proof. Let us consider an arbitrary point $z_0 \in G$ and a small open polydisc Δ with the center at z_0, included in G. Without the loss of generality we can assume $z_0 = 0$. Denote by m the normalized Lebesgue measure on the Shilov boundary of Δ and by C_z the n-dimensional Cauchy kernel for z. Then m is a representing measure for z_0 (with respect to the algebra A). The measure $C_z\, dm$ is absolutely continuous with respect to m and consequently is in $\mathcal{M}_G$.

On the other hand, every $u \in A$ is analytic on Δ, so $u(z) = \int uC_z\, dm$ for $u \in A$, $z \in \Delta$, and by the weak-star density of A in $H^\infty(\mathcal{M}_G)$, also $h(z) = \int hC_z\, dm$ for $h \in H^\infty(\mathcal{M}_G)$, $z \in \Delta$. Hence, by Cauchy's theorem, h is analytic near z_0. □

In the next theorem we consider a bounded domain of holomorphy $G \subset \mathbb{C}^n$ such that its closure, $\overline{G}$, is the spectrum of $A(G)$, which plays the role of our initial uniform algebra A. For this it suffices to assume either that $\overline{G}$ is an intersection of a sequence of domains of holomorphy, that is, $\overline{G}$ has a Stein neighbourhoods basis [12], or that it has a smooth boundary [13].

For the purpose of the proof we assume additionally that G is a star-shaped domain.

Theorem 17. *If G is a domain in $\mathbb{C}^n$ satisfying the above conditions, then the algebras $H^\infty(G)$ and $H^\infty(\mathcal{M}_G)$ are isometrically isomorphic.*

Proof. We need to check whether $H^\infty(G) \subset H^\infty(\mathcal{M}_G)$ in the sense of isometric embedding. Without loss of generality we can assume that G is star-shaped with respect to 0. For $f \in H^\infty(G)$ and $0 < r < 1$ define $f_r \in H^\infty(G)$ as follows: $f_r(z) := f(rz)$ $(z \in G)$. The directed family $\{f_r\}_{r \to 1^-}$ has an adherent point $F \in H^\infty(\mathcal{M}_G)$. Passing to a suitable subnet, we can write $\langle \mu, f_r \rangle \to \langle F, \mu \rangle$ for $\mu \in \mathcal{M}_G$. In particular $\langle \nu_z, f_r \rangle \to \langle F, \nu_z \rangle$ for any measure ν_z representing any $z \in G$. On the other hand $\langle \nu_z, f_r \rangle = f_r(z) \to f(z)$ for $z \in G$. Hence F and f agree on G. By Proposition 15, the mapping $f \to F$ is isometric. The surjectivity follows from Proposition 16. □

Note that this provides also a direct representation of a predual to $H^\infty(G)$ and since the multiplication is weak-star continuous, it shows that $H^\infty(G)$ is a dual algebra. For the relevance of dual algebras to operator theory see [14].

6. A-Measures

Let Q be a Borel subset of X which is a union of some Gleason parts of A:

$$Q = \bigcup_\alpha G_\alpha, \quad \text{where each } G_\alpha \text{ is a Gleason part of } A. \tag{35}$$

We say that a measure $\mu \in M(X)$ is an *analytic measure for the algebra A at the points of Q* or, shortly, an *A-measure at Q*, if $\int u_n\, d\mu \to 0$ whenever $\{u_n\}_{n=1}^\infty \subset A$ is a bounded sequence converging to 0 pointwise on Q. The notion was introduced under the name "L-measures" by Henkin in [1] for $A = A(G)$, $Q = G$. This concept was useful not only in studying the isomorphisms between algebras of analytic functions over various domains G, in approximation theory, but also in operator theory in the construction of analytic functional calculus in a given n-tuple of commuting operators [15, 16].

All representing measures for points in Q are trivially A-measures. Our formulation of the so-called *A-measures problem for the algebra A at the points of Q* is as follows.

(†) Does the absolute continuity of a measure μ on X with respect to some representing measure of a point $x \in Q$ imply that μ is an A-measure at Q?

By the "classical case" we mean the situation when G is a domain in $\mathbb{C}^n$ and A is either the algebra $A(G)$ of complex continuous functions on $\overline{G}$, analytic on G, or its subalgebra $R(\overline{G})$ generated by the rational functions having no singularities on $\overline{G}$.

The problem was solved positively (by advanced complex analysis methods) for two special cases: for the algebra $A(G)$ with $Q = G$ by Henkin in [1] and by Cole with Range (see [17]) on strictly pseudoconvex bounded domains in $\mathbb{C}^n$ (resp., on domains in complex manifolds) with C^2 boundaries and by Bekken [18] (and [15]) in the case of polydomains (cartesian product of planar domains). Bekken's results hold also for A equal to $R(K)$ on compact product sets $K = K_1 \times \cdots \times K_n$ with $K_j \subset \mathbb{C}$. All these previous results will be covered by Theorem 19 below. In the latter case one needs to know that there are only countably many nontrivial Gleason parts of $R(K_j)$ [2] VI 3.2.

Proposition 18. *Any nonnegative A-measure at Q belongs to the band generated by Q.*

Proof. By decomposing such a measure μ with respect to $\mathcal{M}_Q$ we may assume that μ belongs to $\mathcal{M}_Q^s$. Applying equality (7) of Lemma 1 to $\mathcal{M}_Q^s$ in place of $\mathcal{M}$, we obtain $\operatorname{dist}(\mu, \mathcal{M}_Q + A^\perp) = \|[\mu]\| = \|\mu\|$. The last equality holds since μ is nonnegative. Now for some $h \in C(X)$ of norm 1 annihilated by $\mathcal{M}_Q + A^\perp$ we have $\int h\, d\mu$ close to 1. But $h \in A$, since it is annihilated by $A^\perp$. Taking the constant sequence $h_n = h$ vanishing on Q, we get a contradiction to the assumption on μ being an A-measure. □

Note that usually the A-measures problem is formulated in a slightly different way.

(‡) Is any measure which is absolutely continuous with respect to a nonnegative A-measure itself an A-measure?

Theorem 19. *If A is a function algebra on X and $Q \subset X$ is equal to a countable union of its Gleason parts, then A-measures problem for the algebra A at the points of Q has a positive solution.*

Proof. Let us begin with the case when Q is equal to exactly one Gleason part G. Let $\{u_n\}_{n=1}^{\infty} \subset A$ be a bounded sequence such that

$$u_n(x) \longrightarrow 0 \quad \text{for } x \in G. \tag{36}$$

Let $\mu \in \mathscr{M}_G$ and $\varepsilon > 0$. By Proposition 15 we can find a finite subset $x_1, \ldots, x_k$ of G and complex numbers $\alpha_1, \ldots, \alpha_k$ such that $\|\alpha_1 \nu_{x_1} + \cdots + \alpha_k \nu_{x_k} - \mu\| < \varepsilon$, where the norm here is the quotient norm in $\mathscr{M}_G/(\mathscr{M}_G \cap A^{\perp})$ (cf. Lemma 1) and ν_{x_i} is an arbitrary representing measure of x_i $(i = 1, \ldots, k)$. By (36), we have $\alpha_1 u_n(x_1) + \cdots + \alpha_k u_n(x_k) \to 0$, and consequently $|\int u_n \, d\mu| < \varepsilon \sup \|u_n\|$ for n big enough. Since ε was chosen arbitrarily we get $\int u_n \, d\mu \to 0$.

In the countable union of parts case, the result follows after applying the Lebesgue-type decomposition. □

Remark 20. To apply the above result for the algebra $A(G)$ over a concrete domain G it suffices to verify that Q is either the only nontrivial Gleason part of $A(G)$ or a countable union of such parts, since the Lebesgue-type decomposition is then applicable. In this case the abstract formulation (†) is equivalent to its classical version (‡). Hence we get some generalizations of the previously known cases.

Corollary 21. *The problem* (†) *at the points of $Q = G$ for $A(G)$ has positive solution if G is either a strictly pseudoconvex set in $\mathbb{C}^n$ or a Cartesian product of a finite number of such domains.*

This includes polydiscs, polydomains (products of bounded plane domains), and also products of balls with polydiscs.

Finally, as A we can take $H^{\infty}(G)$ with the domain $G \subset \mathbb{C}^n$ satisfying the conditions of Theorem 17 to ensure that its Euclidean closure is the spectrum of $A(G)$.

Theorem 22. *The A-measures problem for the algebra $A = H^{\infty}(G)$ at all points of a countable union Q of its arbitrary Gleason parts has positive solution. In particular, this is applicable to the part corresponding to the domain G if this domain satisfies the conditions of Theorem 17.*

Conflict of Interests

The authors declare that they have no conflict of interests regarding the publication of this paper.

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