

Research Article

The Boundedness of Marcinkiewicz Integral Associated with Schrödinger Operator and Its Commutator

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The authors prove that Marcinkiewicz integral operator is not only are bounded on L^p , for $1 < p < \infty$, but also a bounded map from $L^1(\mathbb{R}^n)$ to weak $L^1(\mathbb{R}^n)$. Meanwhile, the BMO_L -boundedness and (H^1_L, L^1) -boundedness are also obtained. Finally, the L^p -boundedness and (L^∞, BMO_L) -boundedness for the commutator of Marcinkiewicz integral of schrödinger type are established.

1. Introduction and Notation

Let us consider the Schrödinger operator

$$L = -\Delta + V \quad (1)$$

in \mathbb{R}^d , $d \geq 3$. The function V is nonnegative, $V \neq 0$, and belongs to a reverse Hölder class RH_{q_0} , for some exponent $q_0 > d/2$; that is, there exists a constant C such that

$$\left(\frac{1}{|B|} \int_B V(y)^{q_0} dy \right)^{1/q_0} \leq \frac{C}{|B|} \int_B V(y) dy, \quad (2)$$

for every ball $B \subset \mathbb{R}^d$.

We introduce the definition of the reverse Hölder index of V as $q_0 = \sup\{q : V \in RH_q\}$. It is known that $V \in RH_{q_0}$ implies $V \in RH_{q_0+\epsilon}$, for some $\epsilon > 0$. Therefore, under the assumption $V \in RH_{d/2}$, we may conclude $q_0 > d/2$.

The classical Marcinkiewicz integral operator μ is defined by

$$\mu f(x) = \left(\int_0^\infty \left| \int_{|x-y| \leq t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y) dy \right|^2 \frac{dt}{t^3} \right)^{1/2}. \quad (3)$$

The above operator was introduced by Stein in [1] as an extension of the notion of Marcinkiewicz function from one dimension to higher dimension. Meanwhile, Stein [1]

showed that if $\Omega \in \text{Lip}_\alpha(S^{n-1})$, for some $0 < \alpha \leq 1$, then μ is a bounded operator on $L^p(\mathbb{R}^n)$, for $1 < p \leq 2$, and is a bounded map from $L^1(\mathbb{R}^n)$ to weak $L^1(\mathbb{R}^n)$. Benedek et al. [2] showed that if Ω is continuously differentiable in $x \neq 0$, then μ is a bounded operator on $L^p(\mathbb{R}^n)$, for $1 < p \leq \infty$. Ding et al. [3] proved that the Marcinkiewicz function μ is bounded from $H^1(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$ with Ω satisfying cancellation condition on S^{n-1} and L^1 -Dini condition.

Similar to the classical Marcinkiewicz function μ , we define the Marcinkiewicz integral μ_j^L associated with the Schrödinger operator L by

$$\mu_j^L f(x) = \left(\int_0^\infty \left| \int_{|x-y| \leq t} K_j^L(x, y) f(y) dy \right|^2 \frac{dt}{t^3} \right)^{1/2}, \quad (4)$$

where $K_j^L(x, y) = \widetilde{K}_j^L(x, y)|x-y|$ and $\widetilde{K}_j^L(x, y)$ is the kernel of $R_j^L = (\partial/\partial x_j)L^{-1/2}$, $j = 1, \dots, d$. In particular, when $V = 0$, $K_j^\Delta(x, y) = \widetilde{K}_j^\Delta(x, y)|x-y| = ((x-y)_j/|x-y|) / |x-y|^{d-1}$ and $\widetilde{K}_j^\Delta(x, y)$ is the kernel of $R_j = (\partial/\partial x_j)\Delta^{-1/2}$, $j = 1, \dots, d$.

We also give the definition of the commutator generalized by μ_L^j and b by

$$\begin{aligned} [b, \mu_L^j](f)(x) &= \left(\int_0^\infty \left| \int_{|x-y|\leq t} K_j^L(x, y)(b(x) - b(y)) f(y) dy \right|^2 \frac{dt}{t^3} \right)^{1/2}. \end{aligned} \quad (5)$$

In this paper, we write $K_j(x, y) = K_j^\Delta(x, y)$ and

$$\mu_j f(x) = \left(\int_0^\infty \left| \int_{|x-y|\leq t} K_j(x, y) f(y) dy \right|^2 \frac{dt}{t^3} \right)^{1/2}. \quad (6)$$

For a given potential $V \in RH_q$, with $q > d/2$, we introduce the auxiliary function

$$\rho(x) = \sup \left\{ r > 0 : \frac{1}{r^{d-2}} \int_{B(x,r)} V \leq 1 \right\}, \quad x \in R^d. \quad (7)$$

The above assumptions $\rho(x)$ are finite, for all $x \in R^d$.

Proposition 1 (see [4]). *There exist C and $k_0 \geq 1$ such that*

$$\begin{aligned} C^{-1} \rho(x) \left(1 + \frac{|x-y|}{\rho(x)} \right)^{-k_0} \\ \leq \rho(y) \leq C \rho(x) \left(1 + \frac{|x-y|}{\rho(x)} \right)^{k_0/(k_0+1)}, \end{aligned} \quad (8)$$

for all $x, y \in R^d$.

In particular, $\rho(x) = \rho(y)$, if $|x-y| < C\rho(x)$. A ball $B(x, \rho(x))$ is called critical.

Proposition 2 (see [5]). *There exist a sequence of points $\{x_k\}_{k=1}^\infty$ in R^d , so that the family $B_k = B(x_k, \rho(x_k))$, $k \geq 1$, satisfies the following:*

- (1) $\bigcup_k B_k = R^d$;
- (2) *there exists N such that, for every $k \in \mathbb{N}$, $\text{card} \{j : 4B_j \cap 4B_k \neq \emptyset\} \leq N$.*

Function f is an element of BMO_L , if $\exists C > 0$ such that, for every ball $B = B(x, r)$,

$$\int_B |f - f_B| \leq C|B|, \quad (9)$$

if $r < \rho(x)$, and

$$\int_B |f| \leq C|B|, \quad (10)$$

if $r \geq \rho(x)$, where $f_B = (1/|B|) \int_B f(x) dx$. Let $\|f\|_{BMO_L(R^d)}$ be the smallest C in the inequalities above. It is easy to verify that $\|f\|_{BMO(R^d)} \leq 2\|f\|_{BMO_L(R^d)}$, for all $f \in BMO_L(R^d)$.

Lemma 3 (see [6]). *For any $l > 0$, there exists $C_l > 0$ such that*

$$\begin{aligned} |K_j^L(x, y)| &\leq \frac{C_l}{(1 + (|x-y|/\rho(y)))^l} \frac{1}{|x-y|^{d-1}}, \\ |K_j^L(x, y) - K_j(x, y)| &\leq C \frac{\rho(x)}{|x-y|^{d-2}}. \end{aligned} \quad (11)$$

Tang and Dong [6] have shown that Marcinkiewicz integral μ_j^L is bounded on $L^p(R^n)$, for $1 < p < \infty$, and are bounded from $L^1(R^n)$ to weak $L^1(R^n)$. Meanwhile, they also proved that μ_j^L are bounded on $BMO_L(R^n)$ and are also mapped from $H_L^1(R^n)$ to $L^1(R^n)$ under the assumption that K_j^L satisfy the condition in Lemma 3.

Now, we introduce a new BMO-type space of schrödinger operator. Let $\theta > 0$; we define the class $BMO_\theta(\rho)$ of locally integrable functions b such that

$$\frac{1}{|B(x, r)|} \int_{B(x, r)} |b(y) - b_B| dy \leq C \left(1 + \frac{r}{\rho(x)} \right)^\theta, \quad (12)$$

for all $x \in R^d$ and $r > 0$, where $b_B = (1/|B|) \int_B b(y) dy$. A norm for $b \in BMO_\theta(\rho)$, denoted by $[b]_\theta$, is given by the infimum of the constants in the inequalities above. Notice that if we let $\theta = 0$, we obtain the John-Nirenberg space BMO.

With the above definition, we define $BMO_\infty(\rho) = \bigcup_{\theta > 0} BMO_\theta(\rho)$. Clearly, $BMO \subset BMO_\theta(\rho) \subset BMO_{\theta'}(\rho)$, for $0 < \theta < \theta'$, and hence $BMO \subset BMO_\infty(\rho)$. Moreover, it is in general a larger class.

For $\theta > 0$, we denote by $BMO_\theta^{\log}(\rho)$ the set of functions b such that

$$\frac{1}{|B(x, r)|} \int_{B(x, r)} |b(y) - b_B| dy \leq C \frac{(1 + r/\rho(x))^\theta}{1 + \log^+(\rho(x)/r)}, \quad (13)$$

for all $x \in R^d$ and $r > 0$. A norm for $b \in BMO_\theta^{\log}(\rho)$, denoted by $\|b\|_{* \log}$, is given by the infimum of the constants in the inequalities above. Correspondingly, we define $BMO_\infty^{\log}(\rho) = \bigcup_{\theta > 0} BMO_\theta^{\log}(\rho)$.

Next, we give some information on the Hardy space associated with Schrödinger operator L . We say that a function $f \in L^1$ is said to belong to H_L^1 , if the semigroup maximal operator T_L^* is bounded on L^1 . The H_L^1 -norm of f is given by $\|f\|_{H_L^1} = \|T_L^* f\|_{L^1}$, where $T_L^* f(x) = \sup_{t>0} |T_t^L f(x)|$, $\{T_t^L\} = \{e^{-tL}\}$ is a semigroup generated by the Schrödinger operator L (see [7]).

Shen [4] gave the following kernel estimate that we needed.

Lemma 4. *If $V \in RH_d$, then,*

- (i) *for every N , there exists a constant C such that*

$$|K_j^L(x, z)| \leq \frac{C(1 + (|x-z|/\rho(x)))^{-N}}{|x-z|^{d-1}}, \quad (14)$$

(ii) for every N and $0 < \delta < \min\{1, 1 - d/q_0\}$, there exists a constant C such that

$$\begin{aligned} & |K_j^L(x, z) - K_j^L(y, z)| \\ & \leq \frac{C|x - y|^\delta (1 + (|x - z|/\rho(x)))^{-N}}{|x - z|^{d-1+\delta}}, \end{aligned} \quad (15)$$

where $|x - y| < (2/3)|x - z|$,

(iii) if K denotes the R^d vector valued kernel of the classical Riesz operator, for every $0 < \delta < 2 - d/q_0$, we have

$$|K_j^L(x, z) - K_j(x, z)| \leq \frac{C}{|x - z|^{d-1}} \left(\frac{|x - z|}{\rho(z)} \right)^\delta, \quad (16)$$

where $K_j(x, z) = K(x, z)|x - z|$.

Inspired by [6], we consider the same boundedness of Marcinkiewicz integral whose kernel satisfies Lemma 4 and L^p -boundedness of its commutator for $1 < p < \infty$ and (L^∞, BMO_L) -boundedness.

Theorem 5. The operators μ_j^L are bounded on $L^p(R^d)$, for $1 < p < \infty$, and are bounded from $L^1(R^d)$ to weak $L^1(R^d)$. On the other hand, μ_j^L are bounded on $BMO_L(R^d)$ and are bounded from $H_L^1(R^d)$ to $L^1(R^d)$.

In the note, we devote ourselves to establish the following boundedness of the commutators of Marcinkiewicz integral of schrodinger operator type.

Theorem 6. Let $V \in RH_d$, $b \in BMO_\infty(\rho)$, and p_0 such that $1/p_0 = (1/q_0 - 1/d)^+$, where q_0 is the reverse Hölder index of V ; then, for all $f \in L^p$, we have

$$\| [b, \mu_j^L] f \|_p \leq C [b]_\theta \|f\|_p, \quad (17)$$

where $1 < p < p_0$.

Theorem 7. Let $V \in RH_d$ and $b \in BMO_\infty(\rho)$; if b satisfies the stronger condition $b \in BMO_\infty^{\log}(\rho)$, then $[b, \mu_j^L]$ are bounded from $L^\infty(R^d)$ to $BMO_L(R^d)$.

Throughout this paper, C denotes the constants that are independent of the main parameters involved but whose value may differ from line to line. By $A \asymp B$, we mean that there exists a constant $C > 1$ such that $1/C \leq A/B \leq C$. We use the symbol $A \lesssim B$ to denote that there exists a positive constant C such that $A \leq CB$.

2. Notation and Preliminaries

Denote the following maximal functions for $g \in L^1_{loc}(R^d)$, $\alpha > 0$, and $x \in R^d$:

$$M_{\rho, \alpha} g(x) = \sup_{x \in B \in \mathbf{B}_{\rho, \alpha}} \frac{1}{|B|} \int_B |g|, \quad (18)$$

$$M_{\rho, \alpha}^\# g(x) = \sup_{x \in B \in \mathbf{B}_{\rho, \alpha}} \frac{1}{|B|} \int_B |g - g_B|,$$

where $\mathbf{B}_{\rho, \alpha} = \{B(y, r) : y \in R^d \text{ and } r \leq \alpha\rho(y)\}$.

Lemma 8 (Fefferman-Stein type inequality [8]). For $1 < p < \infty$, there exist β and γ such that if $\{Q_k\}_{k=1}^\infty$ is a sequence of balls as that in Proposition 2, then

$$\begin{aligned} & \int_{R^d} |M_{\rho, \beta}(g)|^p \\ & \lesssim \int_{R^d} |M_{\rho, \gamma}^\#(g)|^p + \sum_k |Q_k| \left(\frac{1}{|Q_k|} \int_{2Q_k} |g| \right)^p, \end{aligned} \quad (19)$$

for all $g \in L^1_{loc}(R^d)$.

Lemma 9 (see [9]). Let $\theta > 0$ and $1 \leq s < \infty$. If $b \in BMO_\theta(\rho)$, then

$$\left(\frac{1}{|B|} \int_B |b - b_B|^s \right)^{1/s} \lesssim [b]_\theta \left(1 + \frac{r}{\rho(x)} \right)^{\theta'}, \quad (20)$$

for all $B = B(x, r)$, with $x \in R^d$ and $r > 0$, where $\theta' = (k_0 + 1)\theta$ and k_0 is the constant appearing in Proposition 1.

Lemma 10 (see [9]). Let $b \in BMO_\theta(\rho)$, $B = B(x_0, r)$, and $s \geq 1$; then

$$\left(\frac{1}{|2^k B|} \int_{2^k B} |b - b_B|^s \right)^{1/s} \lesssim [b]_\theta k \left(1 + \frac{2^k r}{\rho(x_0)} \right)^{\theta'}, \quad (21)$$

for all $k \in N$, with θ' as in Lemma 9.

Next, we introduce our important Lemmas.

We denoted by M the Hardy-Littlewood maximal function and, for $s > 1$, by M_s , the operator is defined as $M_s = (M(f^s))^{1/s}$.

Lemma 11. Let $V \in RH_d$, $1/p_0 = (1/q_0 - 1/d)^+$, and $b \in BMO_\theta(\rho)$. Then, for any $1 < s < p_0$, there exists a constant C such that

$$\frac{1}{|Q|} \int_Q |[b, \mu_j^L] f| \leq C [b]_\theta \inf_{y \in Q} M_s f(y), \quad (22)$$

for all $f \in L^s_{loc}(R^d)$ and every ball $Q = B(x_0, \rho(x_0))$.

Proof. Let $f \in L^p(\mathbb{R}^d)$ and $Q = B(x_0, \rho(x_0))$. We first observe

$$[b, \mu_j^L] f \leq |b - b_Q| \mu_j^L f + \mu_j^L (f(b - b_Q)), \quad (23)$$

and so we have to deal with the average on Q of each term.

Thanks to Hölder's inequality with $1 < s < p_0$ and Lemma 10, one has

$$\begin{aligned} & \frac{1}{|Q|} \int_Q |(b - b_Q) \mu_j^L f| \\ & \leq \left(\frac{1}{|Q|} \int_Q |b - b_Q|^{s'} \right)^{1/s'} \left(\frac{1}{|Q|} \int_Q |\mu_j^L f|^s \right)^{1/s} \\ & \leq [b]_\theta \left(\frac{1}{|Q|} \int_Q |\mu_j^L f|^s \right)^{1/s}. \end{aligned} \quad (24)$$

Let $f = f_1 + f_2$ with $f_1 = f \chi_{2Q}$; then $L^s(\mathbb{R}^d)$ -boundedness of μ_j^L , for $1 < s < p_0$, says that

$$\begin{aligned} \left(\frac{1}{|Q|} \int_Q |\mu_j^L f_1|^s \right)^{1/s} & \leq \left(\frac{1}{|Q|} \int_{2Q} |f|^s \right)^{1/s} \\ & \leq \inf_{y \in Q} M_s f(y). \end{aligned} \quad (25)$$

Now, for $x \in Q$, $\rho(x) = \rho(x_0)$ and using Lemma 4 and Minkowski's inequality, we get

$$\begin{aligned} |\mu_j^L f_2(x)| & = \left(\int_0^\infty \left| \int_{|x-y| \leq t} K_j^L(x, y) f_2(y) dy \right|^2 \frac{dt}{t^3} \right)^{1/2} \\ & \leq \left(\int_0^\infty \left| \int_{|x-y| \leq t} \frac{(1 + (|x-y|/\rho(x)))^{-N}}{|x-y|^{d-1}} \right. \right. \\ & \quad \left. \left. \times |f_2(y)| dy \right|^2 \frac{dt}{t^3} \right)^{1/2} \\ & \leq \int_{\mathbb{R}^d} \frac{|f_2(y)| \rho(x_0)^N}{|x-y|^{d-1+N}} \left(\int_{|x-y| \leq t} \frac{dt}{t^3} \right)^{1/2} dy \\ & \leq \rho(x_0)^N \int_{\mathbb{R}^d} \frac{|f_2(y)|}{|x-y|^{d+N}} dy \\ & \leq \rho(x_0)^N \int_{(2Q)^c} \frac{|f(y)|}{|x-y|^{d+N}} dy \\ & \leq \rho(x_0)^N \sum_{k=1}^\infty \int_{2^{k+1}Q \setminus 2^kQ} \frac{|f(y)|}{|x-y|^{d+N}} dy \\ & \leq \rho(x_0)^N \sum_{k=1}^\infty \frac{(2^k \rho(x_0))^{-N}}{|2^{k+1}Q|} \int_{2^{k+1}Q} |f(y)| dy \\ & \leq \inf_{y \in Q} M_s f(y). \end{aligned} \quad (26)$$

For the second term, we split again $f = f_1 + f_2$. By choosing $1 < \bar{s} < s < p_0$ and denoting $\nu = s\bar{s}/(s - \bar{s})$, by the boundedness of μ_j^L on $L^{\bar{s}}(\mathbb{R}^d)$ and Hölder's inequality, we obtain

$$\begin{aligned} & \frac{1}{|Q|} \int_Q |\mu_j^L (f_1(b - b_B))| \\ & \leq \left(\frac{1}{|Q|} \int_Q |\mu_j^L (f_1(b - b_B))|^{\bar{s}} \right)^{1/\bar{s}} \\ & \leq \left(\frac{1}{|Q|} \int_{2Q} |f(b - b_B)|^{\bar{s}} \right)^{1/\bar{s}} \\ & \leq \left(\frac{1}{|Q|} \int_{2Q} |f|^s \right)^{1/s} \left(\frac{1}{|Q|} \int_{2Q} |b - b_B|^\nu \right)^{1/\nu} \\ & \leq [b]_\theta \inf_{y \in Q} M_s f(y), \end{aligned} \quad (27)$$

where, in the last inequality, we have used Lemma 9.

For the remaining term, let $1 = 1/s + 1/\bar{s}$, $\rho(x) = \rho(x_0)$ using Lemma 9, Lemma 10, and Minkowski's inequality, we arrive to

$$\begin{aligned} & |\mu_j^L (f_2(b - b_Q))(x)| \\ & = \left(\int_0^\infty \left| \int_{|x-y| \leq t} K_j^L(x, y) f_2(y) (b(y) - b_Q) dy \right|^2 \frac{dt}{t^3} \right)^{1/2} \\ & \leq \left(\int_0^\infty \left| \int_{|x-y| \leq t} \frac{|f_2(y)| |b(y) - b_Q|}{|x-y|^{d-1} (1 + (|x-y|/\rho(x)))^N} dy \right|^2 \frac{dt}{t^3} \right)^{1/2} \\ & \leq \int_{\mathbb{R}^d} \frac{|f_2(y)| |b(y) - b_Q|}{|x-y|^{d-1} (1 + (|x-y|/\rho(x)))^N} \\ & \quad \times \left(\int_{|x-y| \leq t} \frac{dt}{t^3} \right)^{1/2} dy \\ & \leq \rho(x_0)^N \int_{(2Q)^c} \frac{|f(y)| |b(y) - b_Q|}{|x-y|^{d+N}} dy \\ & \leq \rho(x_0)^N \sum_{k=1}^\infty \frac{(2^k \rho(x_0))^{-N}}{|2^{k+1}Q|} \\ & \quad \times \int_{2^{k+1}Q \setminus 2^kQ} |f(y)| |b(y) - b_Q| dy \\ & \leq \sum_{k=1}^\infty \frac{2^{-kN}}{|2^{k+1}Q|} \int_{2^{k+1}Q} |f(y)| |b(y) - b_Q| dy \end{aligned}$$

$$\begin{aligned}
&\leq \sum_{k=1}^{\infty} 2^{-kN} \left(\frac{1}{|2^{k+1}Q|} \int_{2^{k+1}Q} |b(y) - b_Q|^{s'} dy \right)^{1/s'} \\
&\quad \times \left(\frac{1}{|2^{k+1}Q|} \int_{2^{k+1}Q} |f(y)|^s dy \right)^{1/s} \\
&\leq \sum_{k=1}^{\infty} 2^{-kN} k \left(1 + \frac{2^k \rho(x_0)}{\rho(x_0)} \right)^{\theta'} [b]_{\theta} \inf_{y \in Q} M_s f(y) \\
&\leq \sum_{k=1}^{\infty} k 2^{k(-N+\theta')} [b]_{\theta} \inf_{y \in Q} M_s f(y) \\
&\leq [b]_{\theta} \inf_{y \in Q} M_s f(y).
\end{aligned} \tag{28}$$

Since N can be chosen large enough, the last series converges. Thus, we finished the proof. \square

Remark 12. It is easy to check that if the critical ball Q is replaced by $2Q$, the above lemma also holds.

Lemma 13. *Let $V \in RH_d$ and $b \in BMO_{\infty}(\rho)$, then, for any $1 < s < p_0$ and $\gamma \geq 1$, there exists a constant C such that*

$$\begin{aligned}
&|\mu_j^L(f_2(b - b_B))(u) - \mu_j^L(f_2(b - b_B))(z)| \\
&\leq [b]_{\theta} \inf_{x \in B} M_s f(x),
\end{aligned} \tag{29}$$

for all $f, u, z \in B = B(x_0, r)$, with $r < \gamma\rho(x_0)$ and f_2 as in Lemma 11.

Proof. We write

$$\begin{aligned}
&|\mu_j^L(f_2(b - b_B))(u) - \mu_j^L(f_2(b - b_B))(z)| \\
&\leq \left(\int_0^{\infty} \left| \int_{|u-y| \leq t < |z-y|} |K_j^L(u, y)| \right. \right. \\
&\quad \left. \left. \times f_2(y) (b(y) - b_B) |dy| \frac{dt}{t^3} \right)^{1/2} \\
&\quad + \left(\int_0^{\infty} \left| \int_{|z-y| \leq t < |u-y|} |K_j^L(z, y)| \right. \right. \\
&\quad \left. \left. \times f_2(y) (b(y) - b_B) |dy| \frac{dt}{t^3} \right)^{1/2} \\
&\quad + \left(\int_0^{\infty} \left| \int_{\{|u-y| \leq t, |z-y| \leq t\}} |K_j^L(u, y) - K_j^L(z, y)| \right. \right. \\
&\quad \left. \left. \times |f_2(y) (b(y) - b_B)| dy \frac{dt}{t^3} \right)^{1/2} \\
&= I_1 + I_2 + I_3.
\end{aligned} \tag{30}$$

Since the estimates for I_1 and I_2 follow along similar lines, we only consider I_3 . By denoting $Q = B(x_0, \gamma\rho(x_0))$, since in our situation $\rho(u) = \rho(x_0)$ and $|u - y| = |z - y|$, by Minkowski's inequality and Lemma 4, we have

$$\begin{aligned}
I_1 &\leq \int_{\mathbb{R}^d} |f_2(y)| |b(y) - b_B| |K_j^L(u, y)| \\
&\quad \times \left(\int_{|u-y| \leq t < |z-y|} \frac{dt}{t^3} \right)^{1/2} dy \\
&\leq r^{1/2} \int_{(2B)^c} \frac{|f(y)| |b(y) - b_B| |K_j^L(u, y)|}{|u-y|^{3/2}} dy \\
&\leq r^{1/2} \int_{Q \setminus 2B} \frac{|f(y)| |b(y) - b_B|}{|u-y|^{d+1/2}} dy \\
&\quad + r^{1/2} \rho(x_0)^N \int_{Q^c} \frac{|f(y)| |b(y) - b_B|}{|u-y|^{d+1/2+N}} dy \\
&= I_{11} + I_{12}.
\end{aligned} \tag{31}$$

Splitting into annuli, we have

$$I_{11} \leq \frac{1}{r^d} \sum_{j=2}^{j_0} 2^{-j(d+1/2)} \int_{2^j B} |f(y)| |b(y) - b_B| dy, \tag{32}$$

where j_0 is the least integer such that $2^{j_0} \geq \gamma\rho(x_0)/r$.

By Hölder's inequality and Lemma 10, we obtain, for $j \leq j_0$,

$$\int_{2^j B} |f(y)| |b(y) - b_B| dy \leq j [b]_{\theta} |2^j B| \inf_{x \in B} M_s f(x). \tag{33}$$

Then,

$$\begin{aligned}
I_{11} &\leq [b]_{\theta} \inf_{x \in B} M_s f(x) \sum_{j=2}^{j_0} j 2^{-j/2} \\
&\leq [b]_{\theta} \inf_{x \in B} M_s f(x).
\end{aligned} \tag{34}$$

To deal with I_{12} , by using Lemma 10 and choosing $N > \theta'$, we have

$$\begin{aligned}
I_{12} &\leq \frac{\rho(x_0)^N}{r^{d+N}} \sum_{j=j_0-1}^{\infty} 2^{-j(d+1/2+N)} \int_{2^j B} |f(y)| |b(y) - b_B| dy \\
&\leq [b]_{\theta} \inf_{x \in B} M_s f(x) \left(\frac{\rho(x_0)}{r} \right)^{N-\theta'} \sum_{j=j_0-1}^{\infty} j 2^{-j(1/2+N-\theta')} \\
&\leq [b]_{\theta} \inf_{x \in B} M_s f(x) \sum_{j=j_0-1}^{\infty} j 2^{-j/2} \\
&\leq [b]_{\theta} \inf_{x \in B} M_s f(x).
\end{aligned} \tag{35}$$

For I_3 , again by Minkowski's inequality and in our situation $\rho(u) = \rho(x_0)$ and $|u - y| = |z - y|$, we get

$$\begin{aligned}
I_3 &\leq \int_{\mathbb{R}^d} |f_2(y)| |b(y) - b_B| |K_j^L(u, y) - K_j^L(z, y)| \\
&\quad \times \left(\int_{\{|u-y|\leq t, |z-y|\leq t\}} \frac{dt}{t^3} \right)^{1/2} dy \\
&\leq \int_{(2B)^c} \frac{|f(y)| |b(y) - b_B| |K_j^L(u, y) - K_j^L(z, y)|}{|u - y|} dy \\
&\leq r^\delta \int_{Q \setminus 2B} \frac{|f(y)| |b(y) - b_B|}{|u - y|^{d+\delta}} dy \\
&\quad + r^\delta \rho(x_0)^N \int_{Q^c} \frac{|f(y)| |b(y) - b_B|}{|u - y|^{d+\delta+N}} dy \\
&= I_{31} + I_{32}.
\end{aligned} \tag{36}$$

Thanks to Hölder's inequality and Lemma 10, we have

$$\begin{aligned}
I_{31} &\leq \frac{1}{r^d} \sum_{j=2}^{j_0} 2^{-j(d+\delta)} \int_{2^j B} |f(y)| |b(y) - b_B| dy \\
&\leq \frac{1}{r^d} \sum_{j=2}^{j_0} 2^{-j(d+\delta)} j [b]_\theta |2^j B| \inf_{x \in B} M_s f(x) \\
&\leq [b]_\theta \inf_{x \in B} M_s f(x) \sum_{j=2}^{j_0} j 2^{-j\delta} \\
&\leq [b]_\theta \inf_{x \in B} M_s f(x).
\end{aligned} \tag{37}$$

To deal with I_{22} , similarly as I_{12} , we have

$$\begin{aligned}
I_{32} &\leq \frac{\rho(x_0)^N}{r^{d+N}} \sum_{j=j_0-1}^{\infty} 2^{-j(d+\delta+N)} \int_{2^j B} |f(y)| |b(y) - b_B| dy \\
&\leq [b]_\theta \inf_{x \in B} M_s f(x) \left(\frac{\rho(x_0)}{r} \right)^{N-\theta'} \sum_{j=j_0-1}^{\infty} j 2^{-j(\delta+N-\theta')} \\
&\leq [b]_\theta \inf_{x \in B} M_s f(x) \sum_{j=j_0-1}^{\infty} j 2^{-j\delta} \\
&\leq [b]_\theta \inf_{x \in B} M_s f(x).
\end{aligned} \tag{38}$$

We have completed the proof of the lemma. \square

3. The Boundedness of Marcikiewicz Integral and Its Commutator

In this section, we first employ the same technique in [6] to prove Theorem 5.

Proof. Similarly as [6], it suffices to prove the following pointwise estimate:

$$\mu_j^L f(x) \leq \mu_j f(x) + CMf(x), \quad \text{a.e. } x \in \mathbb{R}^d, \tag{39}$$

where M denotes the Hardy-Littlewood operator.

Fix $x \in \mathbb{R}^d$ and let $r = \rho(x)$. Notice that

$$\begin{aligned}
\mu_j^L f(x) &\leq \left(\int_0^r \left| \int_{|x-y|\leq t} K_j^L(x, y) f(y) dy \right|^2 \frac{dt}{t^3} \right)^{1/2} \\
&\quad + \left(\int_r^\infty \left| \int_{|x-y|\leq r} K_j^L(x, y) f(y) dy \right|^2 \frac{dt}{t^3} \right)^{1/2} \\
&\quad + \left(\int_r^\infty \left| \int_{r<|x-y|\leq t} K_j^L(x, y) f(y) dy \right|^2 \frac{dt}{t^3} \right)^{1/2} \\
&\leq \left(\int_0^r \left| \int_{|x-y|\leq t} [K_j^L(x, y) - K_j(x, y)] f(y) dy \right|^2 \frac{dt}{t^3} \right)^{1/2} \\
&\quad + \left(\int_0^r \left| \int_{|x-y|\leq t} K_j(x, y) f(y) dy \right|^2 \frac{dt}{t^3} \right)^{1/2} \\
&\quad + \left(\int_r^\infty \left| \int_{|x-y|\leq r} K_j^L(x, y) f(y) dy \right|^2 \frac{dt}{t^3} \right)^{1/2} \\
&\quad + \left(\int_r^\infty \left| \int_{r<|x-y|\leq t} K_j^L(x, y) f(y) dy \right|^2 \frac{dt}{t^3} \right)^{1/2} \\
&= E_1 + E_2 + E_3 + E_4.
\end{aligned} \tag{40}$$

For E_1 , by Lemma 4, we have

$$\begin{aligned}
E_1 &\leq \left(\int_0^r \left| \int_{|x-y|\leq t} \frac{1}{|x-y|^{d-1}} \left(\frac{|x-y|}{\rho(x)} \right)^\delta f(y) dy \right|^2 \frac{dt}{t^3} \right)^{1/2} \\
&\leq r^{-\sigma} \left(\int_0^r \left| \sum_{k=-\infty}^0 \frac{1}{(2^{k-1}t)^{d-\delta-1}} \int_{|x-y|\leq 2^k t} |f(y)| dy \right|^2 \frac{dt}{t^3} \right)^{1/2} \\
&\leq r^{-\sigma} \left(\int_0^r \left| \sum_{k=-\infty}^0 \frac{(2^k)^\delta t^{\delta+1}}{(2^k t)^d} \int_{|x-y|\leq 2^k t} |f(y)| dy \right|^2 \frac{dt}{t^3} \right)^{1/2} \\
&\leq r^{-\sigma} \left(\int_0^r \left| \sum_{k=-\infty}^0 (2^k)^{\delta+1} t^{\delta+1} Mf(x) \right|^2 \frac{dt}{t^3} \right)^{1/2} \\
&\leq r^{-\sigma} \left(\int_0^r t^{2\delta-1} dt \right)^{1/2} Mf(x) \\
&\leq Mf(x).
\end{aligned} \tag{41}$$

(41)

Obviously,

$$E_2 \leq \mu_j f(x). \tag{42}$$

For E_3 , using Lemma 4 again, we get

$$\begin{aligned} E_3 &\leq \left(\int_r^\infty \left| \int_{|x-y|\leq r} \frac{1}{|x-y|^{d-1}} f(y) dy \right|^2 \frac{dt}{t^3} \right)^{1/2} \\ &\leq \left(\int_r^\infty \left| \sum_{k=-\infty}^0 \frac{1}{(2^{k+1}r)^{d-1}} \int_{|x-y|\leq 2^k r} f(y) dy \right|^2 \frac{dt}{t^3} \right)^{1/2} \\ &\leq \left(\int_r^\infty \left| \sum_{k=-\infty}^0 \frac{2^k r}{(2^k r)^d} \int_{|x-y|\leq 2^k r} f(y) dy \right|^2 \frac{dt}{t^3} \right)^{1/2} \\ &\leq \left(\int_r^\infty \left| \sum_{k=-\infty}^0 2^k r Mf(x) \right|^2 \frac{dt}{t^3} \right)^{1/2} \\ &\leq r \left(\int_r^\infty \frac{dt}{t^3} \right)^{1/2} Mf(x) \\ &\leq Mf(x). \end{aligned} \tag{43}$$

It remains to estimate E_4 . From Lemma 4, we obtain

$$\begin{aligned} E_4 &\leq \left(\int_r^\infty \left| r \int_{r<|x-y|\leq t} \frac{|f(y)|}{|x-y|^n} dy \right|^2 \frac{dt}{t^3} \right)^{1/2} \\ &\leq r \left(\int_r^\infty \left| \sum_{k=0}^{[\log_2 t/r]+1} (2^k r)^{-n} \int_{|x-y|\leq 2^k r} f(y) dy \right|^2 \frac{dt}{t^3} \right)^{1/2} \\ &\leq r \left(\int_r^\infty \left| \left(\left[\log_2 \frac{t}{r} \right] + 1 \right) Mf(x) \right|^2 \frac{dt}{t^3} \right)^{1/2} \\ &\leq r \left(\int_r^\infty \frac{t}{r} Mf(x)^2 \frac{dt}{t^3} \right)^{1/2} \\ &\leq Mf(x). \end{aligned} \tag{44}$$

With the help of the L^p -boundedness and weak $(1, 1)$ -boundedness of μ_j and M , we can get the same boundedness of μ_j^L .

For the BMO_L -boundedness and (H_L^1, L^1) -boundedness of μ_j^L , we only make similar modification in the procedure of the same estimate in [6]. Here, we omit it. \square

Next we will establish some boundedness for the commutator of Marcinkiewicz integral of Schrödinger operator type.

We start with the proof of Theorem 6.

Proof. For any function $f \in L^p(\mathbb{R}^d)$ with $1 < p < p_0$, we notice that, due to Lemma 11, we have $[b, \mu_j^L]f \in L_{loc}^1(\mathbb{R}^d)$.

By using Lemma 8 and Lemma 11 with $1 < p < p_0$, and Remark 12, we have

$$\begin{aligned} \|[b, \mu_j^L]f\|_p^p &\leq \int_{\mathbb{R}^d} |M_{\rho, \beta}([b, \mu_j^L]f)|^p \\ &\leq \int_{\mathbb{R}^d} |M_{\rho, \gamma}^\#([b, \mu_j^L]f)|^p \\ &\quad + \sum_k |Q_k| \left(\frac{1}{|Q_k|} \int_{2Q_k} |[b, \mu_j^L]f| \right)^p \\ &\leq \int_{\mathbb{R}^d} |M_{\rho, \gamma}^\#([b, \mu_j^L]f)|^p + [b]_\theta^p \sum_k \int_{2Q_k} |M_s f|^p. \end{aligned} \tag{45}$$

By the finite overlapping property given by Proposition 2 and the boundedness of M_s in $L^p(\mathbb{R}^d)$, the second term is controlled by $[b]_\theta^p \|f\|_p^p$. Thus, we have to take care of the first term.

Our goal is to find a point-wise estimate of $M_{\rho, \gamma}^\#([b, \mu_j^L]f)$. Let $x \in \mathbb{R}^d$ and $B = B(x_0, r)$, with $r < \gamma\rho(x_0)$ such that $x \in B$. If $f = f_1 + f_2$, with $f_1 = f\chi_{2B}$, then we write

$$\begin{aligned} \frac{1}{|B|} \int_B [b, \mu_j^L]f(x) dx &\leq \frac{1}{|B|} \int_B |b - b_B| \mu_j^L f(x) dx \\ &\quad + \frac{1}{|B|} \int_B \mu_j^L(f_1(b - b_B))(x) dx \\ &\quad + \frac{1}{|B|} \int_B \mu_j^L(f_2(b - b_B))(x) dx \\ &= U_1 + U_2 + U_3. \end{aligned} \tag{46}$$

Let $1 < s < p_0$; an application of Hölder's inequality and Lemma 10 gives

$$\begin{aligned} U_1 &= \frac{1}{|B|} \int_B |b - b_B| \mu_j^L f - \frac{1}{|B|} \int_B |b - b_B| \mu_j^L f \\ &\leq \frac{2}{|B|} \int_B |(b - b_B) \mu_j^L f| \\ &\leq \left(\frac{1}{|B|} \int_B |b - b_B|^{s'} \right)^{1/s'} \left(\frac{1}{|B|} \int_B |\mu_j^L f|^s \right)^{1/s} \\ &\leq [b]_\theta M_s \mu_j^L f(x), \end{aligned} \tag{47}$$

since $r/\rho(x_0) < \gamma$.

To estimate U_2 , let $1 < \bar{s} < s < p_0$ and $\nu = \bar{s}s/(s - \bar{s})$. Then,

$$\begin{aligned}
U_2 &= \frac{1}{|B|} \int_B \left| \mu_j^L((b - b_B) f_1) - \frac{1}{|B|} \int_B \mu_j^L((b - b_B) f_1) \right| \\
&\leq \frac{2}{|B|} \int_B \left| \mu_j^L((b - b_B) f_1) \right| \\
&\leq \left(\frac{1}{|B|} \int_B \left| \mu_j^L((b - b_B) f_1) \right|^{\bar{s}} \right)^{1/\bar{s}} \\
&\leq \left(\frac{1}{|B|} \int_{2B} |(b - b_B) f|^{\bar{s}} \right)^{1/\bar{s}} \\
&\leq \left(\frac{1}{|B|} \int_{2B} |b - b_B|^\nu \right)^{1/\nu} \left(\frac{1}{|B|} \int_{2B} |f|^s \right)^{1/s} \\
&\leq [b]_\theta M_s f(x).
\end{aligned} \tag{48}$$

For U_3 , using Lemma 13, we get

$$\begin{aligned}
U_3 &= \frac{1}{|B|} \int_B \left| \mu_j^L(f_2(b - b_B))(u) \right. \\
&\quad \left. - \frac{1}{|B|} \int_B \mu_j^L(f_2(b - b_B))(z) \right| du dz \\
&\leq \frac{1}{|B|^2} \iint_B \left| \mu_j^L(f_2(b - b_B))(u) \right. \\
&\quad \left. - \mu_j^L(f_2(b - b_B))(z) \right| du dz \\
&\leq \frac{1}{|B|^2} \iint_B [b]_\theta M_s f(x) du dz \\
&\leq [b]_\theta M_s f(x).
\end{aligned} \tag{49}$$

Therefore, we have proved that

$$\left| M_{\rho, \gamma}^\#([b, \mu_j^L] f) \right| \leq [b]_\theta [M_s \mu_j^L f(x) + M_s f(x)]. \tag{50}$$

Since $s < p$, we obtain the desired result. \square

Finally, we will prove Theorem 7.

Proof. Let $f \in L^\infty(\mathbb{R}^d)$ and $Q = B(x_0, \gamma\rho(x_0))$. From Proposition 2, we only consider averages over critical ball. Thanks to Lemma 11, one gets

$$\frac{1}{|Q|} \int_Q |[b, \mu_j^L] f| \leq [b]_\theta \inf_{y \in Q} M_s f(y) \leq [b]_\theta \|f\|_\infty. \tag{51}$$

Next, we deal with the oscillations; let $B = B(x_0, r)$ with $r < \gamma\rho(x_0)$. Lemma 11 states that the function $[b, \mu_j^L] f$ belongs to $L^1_{\text{loc}}(\mathbb{R}^d)$.

We write

$$\begin{aligned}
&[b, \mu_j^L] f \\
&\leq |b - b_B| \mu_j^L f + \mu_j^L(f_1(b - b_B)) + \mu_j^L(f_2(b - b_B)),
\end{aligned} \tag{52}$$

and its mean oscillations on B as E_1, E_2 , and E_3 .

The estimates for the terms E_2 and E_3 are the same with U_2 and U_3 in Theorem 6. Due to the boundedness of μ_j^L in $L^{\bar{s}}(\mathbb{R}^d)$, for $\bar{s} > 1$, and Lemma 13, we get that E_2 and E_3 are bounded by $[b]_\theta \|f\|_\infty$.

To deal with E_1 , we fix $u \in B$ and write

$$\begin{aligned}
|b - b_B| \mu_j^L f &\leq |b - b_B| \mu_j^L f_1 + |b - b_B| (\mu_j^L f_2 - \mu_j^L f_2(u)) \\
&\quad + \mu_j^L f_{21}(u) |b - b_B| + \mu_j^L f_{22}(u) |b - b_B|,
\end{aligned} \tag{53}$$

where $f_2 = f_{21} + f_{22}$, with $f_{22} = f \chi_{4Q \setminus 2B}$ and $Q = B(x_0, \gamma\rho(x_0))$. We denoted each oscillation E_{11}, E_{12}, E_{13} , and E_{14} .

We observe that $\mu_j^L f_{21}(u)$ and $\mu_j^L f_{22}(u)$ are finite, for any $u \in B$, since $f \in L^\infty$ and

$$\int_{(2B)^c} |K_j^L(u, z)| dz < \infty. \tag{54}$$

We will see that E_{11}, E_{12} , and E_{13} are bounded under the condition $b \in \text{BMO}_\infty(\rho)$.

For E_{11} , by choosing s such that μ_j^L is bounded on $L^s(\mathbb{R}^d)$, we have

$$\begin{aligned}
E_{11} &\leq \frac{2}{|B|} \int_B |(b - b_B) \mu_j^L f_1| \\
&\leq \left(\frac{1}{|B|} \int_B |b - b_B|^{s'} \right)^{1/s'} \left(\frac{1}{|B|} \int_B |\mu_j^L f_1|^s \right)^{1/s} \\
&\leq \left(\frac{1}{|B|} \int_B |b - b_B|^{s'} \right)^{1/s'} \left(\frac{1}{|B|} \int_{2B} |f|^s \right)^{1/s} \\
&\leq [b]_\theta \|f\|_\infty.
\end{aligned} \tag{55}$$

To deal with E_{12} , we claim

$$|\mu_j^L f_2(x) - \mu_j^L f_2(u)| \leq \|f\|_\infty, \tag{56}$$

for any x and u in B .

It is easy to see that

$$\begin{aligned}
 & \left| \mu_j^L f_2(x) - \mu_j^L f_2(u) \right| \\
 & \leq \left(\int_0^\infty \left| \int_{|x-y| \leq t < |u-y|} |K_j^L(x, y) f_2(y)| dy \right|^2 \frac{dt}{t^3} \right)^{1/2} \\
 & \quad + \left(\int_0^\infty \left| \int_{|u-y| \leq t < |x-y|} |K_j^L(u, y) f_2(y)| dy \right|^2 \frac{dt}{t^3} \right)^{1/2} \\
 & \quad + \left(\int_0^\infty \left| \int_{\{|x-y| \leq t, |u-y| \leq t\}} |K_j^L(x, y) - K_j^L(u, y)| \right. \right. \\
 & \quad \quad \left. \left. \times |f_2(y)| dy \right|^2 \frac{dt}{t^3} \right)^{1/2} \\
 & = I_1 + I_2 + I_3.
 \end{aligned} \tag{57}$$

Since the estimates for I_1 and I_2 follow along similar lines, we only consider I_3 . Since $|x - y| = |u - y|$, by Minkowski's inequality and Lemma 4, we have

$$\begin{aligned}
 I_1 & \leq \int_{\mathbb{R}^n} |f_2(y)| |K_j^L(x, y)| \left(\int_{|x-y| \leq t < |u-y|} \frac{dt}{t^3} \right)^{1/2} dy \\
 & \leq r^{1/2} \int_{(2B)^c} \frac{|f(y)| |K_j^L(x, y)|}{|x-y|^{3/2}} dy \\
 & \leq r^{1/2} \int_{(2B)^c} \frac{|f(y)|}{|x-y|^{n+1/2}} dy \\
 & \leq \frac{1}{r^n} \sum_{j=1}^\infty 2^{-j(n+1/2)} \int_{2^{j+1}B} |f(y)| dy \\
 & \leq \|f\|_\infty.
 \end{aligned} \tag{58}$$

For I_3 , again thanks to the Minkowski's inequality, Lemma 4, and $|x - y| = |u - y|$, we obtain

$$\begin{aligned}
 I_3 & \leq \int_{\mathbb{R}^n} |f_2(y)| |K_j^L(x, y) - K_j^L(u, y)| \\
 & \quad \times \left(\int_{\{|x-y| \leq t, |u-y| \leq t\}} \frac{dt}{t^3} \right)^{1/2} dy \\
 & \leq \int_{(2B)^c} \frac{|f(y)| |K_j^L(x, y) - K_j^L(u, y)|}{|x-y|} dy \\
 & \leq r^\delta \int_{(2B)^c} \frac{|f(y)|}{|x-y|^{n+\delta}} dy \\
 & \leq \frac{1}{r^n} \sum_{j=1}^\infty 2^{-j(n+\delta)} \int_{2^{j+1}B} |f(y)| dy \\
 & \leq \|f\|_\infty.
 \end{aligned} \tag{59}$$

Thus, the claim is completed.

Then,

$$\begin{aligned}
 E_{12} & \leq \frac{2}{|B|} \int_B |b - b_B| |\mu_j^L f_2(x) - \mu_j^L f_2(u)| dx \\
 & \leq [b]_\theta \|f\|_\infty.
 \end{aligned} \tag{60}$$

That $E_{13} \leq [b]_\theta \|f\|_\infty$ is a consequence of (54).

Therefore, the theorem will follow, if there exists a constant C such that, for any $B \in \mathbf{B}_{\rho, \gamma}$ and $u \in B$,

$$\left(\frac{1}{|B|} \int_B |b(y) - b_B| dy \right) |\mu_j^L f_{22}(u)| \leq C \|b\|_* \log \|f\|_\infty. \tag{61}$$

Using Minkowski's inequality,

$$\begin{aligned}
 |\mu_j^L f_{22}(u)| & = \left(\int_0^\infty \left| \int_{|x-y| \leq t} K_j^L(x, y) f_{22}(y) dy \right|^2 \frac{dt}{t^3} \right)^{1/2} \\
 & \leq \int_{\mathbb{R}^d} \frac{|f_{22}(y)|}{|u-y|^{d-1}} \left(\int_{|u-y| \leq t} \frac{dt}{t^3} \right)^{1/2} dy \\
 & \leq \int_{4Q \setminus 2B} \frac{|f(y)|}{|u-y|^d} dy \\
 & \leq \|f\|_\infty \log \frac{\rho(x_0)}{r}.
 \end{aligned} \tag{62}$$

Since $b \in \text{BMO}_\infty^{\log}(\rho)$, we conclude that (61) holds proving the boundedness of $[b, \mu_j^L]$. \square

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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