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Research Article

Locally Lipschitz Composition Operators in Space of the Functions of Bounded $\kappa\Phi$-Variation

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We give a necessary and sufficient condition on a function $h : \mathbb{R} \to \mathbb{R}$ under which the nonlinear composition operator H, associated with the function h, $Hu(t) = h(u(t))$, acts in the space $\kappa\Phi BV[a,b]$ and satisfies a local Lipschitz condition.

1. Introduction

Given a function $h : \mathbb{R} \to \mathbb{R}$, the composition operator H associated with the function h maps each function $u : [a,b] \to \mathbb{R}$ into the composition function $Hu : [a,b] \to \mathbb{R}$ defined by

$$Hu(t) := h(u(t)), \quad (t \in [a,b]). \tag{1}$$

More generally, given $h : [a,b] \times \mathbb{R} \to \mathbb{R}$, we consider the operator H, defined by

$$Hu(t) := h(t, u(t)), \quad (t \in [a,b]). \tag{2}$$

This operator is also called *superposition operator* or *substitution operator* or *Nemytskij operator* associated with h. In what follows, we will refer to (1) as the *autonomous case* and to (2) as the *nonautonomous case*. For an extensive treatment of composition operator and function spaces we refer to the monographs Appell et al. [1], Appell and Zabrejko [2], and Runst and Sickel [3].

In 1984, Sobolevskij [4] proved the following statement: "the autonomous composition operator associated with $h : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz in the space $\mathrm{Lip}[a,b]$ if and only if the derivative h' exists and is locally Lipschitz." In recent articles Appell et al. [5] and Merentes et al. [6] obtained several results of the Sobolevskij type. As the authors explain in the introduction, the significance of these results lies in the fact that in most applications to many nonlinear problems it is sufficient to impose a local Lipschitz condition, instead of a global Lipschitz condition. In fact they proved that Sobolevskij's result is valid in the spaces $BV_\varphi[a,b]$, $HBV[a,b]$, $RV_\varphi[a,b]$, and $\Phi BV[a,b]$.

Motivated by the work done in the papers [5, 6], we establish a similar result to the one given by Sobolevskij, in the space of functions $\kappa\Phi BV[a,b]$.

Although the composition operator (or Nemytskij operator) is very simple, it turns out to be one of the most interesting and important operators studied in nonlinear functional analysis; the behavior of this operator exhibits many surprising and even pathological features in various function spaces. For example, about 35 years ago Dahlberg [7] proved the following: for $1 \le p \le \infty$ and $1+(1/p) < m < n/p$ integer, if H maps the Sobolev space $W_p^m(\mathbb{R}^n)$ into itself, then h is a linear function. Among these pathologies there is one called *degeneracy phenomenon*, which states that the global Lipschitz condition necessarily leads to affine functions in various functions spaces. This property was first proved in [8] for the space $\mathrm{Lip}[a,b]$. Additional information about the degeneracy phenomena can be found in [9, 10].

This paper is organized as follows: Section 2 contains definitions, notations, and necessary background about the class of functions of bounded $\kappa\Phi$-variation in the sense of Schramm-Korenblum; Section 3 contains the main theorem. Also in this section we state and prove a Helly-type theorem, which plays a crucial role in the demonstration of our Sobolevskij-type result.

2. Some Function Spaces

The concept of functions of bounded variation has been well known since C. Jordan gave the complete characterization of functions of bounded variation as a difference of two increasing functions in 1881. This class of functions exhibits so many interesting properties that it makes a suitable class of functions in a variety of contexts with wide applications in pure and applied mathematics [1, 11].

Definition 1. Let $f : [a,b] \to \mathbb{R}$ be a function. For a given partition $\pi : a = t_0 < t_1 < \cdots < t_n = b$ of the interval $[a,b]$,

$$\sigma(f,\pi) = \sigma(f,\pi;[a,b]) := \sum_{i=1}^{n} \left| f(t_i) - f(t_{i-1}) \right| \tag{3}$$

is called the variation of f on $[a,b]$ with respect to π.

The (possibly infinite) number,

$$V(f;[a,b]) := \sup_{\pi} \sum_{i=1}^{n} \left| f(t_i) - f(t_{i-1}) \right|, \tag{4}$$

where the supremum is taken over all partitions π of the interval $[a,b]$ is called the total variation of f on $[a,b]$. If $V(f;[a,b]) < \infty$, we say that f has bounded variation. The collection of all functions of bounded variation on $[a,b]$ is denoted by $BV[a,b]$.

This notion of a function of bounded variation has been generalized by several authors. One of these generalized versions was given by Korenblum in 1975 [12]. He considered a new kind of variation, called κ-variation, and introduced a function κ for distorting the expression $|t_j - t_{j-1}|$ in the partition itself, rather than the expression $|f(t_j) - f(t_{j-1})|$ in the range. One advantage of this alternative approach is that a function of bounded κ-variation may be decomposed into the difference of two simpler functions called κ-decreasing functions.

Definition 2. A function $\kappa : [0,1] \to [0,1]$ is called a distortion function (κ-function) if κ satisfies the following properties:

(1) κ is continuous with $\kappa(0) = 0$ and $\kappa(1) = 1$;

(2) κ is concave and increasing;

(3) $\lim_{t \to 0^+} (\kappa(t)/t) = \infty$.

Korenblum (see [12]) introduced the definition of bounded κ-variation as follows.

Definition 3. Let κ be a distortion function, f a real function $f : [a,b] \to \mathbb{R}$, and $\pi : a = t_0 < t_1 < \cdots < t_n = b$ a partition of the interval $[a,b]$. Let one consider

$$\begin{gathered} \kappa(f,\pi) := \frac{\sum_{i=1}^{n} \left| f(t_i) - f(t_{i-1}) \right|}{\sum_{i=1}^{n} \kappa\left((t_i - t_{i-1}) / (b-a) \right)}, \\ \kappa V(f) = \kappa V(f;[a,b]) := \sup_{\pi} \kappa(f,\pi), \end{gathered} \tag{5}$$

where the supremum is taken over all partitions π of the interval $[a,b]$. In the case $\kappa V(f;[a,b]) < \infty$ one says that f has bounded κ-variation on $[a,b]$ and one will denote by $\kappa BV[a,b]$ the space of functions of bounded κ-variation on $[a,b]$.

Schramm in 1985 [13] considered a Φ-sequence as follows.

Definition 4 (Φ-sequence). Let $\Phi = \{\phi_n\}_{n\geq 1}$ be a sequence of increasing convex functions, defined on $\mathbb{R}_+ = [0,\infty)$ such that

(1) $\phi_n(0) = 0, n \geq 1$;

(2) $\phi_n(t) > 0$ for $t > 0$.

We will say that Φ is a Φ^*-sequence if $\phi_{n+1}(t) \leq \phi_n(t)$ for all n and t and a Φ-sequence if in addition $\sum_n \phi_n(t)$ diverges for $t > 0$.

From now on, all sequences considered in this work will be Φ-sequences. We will consider a nonoverlapping family of subintervals $I_n = [t_{n-1}, t_n]$ of the interval $I = [a,b]$, $(n = 1,2,\ldots)$; it means that $I_i \cap I_j$ either is empty or contains a single point for $i, j = 1,2,\ldots, i \neq j$.

Definition 5. If Φ is a Φ-sequence, one says that a function $f : [a,b] \to \mathbb{R}$ is of bounded Φ-variation if the Φ-sums $\sum_n \phi_n(|f(t_n) - f(t_{n-1})|) < \infty$ for any nonoverlapping collection $\{I_n\}$ of the interval I.

Definition 6 (condition δ_2 generalized for small values G_{δ_2}). The Φ-sequence $\Phi = \{\phi_n\}_{n\geq 1}$ satisfies condition G_{δ_2} if and only if there exist $t_0 > 0$ and $M(t_0) > 0$ such that

$$\sum_{n=1}^{m} \phi_n(2t) \leq M(t_0) \sum_{n=1}^{m} \phi_n(t) \quad 0 \leq t \leq t_0,\ m \geq 1. \tag{6}$$

We may define, for f of bounded Φ-variation, the total Φ-variation of f by

$$V_\Phi(f) = V_\Phi(f;[a,b]) := \sup \sum_n \phi_n\left(\left| f(t_n) - f(t_{n-1}) \right| \right), \tag{7}$$

where the supremum is taken over all $\{I_n\}, I_n \subseteq [a,b]$. Hernández and Rivas (see [14]) showed that if $\Phi = \{\phi_n\}_{n\geq 1}$ is a Φ-sequence and Φ satisfies condition G_{δ_2}, then $V_\Phi[a,b]$ is a linear space. We denote by $\Phi BV[a,b]$ the collection of all functions f such that cf is of bounded Φ-variation for some $c > 0$.

S. K. Kim and J. Kim in 1986 [15] considered a bounded $\kappa\Phi$-variation as follows.

Definition 7. Let $\kappa : [0,1] \to [0,1]$ be a distortion function and $\Phi = \{\phi_n\}_{n\geq 1}$ a Φ-sequence and let $f : [a,b] \to \mathbb{R}$. One defines

$$\begin{gathered} \kappa\sigma_\phi(f, I_n) := \frac{\sum_{n=1}^{m} \phi_n\left(\left| f(t_n) - f(t_{n-1}) \right| \right)}{\sum_{n=1}^{m} \kappa\left((t_n - t_{n-1}) / (b-a) \right)}, \\ \kappa V_\Phi(f) = \kappa V_\Phi(f;[a,b]) := \sup \kappa\sigma_\phi(f, I_n). \end{gathered} \tag{8}$$

If $\kappa V_\Phi(f;[a,b]) < \infty$, we say that f has bounded $\kappa\Phi$-variation in the interval $[a,b]$ and this number denotes the $\kappa\Phi$-variation of f in Schramm-Korenblum's sense in $[a,b]$. The class of functions that has bounded $\kappa\Phi$-variation in the interval $[a,b]$ is denoted by $\kappa V_\Phi[a,b]$. The vector space generated by this class is denoted by $\kappa\Phi BV[a,b]$.

Let us consider $\kappa V_\Phi(cf)$ as a function of variable c. If $\Phi = \{\phi_n\}_{n\geq 1}$ is a sequence of increasing convex functions, $\phi_n(0) = 0$, $t \geq 0$, we have $\phi_n(ct) \leq c\phi_n(t)$, $0 \leq c \leq 1$. Let $\kappa V_\Phi(f) < \infty$ and let $0 < c \leq 1$. Then $\kappa V_\Phi(cf) \leq c\kappa V_\Phi(f) \to 0$ as $c \to 0$. With this in mind, we define a norm in the space $\kappa\Phi BV_0 = \{f \in \kappa\Phi BV \mid f(a) = 0\}$ as follows:

$$\|f\| = \inf\left\{c > 0 \mid \kappa V_\Phi\left(\frac{f}{c}\right) \leq 1\right\}. \tag{9}$$

We will consider the following norm in the space $\kappa\Phi BV[a,b]$:

$$\|f\|_{\kappa\Phi} = \|f\|_\infty + \mu_\phi(f) \quad (f \in \kappa\Phi BV[a,b]), \tag{10}$$

where $\mu_\phi(f) = \inf\{c > 0 \mid \kappa V_\Phi(f/c) \leq 1\}$ and $\|\cdot\|_\infty$ denotes the supremum norm.

By the above definition, we have the following.

Theorem 8 (see [16]). *Let $\{f_n\} \subset \kappa\Phi BV_0$ be a sequence such that f_n converges to f almost everywhere with $f \in \kappa\Phi BV_0$. Then*

$$\|f\| \leq \lim_{n\to\infty} \inf \|f_n\|; \tag{11}$$

that is, the Luxemburg norm is lower semicontinuous on $\kappa\Phi BV_0$.

Theorem 9 (see [15]). *$(\kappa\Phi BV_0, \|\cdot\|)$ is a Banach space.*

Definition 10 (see [17]). Let $\Phi = \{\phi_n\}_{n\geq 1}$ be a Φ-sequence. A real function $f : [a,b] \to \mathbb{R}$ is said to be $\kappa\Phi$-decreasing on $[a,b]$ if there exists a positive constant c such that for each subinterval I of $[a,b]$

$$\phi_n(|f(I)|) \leq c\kappa\left(\frac{|I|}{b-a}\right). \tag{12}$$

Lemma 11 (see [16]). *For any κ-function and any Φ-sequence $\Phi = \{\phi_n\}_{n\geq 1}$, one has the following:*

(1) $\kappa V_\Phi(f/\|f\|_{\kappa\Phi}) \leq 1$, $f \in \kappa\Phi BV$,

(2) *if $\|f\|_{\kappa\Phi} \leq 1$, then $\kappa V_\Phi(f) \leq \|f\|_{\kappa\Phi}$, $f \in \kappa\Phi BV$.*

Lemma 12 (see [18]). *Let $\kappa : [0,1] \to [0,1]$ be a distortion function and $\Phi = \{\phi_n\}_{n\geq 1}$ a Φ-sequence and let $f \in \kappa\Phi BV[a,b]$ and $c > 0$. Then $\mu_\phi(f) < c$ if and only if $\kappa V_\Phi(f/c) < 1$.*

Theorem 13 (see [15] or [17]). *If a function f is $\kappa\Phi$-decreasing on $[a,b]$, then one has the following properties.*

(1) *f is of bounded $\kappa\Phi$-variation.*

(2) *$f(x_0^+)$ and $f(y_0^-)$ exist for any $a \leq x_0 < b$ and $a < y_0 \leq b$.*

(3) *f is continuous on $[a,b]$.*

Theorem 14 (see [18]). *Let $\kappa : [0,1] \to [0,1]$ be a distortion function, let $\Phi = \{\phi_n\}_{n\geq 1}$ be a Φ-sequence, let $h : \mathbb{R} \to \mathbb{R}$, and let H be the composition operator associated with h. H maps the space* Lip $[0,1]$ *into the space $\kappa\Phi BV[0,1]$ or $\kappa BV[0,1]$ if and only if h is locally Lipschitz. Furthermore, the operator H is bounded.*

The following lemma is basic for our main result.

Lemma 15 (invariance principle). *Let $h : \mathbb{R} \to \mathbb{R}$ be a function. Then the composition operator* (1) *maps the space $\kappa\Phi BV[a,b]$ into itself if and only if it maps, for any other choice of $c < d$, the space $\kappa\Phi BV[c,d]$ into itself.*

Proof. Suppose that the composition operator defined by $Hu = h\circ u$ maps the space $\kappa\Phi BV[a,b]$ into itself. The function $\alpha : [c,d] \to [a,b]$ defined by

$$\alpha(t) := \frac{b-a}{d-c}(t-c) + a \quad (c \leq t \leq d) \tag{13}$$

is a strictly increasing homeomorphism between $[c,d]$ and $[a,b]$ with inverse

$$\alpha^{-1}(s) = \frac{d-c}{b-a}(s-a) + c \quad (a \leq s \leq b) \tag{14}$$

which satisfies $\alpha(c) = a$ and $\alpha(d) = b$. Let $\mathscr{P}([a,b])$ denote the family of all partitions of $[a,b]$. Thus, $\alpha : \mathscr{P}([c,d]) \to \mathscr{P}([a,b])$ with

$$\begin{aligned}\alpha(\{t_0, t_1, \ldots, t_{m-1}, t_m\}) \\ = \{\alpha(t_0), \alpha(t_1), \ldots, \alpha(t_{m-1}), \alpha(t_m)\}\end{aligned} \tag{15}$$

defines a one-to-one correspondence between all partitions of $[c,d]$ and all partitions of $[a,b]$.

Given $v \in \kappa\Phi BV[c,d]$, the function $u := v\circ\alpha^{-1}$ belongs to $\kappa\Phi BV[a,b]$, by the definition of functions of bounded $\kappa\Phi$-variation, and so $Hu = h\circ v\circ\alpha^{-1}$ belongs to $\kappa\Phi BV[a,b]$, by assumption. But for $P \in \mathscr{P}([c,d])$ and $\alpha(P) \in \mathscr{P}([a,b])$ as above we have

$$\begin{aligned}
&\kappa\sigma_\phi(h\circ u, \alpha(P)) \\
&\quad = \kappa\sigma_\phi\left(h\circ v\circ\alpha^{-1}, P\right) \\
&\quad = \frac{\sum_{j=1}^m \phi_j\left(\left|h\left(u\left(\alpha\left(t_j\right)\right)\right) - h\left(u\left(\alpha\left(t_{j-1}\right)\right)\right)\right|\right)}{\sum_{j=1}^m c\kappa\left(\left|u\left(\alpha\left(t_j\right)\right) - u\left(\alpha\left(t_{j-1}\right)\right)\right| / (b-a)\right)} \\
&\quad = \frac{\sum_{j=1}^m \phi_j\left(\left|h\left(v\left(t_j\right)\right) - h\left(v\left(t_{j-1}\right)\right)\right|\right)}{\sum_{j=1}^m c\kappa\left(\left|v\left(t_j\right) - v\left(t_{j-1}\right)\right| / (d-c)\right)} \\
&\quad = \kappa\sigma_\phi(h\circ v, P).
\end{aligned} \tag{16}$$

Passing to the supremum with respect to $P \in \mathscr{P}([c,d])$ and $\alpha(P) \in \mathscr{P}([a,b])$ we conclude that $\kappa V_\Phi(h\circ v;[c,d]) = \kappa V_\Phi(h\circ u;[a,b])$. □

3. Main Results

In the proof of the main result of this paper, we will employ a compactness result, for instance, Helly's selection principle or second Helly's theorem. Helly's theorem for functions of generalized variation has been of some importance for a long time. Helly's selection principle has been the subject of intensive research, and many applications, generalizations, and improvements of them can be found in the literature (see, e.g., [19–21] and the references therein).

In this part we will state and prove our main results. In the proof of our main result we make use of a Helly-type selection theorem for a $\kappa\Phi$-decreasing function.

In the paper [22] Cyphert and Kelingos proved the same result for an arbitrary infinite family of functions defined on $[0,1]$ which is both uniformly bounded and uniformly κ-decreasing.

Theorem 16 (Helly-type selection theorem). *An arbitrary infinite family of functions defined on* $[0,1]$ *which is both uniformly bounded and uniformly* $\kappa\Phi$*-decreasing contains a subsequence which converges at every point of* $[0,1]$ *to a* $\kappa\Phi$*-decreasing function.*

Proof. Let us denote by $\mathcal{F}$ an arbitrary infinite family of functions defined on $[0,1]$, which is both uniformly bounded and uniformly $\kappa\Phi$-decreasing. Then, there exists a constant $c > 0$ such that for every $f \in \mathcal{F}$ and every pair $0 \le x < y \le 1$

$$|f(x)| \le c, \tag{17}$$

$$\phi_n(f(y) - f(x)) \le c\kappa(y - x). \tag{18}$$

Using (17) we can, by means of the standard Cantor diagonalization technique, find a sequence of functions f_k in $\mathcal{F}$ which converges pointwise at each rational point of $[0,1]$, to a function g. Since each f_k satisfies (18), so does g, for all rational numbers $x, y \in [0,1]$.

Define g at irrational points x by

$$g(x) = \lim_{y \to x^-} g(y), \quad y \text{ rational}. \tag{19}$$

The existence of this limit can be seen as follows:

$$A = \liminf_{\mathbb{Q} \ni y \to x^-} g(y) \le \limsup_{\mathbb{Q} \ni y \to x^-} g(y) = B \quad \text{as } y \to x^-, \quad y \text{ rational}. \tag{20}$$

Let $\{y_i\}$ and $\{y_i'\}$ be two sequences of rational points converging to x, arranged so that $y_1 < y_1' < y_2 < y_2' < \cdots < x$ and such that $g(y_i) \to A$ and $g(y_i') \to B$ as $i \to \infty$. Then

$$\begin{aligned} \phi_n\left(g(y_i') - g(y_i)\right) &\le c\kappa\left(y_i' - y_i\right), \\ \phi_n\left(g(y_i') - g(y_i)\right) &= \phi_n\left(\lim_{y \to y_i'} g(y) - \lim_{y \to y_i^-} g(y)\right) \\ &= \phi_n(B - A) \le 0. \end{aligned} \tag{21}$$

Then $\phi_n(B - A) = 0$, and hence $A = B$.

From (19) we obtain, by taking limits of rational points in inequality (18), that g satisfies (18) for all pairs of positive real numbers; that is, g is $\kappa\Phi$-decreasing with constant c on $[0,1]$. By Theorem 13 g is of bounded $\kappa\Phi$-variation and g is continuous. Hence, by another Cantor diagonalization process, a convergent subsequence of the functions f_k can be found.

Now, let us consider $0 < t < 1$ and $\varepsilon > 0$. Then, we fix two rational numbers y_1 and y_2 with $y_1 < t < y_2$ such that

$$\begin{aligned} |g(y_i) - g(t)| &< \frac{\varepsilon}{3}, \quad i = 1,2, \\ c\kappa(|y_i - t|) &< \frac{\varepsilon}{3}, \quad i = 1,2. \end{aligned} \tag{22}$$

Since the sequence $\{f_k\}$, $k = 1,2,\ldots$, converges to g in the rational numbers, there exists $N > 0$ such that

$$|f_k(y_i) - g(y_i)| < \frac{\varepsilon}{3}, \quad i = 1,2, \ k \ge N. \tag{23}$$

Now, from (22) and (23) we obtain

$$\begin{aligned} g(t) - f_k(t) &= (f_k(y_2) - f_k(t)) + (g(t) - g(y_2)) \\ &\quad + (g(y_2) - f_k(y_2)) \\ &\le c\kappa(|y_2 - t|) + (g(t) - g(y_2)) \\ &\quad + (g(y_2) - f_k(y_2)) \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned} \tag{24}$$

Similarly,

$$\begin{aligned} f_k(t) - g(t) &= (f_k(t) - f_k(y_1)) + (g(y_1) - g(t)) \\ &\quad + (f_k(y_1) - g(y_1)) \\ &\le c\kappa(|t - y_1|) \\ &\quad + (g(y_1) - g(t)) + (f_k(y_1) - g(y_1)) \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned} \tag{25}$$

Then, $|f_k(t) - g(t)| < \varepsilon$. □

We are now in a position to formulate and prove our main result.

Theorem 17. *Let us suppose that the composition operator* H *associated with* h *maps the space* $\kappa\Phi BV[a,b]$ *into itself. Then* H *is locally Lipschitz if and only if* h' *exists and is locally Lipschitz in* $\mathbb{R}$.

Proof. First let us assume that h' is locally Lipschitz in $\mathbb{R}$. Given $u \in \kappa\Phi BV[a,b]$, for $r > 0$, we denote by $K_1(r)$ the minimal Lipschitz constant of h' and by $K_2(r)$ the supremum of $|h'|$ on the bounded set

$$B_r := \bigcup_{a \le t \le b} \{u(t) : \|u\|_{\kappa\Phi} \le r\} \subset \mathbb{R}. \tag{26}$$

The finiteness of $K_2(r)$ implies that H satisfies a local Lipschitz condition with respect to the norm $\|\cdot\|_\infty$, so we only have to prove a local Lipschitz condition for H with respect to the $\kappa\Phi$-variation norm. We will prove this by applying twice the mean value theorem.

In fact, let us fix $u, v \in \kappa\Phi BV[a,b]$ with $u \neq v$ and $\|u\|_{\kappa\Phi} \leq r$, $\|v\|_{\kappa\Phi} \leq r$. Given a partition $P = \{t_0, t_1, \ldots, t_m\}$ of $[a,b]$, we split the index set $\{1, \ldots, m\}$ into a union $I \cup J$ of disjoint sets I and J by defining the following:

$j \in I$ if

$$\begin{aligned} &\left|u(t_j) - v(t_j)\right| + \left|u(t_{j-1}) - v(t_{j-1})\right| \\ &\quad \leq \left|u(t_j) - u(t_{j-1})\right| + \left|v(t_j) - v(t_{j-1})\right| \end{aligned} \tag{27}$$

and $j \in J$ if

$$\begin{aligned} &\left|u(t_j) - v(t_j)\right| + \left|u(t_{j-1}) - v(t_{j-1})\right| \\ &\quad > \left|u(t_j) - u(t_{j-1})\right| + \left|v(t_j) - v(t_{j-1})\right|. \end{aligned} \tag{28}$$

By the classical mean value theorem we find α_j between $v(t_j)$ and $u(t_j)$ such that

$$\begin{aligned} Hu(t_j) - Hv(t_j) &= h'(\alpha_j)\left[u(t_j) - v(t_j)\right] \\ &\qquad (j = 1, 2, \ldots, m). \end{aligned} \tag{29}$$

Now, by definition of I we have

$$\begin{aligned} \left|\alpha_j - \alpha_{j-1}\right| &\leq 2\left|u(t_j) - u(t_{j-1})\right| + 2\left|v(t_j) - v(t_{j-1})\right| \\ &\qquad (j \in I). \end{aligned} \tag{30}$$

A straightforward calculation shows then that

$$\begin{aligned} &\left|Hu(t_j) - Hv(t_j) - Hu(t_{j-1}) + Hv(t_{j-1})\right| \\ &= \left|h'(\alpha_j)\left[u(t_j) - v(t_j)\right] - h'(\alpha_{j-1})\left[u(t_{j-1}) - v(t_{j-1})\right]\right| \\ &= \left|\left(h'(\alpha_j) - h'(\alpha_{j-1})\right)\left[u(t_j) - v(t_j)\right]\right. \\ &\quad \left. + h'(\alpha_{j-1})\left[u(t_j) - v(t_j) - u(t_{j-1}) + v(t_{j-1})\right]\right| \\ &\leq K_1(r)\left|\alpha_j - \alpha_{j-1}\right|\|u - v\|_\infty \\ &\quad + K_2(r)\left|u(t_j) - v(t_j) - u(t_{j-1}) + v(t_{j-1})\right| \\ &\leq \left[2K_1(r)\|u - v\|_\infty + K_2(r)\right] \\ &\quad \times \left[\left|u(t_j) - u(t_{j-1})\right| + \left|v(t_j) - v(t_{j-1})\right|\right] \\ &= K_3(r)\left[\left|u(t_j) - u(t_{j-1})\right| + \left|v(t_j) - v(t_{j-1})\right|\right]. \end{aligned} \tag{31}$$

Since $\phi_n(t_1) \leq \phi_n(t_2)$ for $t_1 \leq t_2$, we obtain that

$$\begin{aligned} &\phi_n\left(\left|Hu(t_j) - Hv(t_j) - Hu(t_{j-1}) + Hv(t_{j-1})\right|\right) \\ &\quad \leq \phi_n\left(K_3(r)\left[\left|u(t_j) - u(t_{j-1})\right| + \left|v(t_j) - v(t_{j-1})\right|\right]\right), \end{aligned} \tag{32}$$

and dividing by $\sum_{j=1}^m \kappa(|t_j - t_{j-1}|/(b-a))$ and adding on $j \in I$ we get that

$$\begin{aligned} &\sum_{j\in I}\left(\phi_j\left(\left|Hu(t_j) - Hv(t_j) - Hu(t_{j-1}) + Hv(t_{j-1})\right|\right)\right. \\ &\qquad \left. \times \left(\sum_{j=1}^m \kappa\left(\frac{\left|t_j - t_{j-1}\right|}{b-a}\right)\right)^{-1}\right) \\ &\leq \sum_{j\in I}\left(\phi_j\left(K_3(r)\left[\left|u(t_j) - u(t_{j-1})\right| + \left|v(t_j) - v(t_{j-1})\right|\right]\right)\right. \\ &\qquad \left. \times \left(\sum_{j=1}^m \kappa\left(\frac{\left|t_j - t_{j-1}\right|}{b-a}\right)\right)^{-1}\right) \\ &\leq \sum_{j\in I}\left(\frac{(1/2)\,\phi_j\left(2K_3(r)\left|u(t_j) - u(t_{j-1})\right|\right)}{\sum_{j=1}^m \kappa\left(\left|t_j - t_{j-1}\right|/(b-a)\right)}\right. \\ &\qquad \left. + \frac{(1/2)\,\phi_j\left(2K_3(r)\left|v(t_j) - v(t_{j-1})\right|\right)}{\sum_{j=1}^m \kappa\left(\left|t_j - t_{j-1}\right|/(b-a)\right)}\right) \\ &= \frac{1}{2}\kappa\sigma_\phi\left(2K_3(r)u, P\right) + \frac{1}{2}\kappa\sigma_\phi\left(2K_3(r)v, P\right) \\ &\leq K_3(r)\left(\|u\|_{\kappa\Phi} + \|v\|_{\kappa\Phi}\right) \leq K_4(r)\|u - v\|_{\kappa\Phi}. \end{aligned} \tag{33}$$

Again, by the mean value theorem, we find β_j between $u(t_j)$ and $u(t_{j-1})$ and γ_j between $v(t_j)$ and $v(t_{j-1})$ such that

$$\begin{aligned} Hu(t_j) - Hu(t_{j-1}) &= h'(\beta_j)\left[u(t_j) - u(t_{j-1})\right] \\ &\qquad (j = 1, 2, \ldots, m), \\ Hv(t_j) - Hv(t_{j-1}) &= h'(\gamma_j)\left[v(t_j) - v(t_{j-1})\right] \\ &\qquad (j = 1, 2, \ldots, m). \end{aligned} \tag{34}$$

By definition of J we have

$$\left|\beta_j - \gamma_j\right| < 2\left|u(t_j) - v(t_j)\right| + 2\left|u(t_{j-1}) - v(t_{j-1})\right|. \tag{35}$$

A straightforward calculation shows that

$$\begin{aligned} &\left|Hu(t_j) - Hv(t_j) - Hu(t_{j-1}) + Hv(t_{j-1})\right| \\ &= \left|h'(\beta_j)\left[u(t_j) - u(t_{j-1})\right] - h'(\gamma_j)\left[v(t_j) - v(t_{j-1})\right]\right| \end{aligned}$$

$$
\begin{aligned}
&= \left|\left(h'\left(\beta_j\right) - h'\left(\gamma_j\right)\right)\left[u\left(t_j\right) - u\left(t_{j-1}\right)\right]\right. \\
&\quad \left. + h'\left(\gamma_j\right)\left[u\left(t_j\right) - u\left(t_{j-1}\right) - v\left(t_j\right) + v\left(t_{j-1}\right)\right]\right| \\
&\le K_1(r)\left|\beta_j - \gamma_j\right|\left|u\left(t_j\right) - u\left(t_{j-1}\right)\right| \\
&\quad + K_2(r)\left|u\left(t_j\right) - v\left(t_j\right) - u\left(t_{j-1}\right) + v\left(t_{j-1}\right)\right| \\
&< 2K_1(r)\left[\left|u\left(t_j\right) - v\left(t_j\right)\right| + \left|u\left(t_{j-1}\right) - v\left(t_{j-1}\right)\right|\right] \\
&\quad \times \left|u\left(t_j\right) - u\left(t_{j-1}\right)\right| \\
&\quad + K_2(r)\left[\left|u\left(t_j\right) - u\left(t_{j-1}\right)\right| + \left|v\left(t_j\right) - v\left(t_{j-1}\right)\right|\right] \\
&\le 4K_1(r)\|u - v\|_\infty\left|u\left(t_j\right) - u\left(t_{j-1}\right)\right| \\
&\quad + K_2(r)\left|u\left(t_j\right) - u\left(t_{j-1}\right)\right| \\
&\quad + K_2(r)\left|v\left(t_j\right) - v\left(t_{j-1}\right)\right| \\
&\le \left(4K_1(r)\|u - v\|_\infty + K_2(r)\right)\left|u\left(t_j\right) - u\left(t_{j-1}\right)\right| \\
&\quad + K_2(r)\left|v\left(t_j\right) - v\left(t_{j-1}\right)\right| \\
&\le K_5(r)\left|u\left(t_j\right) - u\left(t_{j-1}\right)\right| + K_2(r)\left|v\left(t_j\right) - v\left(t_{j-1}\right)\right|.
\end{aligned} \tag{36}
$$

Since $\phi_n(t_1) \le \phi_n(t_2)$ for $t_1 \le t_2$, we obtain that

$$
\begin{aligned}
&\phi_n\left(\left|Hu\left(t_j\right) - Hv\left(t_j\right) - Hu\left(t_{j-1}\right) + Hv\left(t_{j-1}\right)\right|\right) \\
&\le \phi_n\left(K_5(r)\left|u\left(t_j\right) - u\left(t_{j-1}\right)\right| + K_2(r)\left|v\left(t_j\right) - v\left(t_{j-1}\right)\right|\right),
\end{aligned} \tag{37}
$$

and dividing by $\sum_{j=1}^m \kappa(|t_j - t_{j-1}|/(b-a))$ and adding on $j \in J$ we get that

$$
\begin{aligned}
&\sum_{j\in J}\left(\frac{\phi_j\left(\left|Hu\left(t_j\right) - Hv\left(t_j\right) - Hu\left(t_{j-1}\right) + Hv\left(t_{j-1}\right)\right|\right)}{\sum_{j=1}^m \kappa\left(\left|t_j - t_{j-1}\right|/(b-a)\right)}\right) \\
&\le \sum_{j\in J}\left(\phi_j\left(K_5(r)\left|u\left(t_j\right) - u\left(t_{j-1}\right)\right|\right.\right. \\
&\qquad \left. + K_2(r)\left|v\left(t_j\right) - v\left(t_{j-1}\right)\right|\right) \\
&\qquad \left. \times\left(\sum_{j=1}^m \kappa\left(\frac{\left|t_j - t_{j-1}\right|}{b-a}\right)\right)^{-1}\right) \\
&\le \sum_{j\in J}\left(\frac{(1/2)\,\phi_j\left(2K_5(r)\left|u\left(t_j\right) - u\left(t_{j-1}\right)\right|\right)}{\sum_{j=1}^m \kappa\left(\left|t_j - t_{j-1}\right|/(b-a)\right)}\right. \\
&\qquad \left. + \frac{(1/2)\,\phi_j\left(2K_2(r)\left|v\left(t_j\right) - v\left(t_{j-1}\right)\right|\right)}{\sum_{j=1}^m \kappa\left(\left|t_j - t_{j-1}\right|/(b-a)\right)}\right) \\
&= \frac{1}{2}\kappa\sigma_\phi\left(2K_5(r)u, P\right) + \frac{1}{2}\kappa\sigma_\phi\left(2K_2(r)v, P\right) \\
&\le K_6(r)\left(\|u\|_{\kappa\Phi} + \|v\|_{\kappa\Phi}\right) \\
&\le K_7(r)\|u - v\|_{\kappa\Phi}.
\end{aligned} \tag{38}
$$

Summing up both partial sums and observing that $K_4(r)$ and $K_7(r)$ do not depend on the partition P we conclude that

$$
\kappa V_\Phi\left(\frac{Hu - Hv}{\left(K_4(r) + K_7(r)\right)\|u - v\|_{\kappa\Phi}}\right) \le 1 \tag{39}
$$

which proves the assertion.

Conversely, suppose that H satisfies a Lipschitz condition. By assumption, the constant

$$
\begin{aligned}
K(r) := \sup\Bigg\{&\frac{\|Hu - Hv\|_{\kappa\Phi}}{\|u - v\|_{\kappa\Phi}} : u, v \in \kappa\Phi BV[a,b], \\
&\|u\|_{\kappa\Phi} \le r, \|v\|_{\kappa\Phi} \le r, u \ne v\Bigg\}
\end{aligned} \tag{40}
$$

is finite for each $r > 0$. Considering, in particular, both functions u and v in (40) constant, we see that

$$
|h(u) - h(v)| \le K(r)|u - v| \quad (u, v \in \mathbb{R}, |u| \le r, |v| \le r). \tag{41}
$$

This shows that h is locally Lipschitz, and so the derivative h' exists almost everywhere in $\mathbb{R}$. It remains to prove that h' exists everywhere in $\mathbb{R}$ and is locally Lipschitz. For the proof of the first claim we show that h' exists in any closed interval $I = [a,b]$.

Given $r > 0$, we consider $z \in \kappa\Phi BV[a,b]$ with $\|z\|_{\kappa\Phi} \le r/2$. Let $\{\alpha_n\}_{n=1}^\infty$ be a decreasing sequence of positive real numbers converging to 0; without loss of generality, we may assume that $\alpha_n \le r/2$ for all $n \in \mathbb{N}$. We define a sequence of functions $h_{\alpha_n,z} : [a,b] \to \mathbb{R}$ by

$$
h_{\alpha_n,z}(t) = \frac{h(z(t) + \alpha_n) - h(z(t))}{\alpha_n} \quad (t \in [a,b]). \tag{42}
$$

Since the composition operator H associated with h acts in the space $\kappa\Phi BV[a,b]$, by assumption, the functions $h_{\alpha_n,z}$ given by (42) belong to $\kappa\Phi BV[a,b]$.

Now, we show that the sequence $\{h_{\alpha_n,z}\}_{n=1}^\infty$ has uniformly bounded $\kappa\Phi$-variation for all $z \in \kappa\Phi BV[a,b]$ with $\|z\|_{\kappa\Phi} \le r/2$. In fact, let $\pi = \{t_0, t_1, \ldots, t_m\}$ be a partition of the interval $[a,b]$. For each $n \in \mathbb{N}$ we define functions u_n and v by

$$
u_n(t) = z(t) + \alpha_n, \quad v(t) = z(t) \quad (t \in [a,b]). \tag{43}
$$

Then $\|u_n\|_{\kappa\Phi} \le r$ and $\|v\|_{\kappa\Phi} \le r$. Furthermore, from Lemma 11, (42), and (43), we obtain the estimates

$$\begin{aligned}
&\frac{\sum_{j=1}^{m}\phi_j\left(\left|\alpha_n\left[h_{\alpha_n,z}\left(t_j\right)-h_{\alpha_n,z}\left(t_{j-1}\right)\right]\right|/\left\|Hu_n-Hv\right\|_{\kappa\Phi}\right)}{\sum_{j=1}^{m}\kappa\left(b-a\right)}\\
&\quad=\frac{\sum_{j=1}^{m}\phi_j\left(\left|h\left(z\left(t_j\right)+\alpha_n\right)-h\left(z\left(t_j\right)\right)-h\left(z\left(t_{j-1}\right)+\alpha_n\right)+h\left(z\left(t_{j-1}\right)\right)\right|/\left\|Hu_n-Hv\right\|_{\kappa\Phi}\right)}{\sum_{j=1}^{m}\kappa\left(b-a\right)}\\
&\quad=\frac{\sum_{j=1}^{m}\phi_j\left(\left|h\left(u_n\left(t_j\right)\right)-h\left(v\left(t_j\right)\right)-h\left(u_n\left(t_{j-1}\right)\right)+h\left(v\left(t_{j-1}\right)\right)\right|/\left\|Hu_n-Hv\right\|_{\kappa\Phi}\right)}{\sum_{j=1}^{m}\kappa\left(b-a\right)}\\
&\quad=\frac{\sum_{j=1}^{m}\phi_j\left(\left(Hu_n-Hv\right)/\left\|Hu_n-Hv\right\|_{\kappa\Phi}\right)}{\sum_{j=1}^{m}\kappa\left(b-a\right)}\\
&\quad\le\kappa V_{\Phi}\left(\frac{Hu_n-Hv}{\left\|Hu_n-Hv\right\|_{\kappa\Phi}};[a,b]\right)\le 1.
\end{aligned}\tag{44}$$

Since the partition $\pi=\{t_0,t_1,\ldots,t_m\}$ was arbitrary, the inequality

$$\kappa V_{\Phi}\left(\frac{\alpha_n h_{\alpha_n,z}}{\left\|Hu_n-Hv\right\|_{\kappa\Phi}};[a,b]\right)\le 1\tag{45}$$

holds for every $n\in\mathbb{N}$ and each $z\in\kappa\Phi BV[a,b]$ with $\|z\|_{\kappa\Phi}\le r/2$. From Lemma 11, the definition of the function $h_{\alpha_n,z}$ in (42), and the definition of the functions u_n and v in (43), we further get

$$\begin{aligned}
\left\|\alpha_n h_{\alpha_n,z}\right\|_{\kappa\Phi}&=\left\|h\left(z+\alpha_n\right)-h\left(z\right)\right\|_{\kappa\Phi}\\
&=\left\|h\left(u_n\right)-h\left(v\right)\right\|_{\kappa\Phi}\\
&\le K\left(r\right)\left\|u_n-v\right\|_{\kappa\Phi}=K\left(r\right)\alpha_n
\end{aligned}\tag{46}$$

and hence $\|h_{\alpha_n,z}\|_{\kappa\Phi}\le K(r)$. By Lemma 11, we conclude that

$$\kappa V_{\Phi}\left(h_{\alpha_n,z}\right)\le K\left(r\right),\tag{47}$$

which shows that the sequence $\{h_{\alpha_{n,z}}\}_{n=1}^{\infty}$ satisfies the hypotheses of Theorem 16.

Theorem 16 ensures the existence of a pointwise convergent subsequence of $\{h_{\alpha_n,z}\}_{n=1}^{\infty}$; without loss of generality we assume that the whole sequence $\{h_{\alpha_n,z}\}_{n=1}^{\infty}$ converges pointwise on $[a,b]$ to some function $f\in\kappa\Phi BV[a,b]$.

Now we define $z(t):=\lambda t$, where $\lambda>0$ is so small that $\|z\|_{\kappa\Phi}\le r/2$. By (43) we see that

$$\begin{aligned}
f\left(t\right)&=\lim_{n\to\infty}\frac{h\left(z\left(t\right)+\alpha_n\right)-h\left(z\left(t\right)\right)}{\alpha_n}\\
&=\lim_{n\to\infty}\frac{h\left(\lambda t+\alpha_n\right)-h\left(\lambda t\right)}{\alpha_n}=\lambda h'\left(\lambda t\right)
\end{aligned}\tag{48}$$

for almost all $t\in[a,b]$. Since the primitive of f and the function $t\mapsto h(\lambda t)$ are both absolutely continuous and have the same derivative on $[a,b]$, we conclude that they differ only by some constant on $[a,b]$, and so h' exists everywhere on $[a,b]$. From the invariance principle (Lemma 15) we deduce that the derivative h' of h exists on any interval and so everywhere in $\mathbb{R}$.

It remains to prove that h' satisfies a local Lipschitz condition. Denoting by F the composition operator associated with the function f from (48), we claim that, for $z\in\kappa\Phi BV[a,b]$ with $\|z\|_{\kappa\Phi}\le r/2$, we have

$$\|Fz\|_{\kappa\Phi}\le K\left(r\right),\tag{49}$$

where $K(r)$ is the Lipschitz constant from (40). In fact, we conclude that

$$\left\|f\right\|_{\kappa\Phi}\le\lim_{n\to\infty}\inf\left\|h_n\right\|_{\kappa\Phi},\tag{50}$$

whenever the sequence $\{h_n\}_{n=1}^{\infty}$ of functions $h_n\in\kappa\Phi BV[a,b]$ converges pointwise on $[a,b]$ to a function f. Combining this with (47) and the observation that the sequence $\{h_{\alpha_n,z}(a)\}$ converges as $n\to\infty$, we obtain (49). We conclude that the composition operator F maps the space $\kappa\Phi BV[a,b]$ into itself, and so the corresponding function f is locally Lipschitz on $\mathbb{R}$. By (48), the same is true for the function h'. □

Conflict of Interests

The authors declare that there is no conflict of interests.

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