Hindawi Publishing Corporation
Journal of Function Spaces
Volume 2014, Article ID 505796, 10 pages
http://dx.doi.org/10.1155/2014/505796

Research Article

Anisotropic Two-Microlocal Spaces and Regularity

Mourad Ben Slimane

Department of Mathematics, College of Science, King Saud University, P.O. Box 2455, Riyadh 11451, Saudi Arabia

Correspondence should be addressed to Mourad Ben Slimane; mbenslimane@ksu.edu.sa

Received 9 May 2014; Accepted 28 July 2014; Published 17 August 2014

Academic Editor: Stanislav Hencl

We define D-$\mathbf{u}$-anisotropic two-microlocal spaces by decay conditions on anisotropic wavelet coefficients on any D-$\mathbf{u}$-anisotropic wavelet basis of $L^2(\mathbb{R}^d)$. We prove that these spaces allow the characterizing of pointwise anisotropic Hölder regularity. We also prove an anisotropic wavelet criterion for anisotropic uniform regularity. We finally prove that both this criterion and anisotropic D-$\mathbf{u}$-two-microlocal spaces are independent of the chosen anisotropic D-$\mathbf{u}$-orthonormal wavelet basis.

1. Introduction

Two-microlocal spaces were introduced by Bony [1] for the study of the propagation of singularities of solutions of hyperbolic PDEs. These spaces became much simpler when Jaffard [2] characterized them by decay conditions on isotropic wavelet coefficients. These spaces yield very accurate information on the local oscillations of a function f near a point x_0 and the regularity of its fractional derivatives and primitives at x_0. The properties of the two-microlocal domain were investigated by Seuret and Véhel [3].

However, many natural mathematical objects, as well as many multidimensional signals and images from real physical problems, present strong anisotropies (see [4, 5] and references therein). For instance, this is the case in the textures of medical images (mammographies, osteoporosis, muscular tissues, etc.), hydrology, fracture surfaces analysis, and so forth (see [4, 6–12]). Anisotropic pointwise Hölder regularity and anisotropic Besov spaces have been introduced (see [13]). Anisotropic Besov spaces have played a central role in the mathematical modeling of anisotropic textures. They also have been used to study some PDEs; see [14] and for the study of semielliptic pseudodifferential operators whose symbols have different degrees of smoothness along different directions see [15]. Two-microlocal spaces have to be changed in order to fit anisotropic behaviors. Let $\mathbf{u} = (u_1, \ldots, u_d)$ be such that

$$0 < u_1 \le u_2 \le \cdots \le u_d, \qquad \sum_{i=1}^{d} u_i = d. \tag{1}$$

The vector $\mathbf{u}$ is called anisotropy. For $t > 0$ and $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$, we call anisotropic dilation the map

$$t^{\mathbf{u}} x = \left(t^{u_1} x_1, \ldots, t^{u_d} x_d\right). \tag{2}$$

In [16] (resp., [17]) Calderón and Torchinsky (resp., Folland and Stein) have developed a theory of anisotropic $\mathscr{H}^p(\mathbb{R}^d)$ spaces by replacing the Euclidean norm by a homogeneous quasinorm $\rho_{\mathbf{u}}$; recall that $\rho_{\mathbf{u}}$ is defined on $\mathbb{R}^d$ by $\rho_{\mathbf{u}}(0) = 0$ and, for all $x \neq 0$, $\rho_{\mathbf{u}}(x)$ is the unique $r > 0$ for which $|r^{-\mathbf{u}} x| = 1$, where $|\cdot|$ is the Euclidean norm on $\mathbb{R}^d$.

The function $\rho_{\mathbf{u}}$ is continuous and homogeneous in the sense that

$$\rho_{\mathbf{u}}\left(r^{\mathbf{u}} x\right) = r \rho_{\mathbf{u}}(x), \quad \forall r > 0 \text{ and all } x \in \mathbb{R}^d. \tag{3}$$

The corresponding $\mathbf{u}$-ball $B_{\mathbf{u}}(x, r) := \{y \in \mathbb{R}^d; \rho_{\mathbf{u}}(x - y) < r\}$ of $\rho_{\mathbf{u}}$-radius r, centered on x, is an ellipse of axis of lengths $2r^{u_1}, \ldots, 2r^{u_d}$, centered on x.

In the isotropic case ($u_i = 1$ for all $1 \le i \le d$), the homogeneous quasinorm $\rho_{\mathbf{u}}$ coincides with the Euclidean norm.

If we set

$$|x|_{\mathbf{u}} = \sum_{i=1}^{d} |x_i|^{1/u_i} \tag{4}$$

then $|\cdot|_{\mathbf{u}}$ is also a homogeneous quasinorm in the sense that $|r^{\mathbf{u}}x|_{\mathbf{u}} = r|x|_{\mathbf{u}}$ and there exists a constant $C > 0$ such that, for all $x, y \in \mathbb{R}^d$,

$$|x+y|_{\mathbf{u}} \le C\left(|x|_{\mathbf{u}} + |y|_{\mathbf{u}}\right). \tag{5}$$

The homogeneous quasinorm $|\cdot|_{\mathbf{u}}$ is equivalent to $\rho_{\mathbf{u}}$ because

$$\frac{1}{d}|x|_{\mathbf{u}} \le \rho_{\mathbf{u}}(x) \le d^{1/2u_1}|x|_{\mathbf{u}}. \tag{6}$$

In [18–20], we adapted the notion of pointwise regularity to the anisotropy. Let $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$. For $I = (i_1, \ldots, i_d) \in \mathbb{N}^d$, we set $d(I) = \sum_{l=1}^{d}(u_l/u_1)i_l$ and $d_{\mathbf{u}}(I) = u_1 d(I) = \sum_{l=1}^{d} u_l i_l$. Thus $d(I)$ is the degree of homogeneity of the differential operator ∂^I, or, as we will say, its homogeneous degree. If $P = \sum_I a_I x^I$, $a_I \in \mathbb{R}$, or $\mathbb{C}$ is a polynomial, we define its homogeneous degree to be $d(P) := \max\{d(I) : a_I \ne 0\}$. We also define its $\mathbf{u}$-homogeneous degree to be

$$d_{\mathbf{u}}(P) := u_1 d(P) = \max\{d_{\mathbf{u}}(I) : a_I \ne 0\}. \tag{7}$$

Definition 1. Let $s > 0$ and $f : \mathbb{R}^d \to \mathbb{R}$ or let $\mathbb{C}$ be a function.

(1) Let $x_0 \in \mathbb{R}^d$. We say that f belongs to $C^s_{\mathbf{u}}(x_0)$ (resp., $C^s_{\mathbf{u},\log}(x_0)$) if $f \in L^\infty(\mathbb{R}^d)$ and if there exist a constant $C > 0$, $0 < \varepsilon < 1$, and a polynomial P of $\mathbf{u}$-homogeneous degree less than s such that

$$\text{if } |x - x_0|_{\mathbf{u}} \le \varepsilon, \qquad |f(x) - P(x - x_0)| \le C|x - x_0|^s_{\mathbf{u}}, \tag{8}$$

respectively,

$$\text{if } |x - x_0|_{\mathbf{u}} \le \varepsilon,$$
$$|f(x) - P(x - x_0)| \le C|x - x_0|^s_{\mathbf{u}} \log\left(\frac{1}{|x - x_0|_{\mathbf{u}}}\right). \tag{9}$$

(2) We say that f belongs to $C^s_{\mathbf{u}}(\mathbb{R}^d)$ (resp., $C^s_{\mathbf{u},\log}(\mathbb{R}^d)$) if $f \in L^\infty(\mathbb{R}^d)$ and if (8) (resp., (9)) holds for any x_0 in $\mathbb{R}^d$ with uniform constant C.

For $j \in \mathbb{Z}$, let I_j be a finite set with cardinality bounded independently of j, and for $\alpha \in I_j$ let

$$D^\alpha_j : x = (x_1, \ldots, x_d) \longmapsto \left(a^\alpha_{j,1}x_1, \ldots, a^\alpha_{j,d}x_d\right) \tag{10}$$

such that there exists $0 < C \le 1 \le C'$ such that for all $j \in \mathbb{Z}$, for all $i \in \{1, \ldots, d\}$, and for all $\alpha \in I_j$,

$$C \le a^\alpha_{j,i} \le C'. \tag{11}$$

Remark that

$$C^{1/u_1}|x|_{\mathbf{u}} \le \left|D^\alpha_j x\right|_{\mathbf{u}} = \sum_{i=1}^{d}\left(a^\alpha_{j,i}|x_i|\right)^{1/u_i} \le C'^{1/u_1}|x|_{\mathbf{u}}. \tag{12}$$

So we write

$$\left|D^\alpha_j x\right|_{\mathbf{u}} \approx |x|_{\mathbf{u}}. \tag{13}$$

For $\beta = (\beta_1, \ldots, \beta_d) \in \mathbb{N}^d$, put $|\beta| = \beta_1 + \cdots + \beta_d$ and $\partial^\beta = \partial^{\beta_1}_{x_1} \cdots \partial^{\beta_d}_{x_d}$. In [13], the following definition was given.

Definition 2. Let $D = (D^\alpha_j)_{j\in\mathbb{Z},\alpha\in I_j}$ be as above. Let $N > 0$. A homogeneous D-$\mathbf{u}$-anisotropic orthonormal wavelet basis of $L^2(\mathbb{R}^d)$ of order N is a family $\{f^\alpha_{j,k,\mathbf{u}}, j \in \mathbb{Z}, \alpha \in I_j, k \in \mathbb{Z}^d\}$, satisfying the following conditions.

(1) $f^\alpha_{j,k,\mathbf{u}} \in C^N(\mathbb{R}^d)$ (regularity condition).

(2) For all $M > 0$, there exists $C > 0$ such that, for all $j \in \mathbb{Z}$, $k \in \mathbb{Z}^d$, $\alpha \in I_j$, and $x \in \mathbb{R}^d$ $|\partial^\beta f^\alpha_{j,k,\mathbf{u}}(x)| \le C2^{(d/2+d_{\mathbf{u}}(\beta))j}(1 + |2^{j\mathbf{u}}D^\alpha_j x - k|_{\mathbf{u}})^{-M}$ if $|\beta| \le N$ (localization condition).

(3) $\int_{\mathbb{R}^d} x^\beta f^\alpha_{j,k,\mathbf{u}}(x)dx = 0$ if $|\beta| < N$ (vanishing moment condition).

(4) $\{f^\alpha_{j,k,\mathbf{u}}, j \in \mathbb{Z}, \alpha \in I_j, k \in \mathbb{Z}^d\}$ is an orthonormal basis of $L^2(\mathbb{R}^d)$.

For the inhomogeneous version we need the following definition.

Definition 3. Let $D = (D^\alpha_j)_{j\in\mathbb{Z},\alpha\in I_j}$ be as above. Let $N > 0$. An inhomogeneous D-$\mathbf{u}$-anisotropic orthonormal wavelet basis of $L^2(\mathbb{R}^d)$ of order N is a collection of union of two families $\{f^\alpha_{j,k,\mathbf{u}}, j \in \mathbb{N}, \alpha \in I_j, k \in \mathbb{Z}^d\}$ and $\{g_k, k \in \mathbb{Z}^d\}$, satisfying the following conditions.

(1) $f^\alpha_{j,k,\mathbf{u}} \in C^N(\mathbb{R}^d)$ and $g_k \in C^N(\mathbb{R}^d)$ (regularity condition).

(2) For all $M > 0$, there exists $C > 0$ such that, for all $j \in \mathbb{Z}, k \in \mathbb{Z}^d, \alpha \in I_j$ and $x \in \mathbb{R}^d$ $|\partial^\beta f^\alpha_{j,k,\mathbf{u}}(x)| \le C2^{(d/2+d_{\mathbf{u}}(\beta))j}(1 + |2^{j\mathbf{u}}D^\alpha_j x - k|_{\mathbf{u}})^{-M}$ and $|\partial^\beta g_k(x)| \le C(1 + |x - k|_{\mathbf{u}})^{-M}$ if $|\beta| \le N$ (localization condition).

(3) $\int_{\mathbb{R}^d} x^\beta f^\alpha_{j,k,\mathbf{u}}(x)dx = 0$ if $|\beta| < N$ (vanishing moment condition).

(4) $\int_{\mathbb{R}^d} g_k(x)dx = 1$.

(5) $\{f^\alpha_{j,k,\mathbf{u}}, j \in \mathbb{N}, \alpha \in I_j, k \in \mathbb{Z}^d\} \bigcup \{g_k, k \in \mathbb{Z}^d\}$ is an orthonormal basis of $L^2(\mathbb{R}^d)$.

The inhomogeneous (resp., homogeneous version of) Triebel family of anisotropic wavelets [21, 22] are examples of inhomogeneous (resp., homogeneous) D-$\mathbf{u}$-anisotropic orthonormal wavelet basis of $L^2(\mathbb{R}^d)$. Let $(v_j)_{j\in\mathbb{Z}}$ be a one-dimensional multiresolution analysis in $L^2(\mathbb{R})$ with a scaling

function (called father wavelet) ψ_F and an associated wavelet (called mother wavelet) ψ_M. Let $I_{j,\mathbf{u}}$, $j \in \mathbb{Z}$, be the set of all elements (G,l) with $G = (G_1,\dots,G_d) \in \{F,M\}^d$ such that at least one component G_i is an M and $l = (l_1,\dots,l_d) \in \mathbb{Z}^d$ such that

$$\begin{gathered} l_i = [ju_i], \quad \text{if } G_i = F, \\ [ju_i] \le l_i < [(j+1)u_i] \quad \text{if } G_i = M,\ [(j+1)u_i] > [ju_i], \\ l_i = [ju_i] \quad \text{if } G_i = M,\ [(j+1)u_i] = [ju_i]. \end{gathered} \tag{14}$$

Remark that the cardinality of $I_{j,\mathbf{u}}$ is bounded, independently of j, by $(2^d-1)\prod_{i=1}^d (u_i+2)$.

We easily get the following result.

Proposition 4. *Let $D_j^{(G,l)}x = (2^{l_1-ju_1}x_1,\dots,2^{l_d-ju_d}x_d)$ and*

$$\Psi_{j,k,\mathbf{u}}^{(G,l)}(x) = 2^{|l|/2}\prod_{i=1}^d \psi_{G_i}\left(2^{l_i}x_i - k_i\right). \tag{15}$$

Let

$$\Phi_k(x) = \prod_{i=1}^d \psi_F(x_i - k_i). \tag{16}$$

Suppose that ψ_F and ψ_M are Daubechies [23] (resp., Lemarié-Rieusset and Meyer [24]) wavelets and let N be the common regularity of ψ_F and ψ_M(resp., $N = \infty$). Then the family of $\{\Psi_{j,k,\mathbf{u}}^{(G,l)}, j \in \mathbb{Z}, (G,l) \in I_{j,\mathbf{u}}, k \in \mathbb{Z}^d\}$(resp., the collection of the union of $\{\Phi_k, k \in \mathbb{Z}^d\}$ and $\{\Psi_{j,k,\mathbf{u}}^{(G,l)}, j \in \mathbb{N}, (G,l) \in I_{j,\mathbf{u}}, k \in \mathbb{Z}^d\}$) is a homogeneous (resp., inhomogeneous) D-$\mathbf{u}$-anisotropic orthonormal wavelet basis of $L^2(\mathbb{R}^d)$ of order N. We call them the Triebel anisotropic orthonormal bases.

Remark 5. Let $\{f_{j,k,\mathbf{u}}^\alpha, j \in \mathbb{N}, \alpha \in I_j, k \in \mathbb{Z}^d\} \bigcup \{g_k, k \in \mathbb{Z}^d\}$ be an inhomogeneous D-$\mathbf{u}$-anisotropic orthonormal wavelet basis of $L^2(\mathbb{R}^d)$ of order N. For $F \in L^2(\mathbb{R}^d)$, let $(c_{j,k,\mathbf{u}}^\alpha) = (\int F(x) f_{j,k,\mathbf{u}}^\alpha(x)dx)$ and $(c_k) = (\int F(x)g_k(x)dx)$ denote the D-$\mathbf{u}$-anisotropic wavelet coefficients of F. Then

$$F(x) = \sum_{k\in\mathbb{Z}^d} c_k g_k(x) + \sum_{j=0}^\infty \sum_{k\in\mathbb{Z}^d}\sum_{\alpha\in I_j} c_{j,k,\mathbf{u}}^\alpha f_{j,k,\mathbf{u}}^\alpha(x). \tag{17}$$

We now define D-$\mathbf{u}$-anisotropic two-microlocal spaces.

Definition 6. Let $s > 0$ and $-s \le s' \le 0$. Let $x_0 \in \mathbb{R}^d$. Define D-$\mathbf{u}$-anisotropic two-microlocal space $C_{D,\mathbf{u}}^{s,s'}(x_0)$ as the space of functions F in $L^2(\mathbb{R}^d)$ such that there exists $C > 0$ satisfying

$$\begin{gathered} \forall j \in \mathbb{N}, \quad k \in \mathbb{Z}^d, \quad \alpha \in I_j, \\ \left|c_{j,k,\mathbf{u}}^\alpha\right| \le C2^{-(d/2+s)j}\left(1 + \left|2^{j\mathbf{u}}D_j^\alpha x_0 - k\right|_{\mathbf{u}}\right)^{-s'}. \end{gathered} \tag{18}$$

In the next section, we will recall the Mean Value Theorem and Taylor's theorem with remainder for the homogeneous quasinorm $\rho_{\mathbf{u}}$.

In the third section, we will prove the following two theorems which characterize uniform anisotropic regularity (resp., pointwise anisotropic regularity) by decay condition on D-$\mathbf{u}$-anisotropic wavelet coefficients (resp., by D-$\mathbf{u}$-anisotropic two-microlocal spaces); we denote by Δ the additive subsemigroup of $\mathbb{R}$ generated by $0, 1, u_2/u_1, \dots$ and u_d/u_1. In other words, Δ is the set of all numbers $d(I)$ as I ranges over $\mathbb{N}^d$.

Theorem 7. *Let $s > 0$. Let $\{f_{j,k,\mathbf{u}}^\alpha, j \in \mathbb{N}, \alpha \in I_j, k \in \mathbb{Z}^d\} \bigcup \{g_k, k \in \mathbb{Z}^d\}$ be any inhomogeneous D-$\mathbf{u}$-anisotropic orthonormal wavelet basis of $L^2(\mathbb{R}^d)$, of order N, and $[N] > s/u_1$. Let $F \in L^\infty(\mathbb{R}^d) \bigcap L^2(\mathbb{R}^d)$.*

(1) *If $F \in C_{\mathbf{u}}^s(\mathbb{R}^d)$, then there exists a constant $C > 0$ such that*

$$\left|c_{j,k,\mathbf{u}}^\alpha\right| \le C2^{-j(d/2+s)}, \quad \forall j,k,\alpha. \tag{19}$$

(2) *Conversely, if (19) holds, then $F \in C_{\mathbf{u}}^s(\mathbb{R}^d)$ if $u_1 s \notin \Delta$ and $F \in C_{\mathbf{u},\log}^s(\mathbb{R}^d)$ if $u_1 s \in \Delta$.*

Theorem 8. *Let $s > 0$. Let $\{f_{j,k,\mathbf{u}}^\alpha, j \in \mathbb{N}, \alpha \in I_j, k \in \mathbb{Z}^d\} \bigcup \{g_k, k \in \mathbb{Z}^d\}$ be any inhomogeneous D-$\mathbf{u}$-anisotropic orthonormal wavelet basis of $L^2(\mathbb{R}^d)$, of order N, and $[N] > s/u_1$. Let $F \in L^\infty(\mathbb{R}^d) \bigcap L^2(\mathbb{R}^d)$.*

(1) *If $F \in C_{\mathbf{u}}^s(x_0)$, then $F \in C_{D,\mathbf{u}}^{s,-s}(x_0)$.*

(2) *Conversely, if $F \in C_{D,\mathbf{u}}^{s,-s}(x_0)$ and if $F \in C_{\mathbf{u}}^\beta(\mathbb{R}^d)$ for $0 < \beta < s$, then $F \in C_{\mathbf{u},\log}^s(x_0)$.*

(3) *If there exist $-s < s' \le 0$ such that $F \in C_{D,\mathbf{u}}^{s,s'}(x_0)$, then $F \in C_{\mathbf{u}}^s(x_0)$ if $u_1 s \notin \Delta$ and $F \in C_{\mathbf{u},\log}^s(x_0)$ if $u_1 s \in \Delta$.*

The spaces $C_{D,\mathbf{u}}^{s,s'}(x_0)$ are defined by conditions on the wavelet coefficients; therefore we should check that this definition is independent of the wavelet basis chosen. This will be done in the fourth section. We will also show that Theorem 7 does not depend on the chosen D-$\mathbf{u}$-anisotropic orthonormal wavelet basis of $L^2(\mathbb{R}^d)$.

2. Mean Value Theorem and Taylor's Theorem

The homogeneous quasinorm $\rho_{\mathbf{u}}$ satisfies the following properties:

$$\begin{gathered} \forall i \in \{1,\dots,d\}, \quad |x_i|^{1/u_i} \le \rho_{\mathbf{u}}(x), \\ \text{there exists } \tilde{i} \in \{1,\dots,d\} \text{ such that } x_{\tilde{i}}^2 \ge \frac{1}{d}\rho_{\mathbf{u}}(x)^{2u_{\tilde{i}}}, \\ \rho_{\mathbf{u}}(x) \le 1 \Longleftrightarrow |x| \le 1, \\ |x|^{1/u_1} \le \rho_{\mathbf{u}}(x) \le |x|^{1/u_d}, \quad \text{if } \rho_{\mathbf{u}}(x) \le 1, \\ |x|^{1/u_d} \le \rho_{\mathbf{u}}(x) \le |x|^{1/u_1}, \quad \text{if } \rho_{\mathbf{u}}(x) \ge 1. \end{gathered} \tag{20}$$

In [16, 17], there are versions of Mean Value Theorem and Taylor's theorem with remainder for the homogeneous quasinorm $\rho_{\mathbf{u}}$. Using the fact that $\rho_{\mathbf{u}}$ and $|\cdot|_{\mathbf{u}}$ are equivalent we deduce the following results.

The $\mathbf{u}$*-Mean Value Theorem*. There exist two positive constants C and ν such that, for all functions f of class C^1 on $\mathbb{R}^d$ and all $x, y \in \mathbb{R}^d$,

$$\left|f(y)-f(x)\right| \le C\sum_{i=1}^{d}\left|y-x\right|_{\mathbf{u}}^{u_i} \sup_{|h|_{\mathbf{u}}\le\nu\left|y-x\right|_{\mathbf{u}}} \left|\partial_{x_i} f(x+h)\right|. \tag{21}$$

The $\mathbf{u}$*-Taylor Inequality*. Suppose $\delta \in \Delta\,(\delta > 0)$, and $k = [\delta]$. There are two constants $C_\delta > 0$ and $\nu > 0$ such that, for all functions f of class $C^{(k+1)}$ on $\mathbb{R}^d$ and all $x, y \in \mathbb{R}^d$,

$$\begin{aligned}&\left|f(y)-P(y-x)\right|\\ &\quad\le C_\delta \sum_{|I|\le k+1, d(I)>\delta} \left|y-x\right|_{\mathbf{u}}^{d_{\mathbf{u}}(I)} \sup_{|h|_{\mathbf{u}}\le\nu^{k+1}\left|y-x\right|_{\mathbf{u}}} \left|\partial^I f(x+h)\right|,\end{aligned} \tag{22}$$

where P is the Taylor polynomial of f at x of homogeneous degree δ as

$$P(y-x) = \sum_{I:d(I)\le\delta} \frac{\partial^I f(x)}{I!}(y-x)^I. \tag{23}$$

3. Proofs of Theorems 7 and 8

Let us first prove Theorem 7.

Proof. (1) Let $F \in C_{\mathbf{u}}^s(\mathbb{R}^d)$. We will prove the decay decreasing (19) for the anisotropic wavelet coefficients.

There exists a polynomial $P = \sum_I a_I x^I$ of $\mathbf{u}$-homogeneous degree $d_{\mathbf{u}}(P) = u_1 d(P) = u_1 \max\{d(I) : a_I \ne 0\}$ less then s such that (8) holds for any x and x_0 in $\mathbb{R}^d$ with uniform constant C. Since $|I| \le d(I)$, then the vanishing moment condition for $f_{j,k,\mathbf{u}}^\alpha$, for $[N] > s/u_1$ implies that

$$\begin{aligned}\left|c_{j,k,\mathbf{u}}^\alpha\right| &= \left|\int_{\mathbb{R}^d} F(x) f_{j,k,\mathbf{u}}^\alpha(x)\,dx\right|\\ &= \left|\int_{\mathbb{R}^d} \left(F(x) - P\left(x - 2^{-j\mathbf{u}}\left(D_j^\alpha\right)^{-1}k\right)\right) f_{j,k,\mathbf{u}}^\alpha(x)\,dx\right|\\ &\le C\int \left|x - 2^{-j\mathbf{u}}\left(D_j^\alpha\right)^{-1}k\right|_{\mathbf{u}}^s \left|f_{j,k,\mathbf{u}}^\alpha(x)\right| dx.\end{aligned} \tag{24}$$

Thanks to the localization condition, we get

$$\begin{aligned}&\left|c_{j,k,\mathbf{u}}^\alpha\right|\\ &\quad\le C2^{dj/2}\int \left|x - 2^{-j\mathbf{u}}\left(D_j^\alpha\right)^{-1}k\right|_{\mathbf{u}}^s \left(1 + \left|2^{j\mathbf{u}}D_j^\alpha x - k\right|_{\mathbf{u}}\right)^{-M} dx.\end{aligned} \tag{25}$$

But $|x - 2^{-j\mathbf{u}}(D_j^\alpha)^{-1}k|_{\mathbf{u}} = |2^{-j\mathbf{u}}(D_j^\alpha)^{-1}(2^{j\mathbf{u}}D_j^\alpha x - k)|_{\mathbf{u}} \approx 2^{-js}|2^{j\mathbf{u}}D_j^\alpha x - k|_{\mathbf{u}}$ in the sense given in (13). It follows that

$$\left|c_{j,k,\mathbf{u}}^\alpha\right| \le C2^{dj/2}2^{-js}. \tag{26}$$

(2) Conversely, assume the decay decreasing (19). We will prove that $F \in C_{\mathbf{u}}^s(\mathbb{R}^d)$ if $u_1 s \notin \Delta$ and $F \in C_{\mathbf{u},\log}^s(\mathbb{R}^d)$ if $u_1 s \in \Delta$.

We know that F is expressed as in (17). Since $F \in L^2$ and $\|g_k\|_2 = 1$, then there exists $C > 0$ such that

$$\left|c_k\right| \le C, \quad \forall k. \tag{27}$$

Then the function is $\sum_{k\in\mathbb{Z}^d} c_k g_k(x) \in C^N(\mathbb{R}^d)$.

Let us study the regularity of $\sum_{j=0}^\infty \sum_{k\in\mathbb{Z}^d} \sum_{\alpha\in I_j} c_{j,k,\mathbf{u}}^\alpha f_{j,k,\mathbf{u}}^\alpha(x)$.

Define

$$F_j(x) = \sum_{k\in\mathbb{Z}^d} \sum_{\alpha\in I_j} c_{j,k,\mathbf{u}}^\alpha f_{j,k,\mathbf{u}}^\alpha(x). \tag{28}$$

Since the cardinality of I_j is bounded independently of j, then from the localization of $f_{j,k,\mathbf{u}}^\alpha$ it follows that

$$\forall j \ge 0, \text{ and all } x \quad \left|F_j(x)\right| \le C2^{-js}, \tag{29}$$

$$\forall j \ge 0, \text{ all } x, \text{ and all } I \quad \left|\partial^I F_j(x)\right| \le C2^{-j(s-d_{\mathbf{u}}(I))}. \tag{30}$$

Let $x_0 \in \mathbb{R}^d$. Denote by $\delta_{\mathbf{u}}$ the largest value of $d_{\mathbf{u}}(I)$ such that $d_{\mathbf{u}}(I) < s$. For $j \ge 0$, denote by $P_j(x - x_0)$ the Taylor polynomial of F_j at x_0 of $\mathbf{u}$-homogeneous degree $\delta_{\mathbf{u}}$ (which was defined in (23)):

$$P_j(x - x_0) = \sum_{I:d_{\mathbf{u}}(I)\le\delta_{\mathbf{u}}} \frac{\partial^I F_j(x_0)}{I!}(x - x_0)^I. \tag{31}$$

Then $P(x - x_0) := \sum_{j=0}^\infty P_j(x - x_0)$ is the $\mathbf{u}$-Taylor polynomial of F. Its convergence follows from (30) and the fact that if $d_{\mathbf{u}}(I) \le \delta_{\mathbf{u}}$ then $d_{\mathbf{u}}(I) < s$ and $\sum_{j=0}^\infty 2^{-j(s-d_{\mathbf{u}}(I))} < \infty$.

Let j_0 be the unique integer such that $2^{-j_0} \le |x - x_0|_{\mathbf{u}} < 2\cdot 2^{-j_0}$. Then

$$\begin{aligned}&\left|F(x) - \sum_{k\in\mathbb{Z}^d} c_k g_k(x) - P(x - x_0)\right|\\ &\quad\le \sum_{j=0}^{j_0} \left|F_j(x) - P_j(x - x_0)\right|\\ &\qquad + \sum_{j>j_0} \left|F_j(x)\right| + \sum_{j>j_0} \left|P_j(x - x_0)\right|.\end{aligned} \tag{32}$$

It follows from (29) that

$$\sum_{j>j_0} \left|F_j(x)\right| \le \sum_{j>j_0} C2^{-js} \le C2^{-j_0 s} \le C\left|x - x_0\right|_{\mathbf{u}}^s. \tag{33}$$

It follows from (30) that

$$\sum_{j>j_0}\left|P_j(x-x_0)\right| \le \sum_{j>j_0}\sum_{d_{\mathbf{u}}(I)\le\delta_{\mathbf{u}}} C2^{-j(s-d_{\mathbf{u}}(I))}\left|(x-x_0)^I\right|. \quad (34)$$

But from the definition of $|\cdot|_{\mathbf{u}}$

$$\left|(x-x_0)^I\right| \le |x-x_0|_{\mathbf{u}}^{d_{\mathbf{u}}(I)}. \quad (35)$$

Hence

$$\sum_{j>j_0}\left|P_j(x-x_0)\right| \le C2^{-j_0 s} \le C|x-x_0|_{\mathbf{u}}^s. \quad (36)$$

Take $\delta = \delta_{\mathbf{u}}/u_1$ and $l = [\delta]$. Then $l < s/u_1 < [N]$ and $l+1 \le [N]$. Since F_j is of class $C^{(l+1)}$, then the $\mathbf{u}$-Taylor inequality implies that

$$\begin{aligned}
&\sum_{j=0}^{j_0}\left|F_j(x)-P_j(x-x_0)\right| \\
&\quad\le \sum_{j=0}^{j_0} C_\delta \sum_{|J|\le l+1, d(J)>\delta} |x-x_0|_{\mathbf{u}}^{d_{\mathbf{u}}(J)} \\
&\qquad\times \sup_{|h|_{\mathbf{u}}\le \gamma^{l+1}|x-x_0|_{\mathbf{u}}} \left|\partial^J F_j(x_0+h)\right| \\
&\quad\le C_\delta \sum_{|J|\le l+1, d(J)>\delta} |x-x_0|_{\mathbf{u}}^{d_{\mathbf{u}}(J)} \sum_{j=1}^{j_0} 2^{-j(s-d_{\mathbf{u}}(J))}.
\end{aligned} \quad (37)$$

(i) If $u_1 s \in \Delta$, then $d(J) > \delta$ implies that $d_{\mathbf{u}}(J) > \delta_u$. From the definition of $\delta_{\mathbf{u}}$, we get $d_{\mathbf{u}}(J) \ge s$.

(a) If $d_{\mathbf{u}}(J) > s$, then

$$|x-x_0|_{\mathbf{u}}^{d_{\mathbf{u}}(J)} \sum_{j=1}^{j_0} 2^{-j(s-d_{\mathbf{u}}(J))} \le C|x-x_0|_{\mathbf{u}}^s. \quad (38)$$

(b) If $d_{\mathbf{u}}(J) = s$, then

$$\begin{aligned}
&|x-x_0|_{\mathbf{u}}^{d_{\mathbf{u}}(J)} \sum_{j=1}^{j_0} 2^{-j(s-d_{\mathbf{u}}(J))} \\
&\quad= j_0|x-x_0|_{\mathbf{u}}^s \le C|x-x_0|_{\mathbf{u}}^s \log\left(\frac{1}{|x-x_0|_{\mathbf{u}}}\right).
\end{aligned} \quad (39)$$

Hence

$$\sum_{j=0}^{j_0}\left|F_j(x)-P_j(x-x_0)\right| \le C|x-x_0|_{\mathbf{u}}^s \log\left(\frac{1}{|x-x_0|_{\mathbf{u}}}\right). \quad (40)$$

We conclude that $F \in C_{\mathbf{u},\log}^s(\mathbb{R}^d)$.

(ii) If $u_1 s \notin \Delta$, then $d(J) > \delta$ implies that $d_{\mathbf{u}}(J) > \delta_u$. From the definition of $\delta_{\mathbf{u}}$, we get $d_{\mathbf{u}}(J) > s$. Therefore

$$\sum_{j=0}^{j_0}\left|F_j(x)-P_j(x-x_0)\right| \le C|x-x_0|_{\mathbf{u}}^s. \quad (41)$$

We conclude that $F \in C_{\mathbf{u}}^s(\mathbb{R}^d)$. □

Let us now prove Theorem 8.

Proof. (1) Let $F \in C_{\mathbf{u}}^s(x_0)$ and $N > s/u_1$; then as above

$$\begin{aligned}
\left|c_{j,k,\mathbf{u}}^\alpha\right| &= \left|\int (F(x)-P(x-x_0))\, f_{j,k,\mathbf{u}}^\alpha(x)\,dx\right| \\
&\le C\int |x-x_0|_{\mathbf{u}}^s \left|f_{j,k,\mathbf{u}}^\alpha(x)\right| dx \\
&\le C\int \Big(\left|x-2^{-j\mathbf{u}}\left(D_j^\alpha\right)^{-1}k\right|_{\mathbf{u}}^s \\
&\qquad + \left|x_0-2^{-j\mathbf{u}}\left(D_j^\alpha\right)^{-1}k\right|_{\mathbf{u}}^s\Big)\left|f_{j,k,\mathbf{u}}^\alpha(x)\right| dx.
\end{aligned} \quad (42)$$

From the localization of the wavelets, we obtain

$$\left|c_{j,k,\mathbf{u}}^\alpha\right| \le C2^{-j(s+d/2)} + C\left|x_0-2^{-j\mathbf{u}}\left(D_j^\alpha\right)^{-1}k\right|_{\mathbf{u}}^s. \quad (43)$$

Therefore

$$\left|c_{j,k,\mathbf{u}}^\alpha\right| \le C2^{-j(s+d/2)}\left(1+2^j\left|x_0-2^{-j\mathbf{u}}\left(D_j^\alpha\right)^{-1}k\right|_{\mathbf{u}}\right)^s. \quad (44)$$

(2) Conversely, assume that (44) holds; then

$$\begin{aligned}
\left|F_j(x)\right| &\le \sum_{k\in\mathbb{Z}^d}\sum_{\alpha\in I_j} C2^{-j(d/2)} \\
&\qquad\times\left(2^{-js}+\left|x_0-2^{-j\mathbf{u}}\left(D_j^\alpha\right)^{-1}k\right|_{\mathbf{u}}^s\right)\left|f_{j,k,\mathbf{u}}^\alpha(x)\right| \\
&\le \sum_{k\in\mathbb{Z}^d}\sum_{\alpha\in I_j} C2^{-j(d/2)} \\
&\qquad\times\left(2^{-js}+|x-x_0|_{\mathbf{u}}^s+\left|x-2^{-j\mathbf{u}}\left(D_j^\alpha\right)^{-1}k\right|_{\mathbf{u}}^s\right) \\
&\qquad\times\left|f_{j,k,\mathbf{u}}^\alpha(x)\right|.
\end{aligned} \quad (45)$$

Since the cardinality of I_j is bounded independently of j, then from the localization of the wavelets it follows that

$$\left|F_j(x)\right| \le C\left(2^{-js}+|x-x_0|_{\mathbf{u}}^s\right). \quad (46)$$

Similarly we have

$$\left|\partial^I F_j(x)\right| \le C2^{-j(s-d_{\mathbf{u}}(I))}\left(1+2^j|x-x_0|_{\mathbf{u}}\right)^s. \quad (47)$$

As in the first point above, the function is $\sum_{k\in\mathbb{Z}^d} c_k g_k(x) \in C^N(\mathbb{R}^d)$.

Let j_0 be the unique integer such that $2^{-j_0} \le |x-x_0|_{\mathbf{u}} < 2\cdot 2^{-j_0}$ and $j_1 = sj_0/\beta$. As previously

$$\begin{aligned}\left|F(x) - \sum_{k\in\mathbb{Z}^d} c_k g_k(x) - P(x-x_0)\right| &\le \sum_{j=0}^{j_0}\left|F_j(x) - P_j(x-x_0)\right| + \sum_{j=j_0}^{j_1}\left|F_j(x)\right| \\ &\quad + \sum_{j>j_1}\left|F_j(x)\right| + \sum_{j>j_0}\left|P_j(x-x_0)\right|.\end{aligned} \tag{48}$$

Relation (46) implies that

$$\begin{aligned}\sum_{j=j_0}^{j_1}\left|F_j(x)\right| &\le \sum_{j=j_0}^{j_1} C\left(2^{-js} + |x-x_0|_{\mathbf{u}}^s\right) \\ &\le C\left(2^{-j_0 s} + (j_1-j_0)|x-x_0|_{\mathbf{u}}^s\right) \\ &\le C\left(|x-x_0|_{\mathbf{u}}^s + |x-x_0|_{\mathbf{u}}^s \log\frac{1}{|x-x_0|_{\mathbf{u}}}\right) \\ &\le C|x-x_0|_{\mathbf{u}}^s \log\left(\frac{1}{|x-x_0|_{\mathbf{u}}}\right).\end{aligned} \tag{49}$$

The assumption $F \in C_{\mathbf{u}}^{\beta}(\mathbb{R}^d)$ for a $\beta > 0$ implies that

$$\sum_{j>j_1}\left|F_j(x)\right| \le \sum_{j>j_1} C2^{-\beta j} \le C2^{-\beta j_1} \le C|x-x_0|_{\mathbf{u}}^s. \tag{50}$$

So (47) yields

$$\begin{aligned}&\sum_{j>j_0}\left|P_j(x-x_0)\right| \\ &\le \sum_{j>j_0}\sum_{I:d_{\mathbf{u}}(I)\le\delta_{\mathbf{u}}} C2^{-j(s-d_{\mathbf{u}}(I))}\left(1+2^j|x_0-x_0|_{\mathbf{u}}\right)^s\left|(x-x_0)^I\right|.\end{aligned} \tag{51}$$

It follows from (35) that

$$\begin{aligned}&\sum_{j>j_0}\left|P_j(x-x_0)\right| \\ &\quad\le C\sum_{d_{\mathbf{u}}(I)\le\delta_{\mathbf{u}}}|x-x_0|_{\mathbf{u}}^{d_{\mathbf{u}}(I)}\sum_{j>j_0}2^{-j(s-d_{\mathbf{u}}(I))} \le C|x-x_0|_{\mathbf{u}}^s.\end{aligned} \tag{52}$$

Set $\delta = \delta_{\mathbf{u}}/u_1$ and take $l = [\delta]$. Clearly F_j is of class $C^{(l+1)}$. The $\mathbf{u}$-Taylor inequality implies that

$$\begin{aligned}&\sum_{j=0}^{j_0}\left|F_j(x) - P_j(x-x_0)\right| \\ &\quad\le \sum_{j=0}^{j_0} C_\delta \sum_{|J|\le l+1, d(J)>\delta}|x-x_0|_{\mathbf{u}}^{d_{\mathbf{u}}(J)} \\ &\qquad\times \sup_{|h|_{\mathbf{u}}\le\gamma^{j+1}|x-x_0|_{\mathbf{u}}}\left|\partial^J F_j(x_0+h)\right| \\ &\quad\le \sum_{j=0}^{j_0} C_\delta \sum_{|J|\le l+1, d(J)>\delta}|x-x_0|_{\mathbf{u}}^{d_{\mathbf{u}}(J)} \\ &\qquad\times \sup_{|h|_{\mathbf{u}}\le\gamma^{j+1}|x-x_0|_{\mathbf{u}}} 2^{-j(s-d_{\mathbf{u}}(J))}\left(1+2^j|h|_{\mathbf{u}}\right)^s \\ &\quad\le C_\delta \sum_{|J|\le l+1, d(J)>\delta}|x-x_0|_{\mathbf{u}}^{d_{\mathbf{u}}(J)} \\ &\qquad\times \sum_{j=0}^{j_0}\left(2^{-j(s-d_{\mathbf{u}}(J))} + 2^{jd_{\mathbf{u}}(J)}|x-x_0|_{\mathbf{u}}^s\right).\end{aligned} \tag{53}$$

As above, if $u_1 s \in \Delta$, then

$$\sum_{j=0}^{j_0}\left|F_j(x) - P_j(x-x_0)\right| \le C|x-x_0|_{\mathbf{u}}^s \log\left(\frac{1}{|x-x_0|_{\mathbf{u}}}\right). \tag{54}$$

But if $u_1 s \notin \Delta$, then

$$\sum_{j=0}^{j_0}\left|F_j(x) - P_j(x-x_0)\right| \le C|x-x_0|_{\mathbf{u}}^s. \tag{55}$$

We conclude that $F \in C_{\mathbf{u},\log}^s(x_0)$.

(3) The proof is very similar to the proof of the above point 2 and is left to the reader. □

4. Independence of the Wavelet Basis

We will first check that the definition of D-$\mathbf{u}$-anisotropic two-microlocal space $C_{D,\mathbf{u}}^{s,s'}(x_0)$ does not depend on the chosen D-$\mathbf{u}$-anisotropic orthonormal wavelet basis. We will check a stronger (but simpler) requirement which implies that the condition considered has some additional stability; indeed, we will first prove that the matrix of the operator which maps a D-$\mathbf{u}$-anisotropic orthonormal wavelet basis to another D-$\mathbf{u}$-anisotropic orthonormal wavelet basis is invariant under the action of infinite matrices which belong to an algebra $\mathscr{M}_{D^{-1},\gamma}^{\mathbf{u}}$ of almost diagonal matrices which was defined in [13] in order to prove the stability of anisotropic Besov spaces under changes of D-$\mathbf{u}$-anisotropic wavelet bases; therefore, we will then prove that condition (18) is also invariant under this action. Note that $\mathscr{M}_{D^{-1},\gamma}^{\mathbf{u}}$ is an anisotropic version of the class of almost diagonal matrices that have been considered by Frazier and Jawerth [25] in the isotropic setting and the corresponding isotropic operator algebras are in the book of Meyer [26] (resp., Coifman and Meyer [27, 28]).

Definition 9. Let $\gamma > d(u_d - 1)$. Denote by $\mathscr{M}_{D,\gamma}^{\mathbf{u}}$ the set of infinite matrices $M(\lambda_{\mathbf{u}}, \lambda'_{\mathbf{u}})$ indexed by $\lambda_{\mathbf{u}} = (j,k,\alpha)$ and $\lambda'_{\mathbf{u}} = (j',k',\alpha')$ (where $k,k' \in \mathbb{Z}^d$, $j,j' \in \mathbb{Z}$, $\alpha \in I_j$, $\alpha' \in I_{j'}$) and satisfying that

$$\begin{aligned}&\text{there exists } C > 0 \text{ such that } \forall\left(\lambda_{\mathbf{u}}, \lambda'_{\mathbf{u}}\right), \\ &\qquad\left|M\left(\lambda_{\mathbf{u}}, \lambda'_{\mathbf{u}}\right)\right| \le C\omega_\gamma\left(\lambda_{\mathbf{u}}, \lambda'_{\mathbf{u}}\right),\end{aligned} \tag{56}$$

where

$$\omega_\gamma\left(\lambda_{\mathbf{u}},\lambda'_{\mathbf{u}}\right) = 2^{-(d/2+\gamma)|j-j'|} \times\left(\left(1+\left(j-j'\right)^2\right) \times\left(1+2^{\inf\{j,j'\}}\left|2^{-j\mathbf{u}}D_j^\alpha k-2^{-j'\mathbf{u}}D_{j'}^{\alpha'}k'\right|_{\mathbf{u}}\right)^{d+\gamma}\right)^{-1}. \tag{57}$$

The following propositions were proved in [13].

Proposition 10. *For all $\gamma > d(u_d-1)$, $\mathscr{M}^{\mathbf{u}}_{D,\gamma}$ is an algebra of matrices.*

Proposition 11. *Assume that $N \geq 1$. Let $\delta = \max\{\delta' \in \Delta; \delta' < [N]\}$.*

If $\{f^\alpha_{j,k,\mathbf{u}}, j \in \mathbb{Z}, \alpha \in I_j, k \in \mathbb{Z}^d\}$ and $\{\tilde{f}^{\alpha'}_{j,k,\mathbf{u}}, j \in \mathbb{Z}, \alpha' \in I_j, k \in \mathbb{Z}^d\}$ are two homogeneous D-$\mathbf{u}$-anisotropic orthonormal wavelet bases of order N, then the matrix, whose coefficients are given by $\int f^\alpha_{j,k,\mathbf{u}}(x)\tilde{f}^{\alpha'}_{j',k',\mathbf{u}}(x)dx$, belongs to $\mathscr{M}^{\mathbf{u}}_{D^{-1},\gamma}$ for $\gamma < u_1\delta$.

Let us recall the following definition [13].

Definition 12. For every $\gamma > d(u_d-1)$, the algebra $\mathscr{O}p(\mathscr{M}^{\mathbf{u}}_{D,\gamma})$ is the space of bounded operators on $L^2(\mathbb{R}^d)$ whose matrices on a homogeneous D-anisotropic wavelet basis belong to $\mathscr{M}^{\mathbf{u}}_{D,\gamma}$.

Proposition 11 implies that this definition does not depend on the chosen homogeneous D-$\mathbf{u}$-anisotropic wavelet basis of order N, for $\gamma < u_1 \max\{\delta' \in \Delta; \delta' < [N]\}$, with $N \geq 1$.

Proposition 11 and the second point in Lemma 15 below yield the following theorem.

Theorem 13. *Let $s > 0$. If $f \in C^{s,s'}_{D,\mathbf{u}}(x_0)$ with $-s \leq s' < 0$ and $T \in \mathscr{O}p(\mathscr{M}^{\mathbf{u}}_{D^{-1},\gamma})$, then $Tf \in C^{s,s'}_{D,\mathbf{u}}(x_0)$ for all $\gamma > \max(s-s', d(u_d-1))$.*

Anisotropic two-microlocal spaces $C^{s,s'}_{D,\mathbf{u}}(x_0)$ with $-s \leq s' < 0$ are independent of the chosen D-$\mathbf{u}$-anisotropic wavelet basis.

Proof. We will need the following version of Schur Lemma. □

Lemma 14. *Let $m(p,q)$ be an infinite matrix and $\omega(p) > 0$. If*

$$\sum_q |m(p,q)|\,\omega(q) \leq \omega(p), \tag{58}$$

$$\sum_p |m(p,q)|\,\omega(p) \leq \omega(q), \tag{59}$$

then m is bounded on ℓ^2, with norm bounded by 1.

Proof. Let $(x(q))$ be such that $\sum |x(q)|^2 \leq 1$ and $y(p) = \sum_q m(p,q)x(q)$. Then

$$|m(p,q)\,x(q)| = |m(p,q)|^{1/2}\omega(q)^{1/2}\omega(q)^{-1/2}|m(p,q)|^{1/2}|x(q)|. \tag{60}$$

Then, using both Cauchy-Schwartz inequality and (58),

$$\begin{aligned}|y(p)|^2 &\leq \left(\sum_q |m(p,q)|\,\omega(q)\right)\left(\sum_q |m(p,q)|\,\omega^{-1}(q)\,|x(q)|^2\right)\\ &\leq \omega(p)\left(\sum_q |m(p,q)|\,\omega^{-1}(q)\,|x(q)|^2\right).\end{aligned} \tag{61}$$

Using (59), we have

$$\begin{aligned}\sum_p |y(p)|^2 &\leq \sum_q\left(\omega^{-1}(q)\,|x(q)|^2\left(\sum_p |m(p,q)|\,\omega(p)\right)\right)\\ &\leq \sum_q |x(q)|^2.\end{aligned} \tag{62}$$

□

Lemma 15. *If $0 \leq s < \gamma$, then*

$$\sum_{\lambda'_{\mathbf{u}}} 2^{-j'(d/2+s)}\omega_\gamma\left(\lambda_{\mathbf{u}},\lambda'_{\mathbf{u}}\right) \leq C2^{-j(d/2+s)}. \tag{63}$$

If $s' \leq 0$, $s+s' \geq 0$, and $\gamma > s-s'$, then

$$\begin{aligned}&\sum_{\lambda'_{\mathbf{u}}} 2^{-j'(d/2+s)}\left(1+\left|2^{j'\mathbf{u}}D^{\alpha'}_{j'}x-k\right|_{\mathbf{u}}\right)^{-s'}\omega_\gamma\left(\lambda_{\mathbf{u}},\lambda'_{\mathbf{u}}\right)\\ &\qquad\leq C2^{-j(d/2+s)}\left(1+\left|2^{j\mathbf{u}}D^\alpha_j x-k\right|_{\mathbf{u}}\right)^{-s'}.\end{aligned} \tag{64}$$

Proof. We split the sum in the left term in (63) in two sums.

(i) If $j' \geq j$

$$\sum_{\lambda'_{\mathbf{u}}} 2^{-j'(d/2+s)} \times\left(2^{-(d/2+\gamma)|j-j'|} \times\left(\left(1+\left(j-j'\right)^2\right) \times\left(1+2^{\inf\{j,j'\}}\left|2^{-j\mathbf{u}}D^\alpha_j k-2^{-j'\mathbf{u}}D^{\alpha'}_{j'}k'\right|_{\mathbf{u}}\right)^{d+\gamma}\right)^{-1}\right)$$

$$
\begin{aligned}
&= \sum_{\lambda'_{\mathbf{u}}} 2^{-j'(d/2+s)} \frac{2^{-(d/2+\gamma)(j'-j)}}{\left(1+(j'-j)^2\right)} \\
&\qquad \times \left(1+\left|D_j^{\alpha}k - 2^{(j-j')\mathbf{u}} D_{j'}^{\alpha'}k'\right|_{\mathbf{u}}\right)^{-d-\gamma} \\
&= 2^{-j(d/2+s)} \\
&\quad \times \sum_{\lambda'_{\mathbf{u}}} 2^{-(j'-j)(\gamma+s)} \frac{2^{-d(j'-j)}}{\left(1+(j'-j)^2\right)} \\
&\qquad \times \left(1+\left|D_j^{\alpha}k - 2^{(j-j')\mathbf{u}} D_{j'}^{\alpha'}k'\right|_{\mathbf{u}}\right)^{-d-\gamma} \\
&\le 2^{-j(d/2+s)} \\
&\quad \times \sum_{\lambda'_{\mathbf{u}}} 2^{-(j'-j)(\gamma+s)} 2^{-d(j'-j)} \\
&\qquad \times \left(1+\left|D_j^{\alpha}k - 2^{(j-j')\mathbf{u}} D_{j'}^{\alpha'}k'\right|_{\mathbf{u}}\right)^{-d-\gamma} \\
&= 2^{-j(d/2+s)} \\
&\quad \times \sum_{j'\ge j} \left(2^{-(j'-j)(\gamma+s)} 2^{-d(j'-j)} \right. \\
&\qquad \left. \times \left(\sum_{\alpha'\in I_{j'}} \left(\sum_{k'\in\mathbb{Z}^d} \left(1+\left|D_j^{\alpha}k - 2^{(j-j')\mathbf{u}} D_{j'}^{\alpha'}k'\right|_{\mathbf{u}}\right)^{-d-\gamma} \right)\right)\right).
\end{aligned}
\tag{65}
$$

From Lemma 14, if $d+\gamma > du_d$, then $\sum_{k'\in\mathbb{Z}^d}(1+|D_j^{\alpha}k - 2^{(j-j')\mathbf{u}}D_{j'}^{\alpha'}k'|_{\mathbf{u}})^{-d-\gamma} \le C2^{-(j-j')\beta}$, with $d\le\beta<d+\gamma$, and since the cardinality of $I_{j'}$ is bounded independently of j', then if $j'\ge j$

$$
\begin{aligned}
&\sum_{\lambda'_{\mathbf{u}}} 2^{-j'(d/2+s)} \\
&\quad \times \left(2^{-(d/2+\gamma)|j-j'|} \right. \\
&\qquad \times \left(\left(1+(j-j')^2\right) \right. \\
&\qquad\quad \left. \left. \times \left(1+2^{\inf\{j,j'\}}\left|2^{-j\mathbf{u}}D_j^{\alpha}k - 2^{-j'\mathbf{u}}D_{j'}^{\alpha'}k'\right|_{\mathbf{u}}\right)^{d+\gamma}\right)^{-1}\right) \\
&\le C2^{-j(d/2+s)} \sum_{j'\ge j} 2^{-(j'-j)(\gamma+s)} 2^{-d(j'-j)} 2^{-(j-j')\beta} \\
&\le C' 2^{-j(d/2+s)},
\end{aligned}
\tag{66}
$$

because $\beta-d-\gamma-s<0$.

(ii) If $j'<j$, then

$$
\begin{aligned}
&\sum_{\lambda'_{\mathbf{u}}} 2^{-j'(d/2+s)} \\
&\quad \times \left(2^{-(d/2+\gamma)|j-j'|} \right. \\
&\qquad \times \left(\left(1+(j-j')^2\right) \right. \\
&\qquad\quad \left. \left. \times \left(1+2^{\inf\{j,j'\}}\left|2^{-j\mathbf{u}}D_j^{\alpha}k - 2^{-j'\mathbf{u}}D_{j'}^{\alpha'}k'\right|_{\mathbf{u}}\right)^{d+\gamma}\right)^{-1}\right) \\
&= \sum_{\lambda'_{\mathbf{u}}} 2^{-j'(d/2+s)} \\
&\qquad \times \frac{2^{-(d/2+\gamma)(j-j')}}{\left(1+(j-j')^2\right)} \left(1+\left|D_{j'}^{\alpha'}k' - 2^{(j'-j)\mathbf{u}}D_j^{\alpha}k\right|_{\mathbf{u}}\right)^{-d-\gamma} \\
&\le C\sum_{j'<j} 2^{-j'(d/2+s)} 2^{-(d/2+\gamma)(j-j')} \\
&= C2^{-j(d/2+s)} \sum_{j'<j} 2^{-(\gamma-s)(j-j')} \\
&= C2^{-j(d/2+s)} \text{ because } \gamma>s.
\end{aligned}
\tag{67}
$$

Hence (63) holds.

Let us now prove (64). Since $s'<0$, then

$$
\begin{aligned}
&\left(1+\left|2^{j'\mathbf{u}}D_{j'}^{\alpha'}x - k'\right|_{\mathbf{u}}\right)^{-s'} \\
&\quad \le C\left(\left|2^{j'\mathbf{u}}D_{j'}^{\alpha'}x - 2^{j'\mathbf{u}}D_{j'}^{\alpha'}2^{-j\mathbf{u}}\left(D_j^{\alpha}\right)^{-1}k\right|_{\mathbf{u}}^{-s'} \right. \\
&\qquad \left. + \left(1+\left|2^{j'\mathbf{u}}D_{j'}^{\alpha'}2^{-j\mathbf{u}}\left(D_j^{\alpha}\right)^{-1}k - k'\right|_{\mathbf{u}}\right)^{-s'}\right) \\
&\quad \le C\left(2^{-j's'}\left|x - 2^{-j\mathbf{u}}\left(D_j^{\alpha}\right)^{-1}k\right|_{\mathbf{u}}^{-s'} \right. \\
&\qquad \left. + \left(1+2^{j'}\left|2^{-j\mathbf{u}}\left(D_j^{\alpha}\right)^{-1}k - 2^{-j'\mathbf{u}}\left(D_{j'}^{\alpha'}\right)^{-1}k'\right|_{\mathbf{u}}\right)^{-s'}\right),
\end{aligned}
\tag{68}
$$

and then

$$
\begin{aligned}
&\sum_{\lambda'_{\mathbf{u}}} 2^{-j'(d/2+s)} \left(1+\left|2^{j'\mathbf{u}}D_{j'}^{\alpha'}x - k\right|_{\mathbf{u}}\right)^{-s'} \omega_{\gamma}\left(\lambda_{\mathbf{u}},\lambda'_{\mathbf{u}}\right) \\
&\quad \le C\left|x - 2^{-j\mathbf{u}}\left(D_j^{\alpha}\right)^{-1}k\right|_{\mathbf{u}}^{-s'} \sum_{\lambda'_{\mathbf{u}}} 2^{-j'(d/2+s+s')} \omega_{\gamma}\left(\lambda_{\mathbf{u}},\lambda'_{\mathbf{u}}\right)
\end{aligned}
$$

$$+ C\sum_{\lambda'_{\mathbf{u}}} 2^{-j'(d/2+s)}\omega_\gamma\left(\lambda_{\mathbf{u}}, \lambda'_{\mathbf{u}}\right) \times \left(1 + 2^{j'}\left|2^{-j\mathbf{u}}\left(D_j^\alpha\right)^{-1}k - 2^{-j'\mathbf{u}}\left(D_{j'}^{\alpha'}\right)^{-1}k'\right|_{\mathbf{u}}\right)^{-s'}. \tag{69}$$

It follows from (63) and the assumption $s+s' \geq 0$ that the first term is bounded by $C|x - 2^{-j\mathbf{u}}(D_j^\alpha)^{-1}k|_{\mathbf{u}}^{-s'}2^{-j(d/2+s+s')}$ and the second term is bounded by

$$\begin{aligned} &C\sum_{\lambda'_{\mathbf{u}}} 2^{-j'(d/2+s)}\omega_\gamma\left(\lambda_{\mathbf{u}}, \lambda'_{\mathbf{u}}\right) \\ &\quad\times \left(1 + 2^{\inf\{j,j'\}}\left|2^{-j\mathbf{u}}\left(D_j^\alpha\right)^{-1}k - 2^{-j'\mathbf{u}}\left(D_{j'}^{\alpha'}\right)^{-1}k'\right|_{\mathbf{u}}\right)^{-s'} \\ &\quad\times 2^{-s'|j-j'|} \\ &\leq C\sum_{\lambda'_{\mathbf{u}}} 2^{-j'(d/2+s)}\omega_{\gamma+s'}\left(\lambda_{\mathbf{u}}, \lambda'_{\mathbf{u}}\right) \\ &\leq C2^{-j(d/2+s)} \text{ because } \gamma + s' > s. \end{aligned} \tag{70}$$

Hence (64) holds. □

Theorem 7 and the first result in Lemma 15 yield the following result.

Corollary 16. *Let $s > 0$. If $f \in C_{\mathbf{u}}^s(\mathbb{R}^d)$ and $T \in \mathcal{O}p(\mathcal{M}_{D^{-1},\gamma}^{\mathbf{u}})$ then $Tf \in C_{\mathbf{u}}^s(\mathbb{R}^d)$ if $u_1 s \notin \Delta$ and $Tf \in C_{\mathbf{u},\log}^s(\mathbb{R}^d)$ if $u_1 s \in \Delta$, for all $\gamma > \max(s, d(u_d - 1))$.*

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

Acknowledgment

The author would like to extend his sincere appreciation to the Deanship of Scientific Research at King Saud University for its funding of this research through the Research Group no. RGP-350.

References

[1] J. M. Bony, "Second microlocalization and propagation of singularities for semilinear hyperbolic equations," in *Proceedings of the Taniguchi Symposium Hert*, pp. 11–49, Katata, Japan, 1984.

[2] S. Jaffard, "Pointwise smoothness, two-microlocalization and wavelet coefficients," *Publicacions Matemàtiques*, vol. 35, no. 1, pp. 155–168, 1991.

[3] S. Seuret and J. L. Véhel, "2-Microlocal analysis," in *Fractal Geometry and Applications: A Jubilee of Benoit Mandelbrot*, Proceedings of Symposia in Pure Mathematics, American Mathematical Society, Providence, RI, USA.

[4] A. Arneodo, B. Audit, N. Decoster, J. F. Muzy, and C. Vaillant, "Wavelet -based multifractal formalism: applications to DNA sequences, satellite images of the cloud structure and stock market data," in *The Science of Disasters*, A. Bunde, J. Kropp, and H. J. Schellnhuber, Eds., pp. 27–102, Springer, 2002.

[5] P. Guttorp and P. G. Sampson, "Nonparametric estimation of nonstationary spatial covariance structure," *Journal of the American Statistical Association*, vol. 87, pp. 108–119, 1992.

[6] A. Bonami and A. Estrade, "Anisotropic analysis of some Gaussian models," *Journal of Fourier Analysis and Applications*, vol. 9, no. 3, pp. 215–236, 2003.

[7] E. J. Candès and D. L. Donoho, "Ridgelets: a key to higher-dimensional intermittency?" *Philosophical Transactions of The Royal Society of London A: Mathematical, Physical and Engineering Sciences*, vol. 357, no. 1760, pp. 2495–2509, 1999.

[8] S. Jaffard, "Pointwise and directional regularity of nonharmonic Fourier series," *Applied and Computational Harmonic Analysis*, vol. 28, no. 3, pp. 251–266, 2010.

[9] S. Mallat, "Applied mathematics meets signal processing," *Applied Mathematics*, vol. 14, no. 3, pp. 350–366, 1999.

[10] J. Sampo and S. Sumetkijakan, "Estimations of Hölder regularities and direction of singularity by HARt Smith and curvelet transforms," *Journal of Fourier Analysis and Applications*, vol. 15, no. 1, pp. 58–79, 2009.

[11] E. P. Simoncelli, W. T. Freeman, E. H. Adelson, and D. J. Heeger, "Shiftable multiscale transforms," *IEEE Transactions on Information Theory*, vol. 38, no. 2, pp. 587–607, 1992.

[12] A. B. Watson, "The cortex transform: rapid computation of simulated neural images," *Computer Vision, Graphics, and Image Processing*, vol. 39, no. 3, pp. 311–327, 1987.

[13] M. Ben Slimane and H. Ben Braiek, "On the gentle properties of anisotropic Besov spaces," *Journal of Mathematical Analysis and Applications*, vol. 396, no. 1, pp. 21–48, 2012.

[14] H. Triebel, *Interpolation Theory, Function Spaces, Differential Operators*, North-Holland, Amsterdam, The Netherlands, 1978.

[15] H. Aimar and I. Gómez, "Parabolic Besov regularity for the heat equation," *Constructive Approximation*, vol. 36, no. 1, pp. 145–159, 2012.

[16] A. Calderón and A. Torchinsky, "Parabolic maximal functions associated with a distribution," *Advances in Mathematics*, vol. 16, no. 1, pp. 1–64, 1975.

[17] G. B. Folland and E. M. Stein, *Hardy Spaces on Homogeneous Groups*, vol. 28 of *Mathematical Notes*, Princeton University Press, Princeton, NJ, USA, 1982.

[18] M. Ben Slimane, "Multifractal formalism and anisotropic self-similar functions," *Mathematical Proceedings of the Cambridge Philosophical Society*, vol. 124, no. 2, pp. 329–363, 1998.

[19] H. Ben Braiek and M. Ben Slimane, "Directional regularity criteria," *Comptes Rendus de l'Académie des Sciences*, vol. 349, pp. 385–389, 2011.

[20] M. Ben Slimane and H. Ben Braiek, "Directional and anisotropic regularity and irregularity criteria in Triebel wavelet bases," *Journal of Fourier Analysis and Applications*, vol. 18, no. 5, pp. 893–914, 2012.

[21] H. Triebel, *Theory of Function Spaces*, vol. 100, Birkhäuser, Basel, Switzerland, 2006.

[22] H. Triebel, "Wavelet bases in anisotropic function spaces," in *Proceedings of the Conference of Function Spaces, Differential Operators and Nonlinear Analysis*, pp. 370–387, Institute of Mathematics Polish Academy of Sciences, Milovy, Czech Republic, 2004.

[23] I. Daubechies, "Orthonormal bases of compactly supported wavelets," *Communications on Pure and Applied Mathematics*, vol. 41, no. 7, pp. 909–996, 1988.

[24] P. G. Lemarié-Rieusset and Y. Meyer, "Ondelettes et bases hilbertiennes," *Revista Matemática Iberoamericana*, vol. 2, no. 1, pp. 1–18, 1986.

[25] M. Frazier and B. Jawerth, "A discrete transform and decompositions of distribution spaces," *Journal of Functional Analysis*, vol. 93, no. 1, pp. 34–170, 1990.

[26] Y. Meyer, *Ondelettes et Opérateurs. I-II*, Hermann, Paris, France, 1990.

[27] R. R. Coifman and Y. Meyer, *Au delà des opérateurs pseudo-différentiels*, Astérisque 57, Société Mathématique de France, Paris, France, 1978.

[28] R. R. Coifman and Y. Meyer, "Non-linear harmonic analysis, operator theory and PDE," in *Beijing Lectures in Harmonic Analysis*, E. M. Stein, Ed., vol. 112 of *Annals of Mathematics Studies*, 1986.