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Research Article

On Some Geometric Properties of a New Paranormed Sequence Space

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We introduce a new sequence space which is defined by the operator $W = (w_{nk})$ on the sequence space $\ell(p)$. We define a modular functional on this space and investigate structure of this space equipped with Luxemburg norm. Also we study some geometric properties which are called Kadec-Klee, k-NUC, and uniform Opial properties and prove that this new space possesses these properties.

1. Introduction

In literature, there are many papers about geometric properties and their applications on different sequence spaces. Some of them are as follows.

In [1], Opial defined the Opial property with his name mentioned and he proved that ℓ_p $(1 < p < \infty)$ satisfies this property but the space $L_p[0, 2\pi]$ $(p \neq 2, 1 < p < \infty)$ does not.

Franchetti [2] has shown that any infinite dimensional Banach space has an equivalent norm satisfying the Opial property. Later, Prus [3] has introduced and investigated uniform Opial property for Banach spaces.

In [4], the notion of nearly uniform convexity for Banach spaces was introduced by Huff. Also Huff proved that every nearly uniformly convex space is reflexive and it has uniform Kadec-Klee property. However, Kutzarova [5] defined k-nearly uniformly convex Banach spaces.

Shue [6] first defined Cesaro sequence spaces with norm. In [7], it is shown that the Cesaro sequence spaces ces_p $(1 \leq p < \infty)$ have Kadec-Klee and local uniform rotundity properties.

In [8], it was shown that Banach-Saks of type-p property holds in these spaces.

Later, Sanhan and Suantai [9] generalized the normed sequence spaces to the paranormed sequence spaces. He showed that the sequence spaces $\text{ces}(p)$ equipped Luxemburg norm are rotund and have Kadec-Klee property.

Petrot and Suantai [10] studied the uniform Opial property of these spaces. In [9], Sanhan and Suantai have showed that the Cesaro sequence space $\text{ces}(p)$, where the sum runs over $2^r \leq k \leq 2^{r+1}$, equipped with Luxemburg norm has property (H) but it is not rotund.

Karakaya [11] introduced a new sequence space involving lacunary sequences connected with Cesaro sequence space and examined some geometric properties of this space equipped with Luxemburg norm. In [12], Karakaş et al. defined and studied a new difference sequence space involving lacunary sequences by using difference operator.

In [13], Khan and Rahman introduced sequence spaces $\text{ces}[(p_n), (q_n)]$. Afterwards, Mursaleen and Khan [14] generalized this space to the vector-valued sequence space. In the space $\text{ces}[(p_n), (q_n)]$, if we specialize $q_n = 1$ for all $n \in \mathbb{N}$, then we get $\text{ces}[(p_n), (q_n)] = \text{ces}(p)$ defined in [9].

In [15], Şimşek and Karakaya generalized sequence space $\text{ces}[(p_n), (q_n)]$ to vector-valued space $\text{ces}(X, p_n, q_n)$ and investigated some topological and geometrical properties as Kadec-Klee and rotund according to Luxemburg norm of this space.

In [16], Savaş et al. introduced an ℓ_p-type new sequence space and examined some geometrical properties of this space concerning Banach-Saks of type-p and Gurarii's modulus of convexity. Also, in [17], Şimşek et al. investigated the k-nearly uniform convexity (k-NUC) property and some fixed point results in modular space $V_\rho(\lambda; p)$; Şimşek and Karakaya [18] introduced modular sequence space $\ell_\rho(u, v, p)$ obtained from paranormed ones by generalized weighted means on Köthe sequence spaces and investigated Kadec-Klee property of this space.

2. Preliminaries and Notation

Let $(X, \|\cdot\|)$ (for the brevity $X = (X, \|\cdot\|)$) be a normed linear space and let $B(X)$ (resp. $S(X)$) be the closed unit ball (resp. unit sphere) of X. The space of all real sequences is denoted by w. For any sequence $\{x_n\}$ in X, we denote by $\operatorname{conv}(\{x_n\})$ the convex hull of the elements of $\{x_n\}$.

A Banach space X is called *uniformly convex* (UC) if for each $\varepsilon > 0$, there is $\delta > 0$ such that, for $x, y \in S(X)$, the inequality $\|x - y\| > \varepsilon$ implies that

$$\left\| \frac{1}{2}(x + y) \right\| < 1 - \delta. \tag{1}$$

Recall that for a number $\varepsilon > 0$ a sequence $\{x_n\}$ is said to be an *ε-seperated sequence* if

$$\operatorname{sep}(\{x_n\}) = \inf\{\|x_n - x_m\|, n \neq m\} > \varepsilon. \tag{2}$$

A Banach space X is said to have the *Kadec-Klee property (H property)* if every weakly convergent sequence on the unit sphere is convergent in norm.

A Banach space X is said to have the *uniform Kadec-Klee property* (UKK) if for every $\varepsilon > 0$ there exists $\delta > 0$ such that if x is the weak limit of a normalized ε-separated sequence, then $\|x\| < 1 - \delta$ (see [4]). We have that every (UKK) Banach space has the Kadec-Klee property.

A Banach space X is said to be the *nearly uniformly convex* (NUC) if for every $\varepsilon > 0$ there exists $\delta > 0$ such that, for every sequence $\{x_n\} \subset B(X)$ with $\operatorname{sep}(\{x_n\}) > \varepsilon$, we have

$$\operatorname{conv}(\{x_n\}) \cap (1 - \delta) B(X) \neq \emptyset. \tag{3}$$

Let $k \geq 2$ be an integer. A Banach space X is said to be *k-nearly uniformly convex* (k-NUC) if for any $\varepsilon > 0$ there exists $\delta > 0$ such that, for every sequence $\{x_n\} \subset B(X)$ with $\operatorname{sep}(\{x_n\}) > \varepsilon$, there are $n_1, n_2, \ldots, n_k \in \mathbb{N}$ such that

$$\left\| \frac{x_{n_1} + x_{n_2} + \cdots + x_{n_k}}{k} \right\| < 1 - \delta. \tag{4}$$

Of course a Banach space X is (NUC) whenever it is (k-NUC) for some integer $k \geq 2$. Clearly, (k-NUC) Banach spaces are (NUC) but the opposite implication does not hold in general (see [5]).

A Banach space X is said to have the *Opial property* if every sequence $\{x_n\}$ that is weakly convergent to x_0 satisfies

$$\liminf_{n\to\infty} \|x_n - x_0\| < \liminf_{n\to\infty} \|x_n - x\|, \tag{5}$$

for every $x \in X$ and $x \neq x_0$ (see [1]).

A Banach space X is said to have the *uniform Opial property* if every $\varepsilon > 0$ there exists $\delta > 0$ such that, for each weakly null sequence $\{x_n\} \subset S(X)$ and $x \in X$ with $\|x\| \geq \varepsilon$, we have (see [3])

$$1 + \tau \leq \liminf_{n\to\infty} \|x_n + x\|. \tag{6}$$

A point $x \in S(X)$ is called an *extreme point* if for any $y, z \in B(X)$ the equality $2x = y + z$ implies that $y = z$. A Banach space X is said to be *rotund* (abbreviated as (R)) if every point of $S(X)$ is an extreme point. A Banach space X is said to be *fully k-rotund* (written as kR) (see [19]) if for every sequence $\{x_n\} \subset B(X)$

$$\left\| x_{n_1} + x_{n_2} + \cdots + x_{n_k} \right\| \longrightarrow k \quad \text{as } n_1, n_2, \ldots, n_k \longrightarrow \infty \tag{7}$$

implies that $\{x_n\}$ is convergent.

It is well known that (UC) implies (kR) and (kR) implies $((k+1)R)$, and (kR) spaces are reflexive and rotund, and it is easy to see that (k-NUC) implies (kR).

For a real vector space X, a function $\rho : X \to [0, \infty]$ is called a modular if it satisfies the following conditions:

(i) $\rho(x) = 0 \Leftrightarrow x = 0$,

(ii) $\rho(\alpha x) = \rho(x)$ for all $\alpha \in F$ with $|\alpha| = 1$,

(iii) $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$ for all $x, y \in X$ and all $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$. Further, the modular ρ is called convex if

(iv) $\rho(\alpha x + \beta y) \leq \alpha\rho(x) + \beta\rho(y)$ holds for all $x, y \in X$ and all $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$.

For any modular ρ on X, the space

$$X_\rho = \{x \in X : \rho(\lambda x) \longrightarrow 0 \text{ as } \lambda \longrightarrow 0^+\} \tag{8}$$

is called the modular space.

A sequence (x_n) of elements of X_ρ is called modular convergent to $x \in X_\rho$ if there exists a $\lambda > 0$ such that $\rho(\lambda(x_n - x)) \to 0$ as $n \to \infty$. If ρ is a convex modular, then the following formula defines a norm on X_ρ which is called the Luxemburg norm:

$$\|x\|_L = \inf\left\{\lambda > 0 : \rho\left(\frac{x}{\lambda}\right) \leq 1\right\}. \tag{9}$$

A modular ρ is said to satisfy the Δ_2-condition ($\rho \in \Delta_2$) if for any $\varepsilon > 0$ there exist constants $K \geq 2$ and $a > 0$ such that

$$\rho(2x) \leq K\rho(x) + \varepsilon, \tag{10}$$

for all $x \in X_\rho$ with $\rho(x) \leq a$.

If ρ satisfies the Δ_2-condition for all $a > 0$ with $K \geq 2$ dependent on a, we say that ρ satisfies the strong Δ_2-condition ($\rho \in \Delta_2^s$).

Lemma 1. *If $\rho \in \Delta_2^s$, then, for any $L > 0$ and $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$|\rho(u + v) - \rho(u)| < \varepsilon \tag{11}$$

whenever $u, v \in X_\rho$ with $\rho(u) \leq L$ and $\rho(v) \leq \delta$.

Proof. See [20]. □

Lemma 2. *If $\rho \in \Delta_2^s$, then, for any $\varepsilon > 0$, there exists $\delta > 0$ such that $\|x\| \leq 1-\delta$ whenever $\rho(x) \leq 1-\varepsilon$.*

See [21].

Lemma 3. *If $\rho \in \Delta_2^s$, then for any $(x_n) \in X_\rho$*

$$\|x_n\| \longrightarrow 0 \Longleftrightarrow \rho(x_n) \longrightarrow 0. \tag{12}$$

See [20].

Lemma 4. *If $\rho \in \Delta_2^s$, then for any $\varepsilon > 0$ there exists $\delta > 0$ such that $\|x\| \geq 1-\delta$ whenever $\rho(x) \geq 1-\varepsilon$.*

See [20].

In this paper, we will need the following inequalities in the sequel:

$$|a_k + b_k|^p \leq 2^{p-1}\left(|a_k|^p + |b_k|^p\right), \tag{13}$$

for $p \geq 1$.

In [22], Polat et al. defined the matrix $V = (v_{nk})$ by

$$v_{nk} = \begin{cases} u_n(v_k - v_{k+1}), & k < n \\ u_n v_n, & k = n \\ 0, & k > n. \end{cases} \tag{14}$$

Here, for all $n \in \mathbb{N}$, $u_n \neq 0$, $v_n \neq 0$, and (u_n) depend on n; (v_k) depend on k.

Using this matrix, we define $\ell_\Delta(u,v,p)$ sequence space as follows:

$$\ell_\Delta(u,v,p) = \Bigg\{x = (x_n) \in w : (y_n) = \left(\sum_{k=1}^{n-1} u_n(v_k - v_{k+1})x_k + u_n v_n x_n\right) \in \ell(p),\ n \in \mathbb{N}.\Bigg\}, \tag{15}$$

where $\Delta x_k = x_k - x_{k-1}$ is back difference, and $\Delta v_k = v_k - v_{k+1}$ is forward difference. Throughout this study, $p = (p_k)$ is a bounded sequence of positive real numbers; $H = \sup p_r$ and $M = \max\{1, H\}$. We denote $(Vx)_n = (\sum_{k=1}^{n-1} u_n(v_k - v_{k+1})x_k + u_n v_n x_n)_n$ for short in proof.

Theorem 5. *The sequence space $\ell_\Delta(u,v,p)$ is a complete metric space of nonabsolute type with respect to the paranorm defined by*

$$g(x) = \left(\sum_n \left|\sum_{k=1}^{n-1} u_n(v_k - v_{k+1})x_k + u_n v_n x_n\right|^{p_k}\right)^{1/M}. \tag{16}$$

Proof. The linearity of $\ell_\Delta(u,v,p)$ with respect to the coordinatewise and scalar multiplication follows from the following inequalities which are satisfied for $x, y \in \ell_\Delta(u,v,p)$:

$$\begin{aligned}&\left(\sum_n \left|\sum_{k=1}^{n-1} u_n(v_k - v_{k+1})(x_k + y_k) + u_n v_n(x_n + y_n)\right|^{p_k}\right)^{1/M} \\ &\quad\leq \left(\sum_n \left|\sum_{k=1}^{n-1} u_n(v_k - v_{k+1})x_k + u_n v_n x_n\right|^{p_k}\right)^{1/M} \\ &\quad\quad+ \left(\sum_n \left|\sum_{k=1}^{n-1} u_n(v_k - v_{k+1})y_k + u_n v_n y_n\right|^{p_k}\right)^{1/M},\end{aligned} \tag{17}$$

and for any $\alpha \in \mathbb{R}$

$$|\alpha|^{p_k} \leq \max\left\{1, |\alpha|^M\right\}. \tag{18}$$

It is clear that $g(\theta) = 0$ and $g(x) = g(-x)$ for all $x \in \ell_\Delta(u,v,p)$. From (17), it can be seen the subadditivity of g and $g(\alpha x) \leq \max\{1, |\alpha|^M\}g(x)$.

Let (x^m) be any sequence in $\ell_\Delta(u,v,p)$ such that $g(x^m - x) \to 0$ and (α_m) are any sequence of scalars such that $\alpha_m \to \alpha$. Then, since the inequality

$$g(x^m) \leq g(x) + g(x^m - x) \tag{19}$$

holds, the subadditivity of g, $(g(x^m))$, is bounded and thus we have

$$\begin{aligned}&g(\alpha_m x^m - \alpha x) \\ &\quad= \left(\sum_n \left|\sum_{k=1}^{n-1} u_n(v_k - v_{k+1})(\alpha_m x_k^m - \alpha x_k) + u_n v_n(\alpha_m x_n^m - \alpha x_n)\right|^{p_k}\right)^{1/M} \\ &\quad\leq |\alpha_m - \alpha|^{p_k/M} g(x^m) + |\alpha|^{p_k/M} g(x^m - x),\end{aligned} \tag{20}$$

which tends to zero as $n \to \infty$. Therefore, the scalar multiplication is continuous. Hence, g is a paranorm on the space $\ell_\Delta(u,v,p)$. It remains to prove the completeness of the space $\ell_\Delta(u,v,p)$. Let (x^j) be any Cauchy sequence in the space $\ell_\Delta(u,v,p)$. Then, for a given $\varepsilon > 0$, there exists a positive integer $m_0(\varepsilon) \in \mathbb{N}$ such that $g(x^i - x^j) < \varepsilon/2$ for all $i, j \geq m_0(\varepsilon)$. Using definition of g, we obtain for each fixed $n \in \mathbb{N}$ that

$$\left|\left(Vx^i\right)_n - \left(Vx^j\right)_n\right|^{p_k/M} \leq \left(\sum_n \left|\left(Vx^i\right)_n - \left(Vx^j\right)_n\right|^{p_k}\right)^{1/M} < \frac{\varepsilon}{2}, \tag{21}$$

for every $i, j \geq m_0(\varepsilon)$ which leads us to the fact that $\{(Vx^0)_n, (Vx^1)_n, \ldots\}$ is a Cauchy sequence of real numbers for every fixed $n \in \mathbb{N}$. Since $\mathbb{R}$ is complete,

$(Vx^i)_n \to V(x)$ as $i \to \infty$. Using these infinitely many limits $(Vx)_0, (Vx)_1, (Vx)_2, \ldots$, we may write the sequence $\{(Vx)_0, (Vx)_1, \ldots\}$.

For all $j \geq m_0(\varepsilon)$ and every fixed $n \in \mathbb{N}$

$$\left|\left(Vx^j\right)_n - (Vx)_n\right| < \frac{\varepsilon}{2}. \tag{22}$$

Now, we have to show $x \in \ell_\Delta(u, v, p)$. To do this, we have

$$\begin{aligned} \left|(Vx)_n\right|^{p_k} &= \left|(Vx)_n - \left(Vx^i\right)_n + \left(Vx^i\right)_n\right|^{p_k} \\ \left(\sum_n \left|(Vx)_n\right|^{p_n}\right)^{1/M} &\leq \left(\sum_n \left|(Vx)_n - \left(Vx^i\right)_n\right|^{p_n}\right)^{1/M} \\ &\quad + \left(\sum_n \left|\left(Vx^i\right)_n\right|^{p_n}\right)^{1/M}. \end{aligned} \tag{23}$$

Hence, we get $x \in \ell_\Delta(u, v, p)$. As a result $\ell_\Delta(u, v, p)$ is a complete metric space. □

We introduce a modular sequence space $\ell_{\rho\Delta}(u, v, p)$ by

$$\begin{aligned} &\ell_{\rho\Delta}(u, v, p) \\ &= \left\{x = (x_n) \in w : \sum_{k=1}^{\infty} \left(\sum_{i=1}^{k-1} u_k \Delta v_i \left|x_i\right| + u_k v_k \left|x_k\right|\right)^{p_k} \right. \\ &\qquad \left. < \infty. \right\}. \end{aligned} \tag{24}$$

The Luxemburg norm on the sequence space $\ell_{\rho\Delta}(u, v, p)$ is defined as follows:

$$\begin{gathered} \|x\|_L = \inf\left\{\lambda > 0 : \rho\left(\frac{x}{\lambda}\right) \leq 1\right\} \\ \text{for every } x \in \ell_{\rho\Delta}(u, v, p). \end{gathered} \tag{25}$$

Here, the modular defined by

$$\rho(x) = \sum_{k=1}^{\infty} \left(\sum_{i=1}^{k-1} u_k \Delta v_i \left|x_i\right| + u_k v_k \left|x_k\right|\right)^{p_k} \tag{26}$$

is a convex modular on $\ell_{\rho\Delta}(u, v, p)$.

3. Main Results

In this section, we will give some basic properties of the modular ρ on the space $\ell_{\rho\Delta}(u, v, p)$. Also, we will investigate some relationships between the modular ρ and the Luxemburg norm on $\ell_{\rho\Delta}(u, v, p)$. Finally, we study some geometric properties on this space.

Let us start with some lemmas which will be used in the proof of the theorems about geometric properties of this space.

Lemma 6. *The functional ρ is a convex modular on $\ell_{\rho\Delta}(u, v, p)$.*

Proof. Let $x, y \in \ell_{\rho\Delta}(u, v, p)$. It is obvious that

(i) $\rho(x) = 0 \Leftrightarrow x = 0$.

(ii) $\rho(\lambda x) = \rho(x)$ for all scalar λ with $|\lambda| = 1$

$$\begin{aligned} \rho(\lambda x) &= \sum_{k=1}^{\infty} \left(\sum_{i=1}^{k-1} u_k \Delta v_i \left|\alpha x_i\right| + u_k v_k \left|\alpha x_k\right|\right)^{p_k} \\ &= \sum_{k=1}^{\infty} |\alpha|^{p_k} \left(\sum_{i=1}^{k-1} u_k \Delta v_i \left|x_i\right| + u_k v_k \left|x_k\right|\right)^{p_k} \\ &= \rho(x). \end{aligned} \tag{27}$$

(iii) For $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$, by the convexity of $t \to |t|^{p_k}$ for every $k \in \mathbb{N}$, we have

$$\begin{aligned} \rho(\lambda x + \beta y) &= \sum_{k=1}^{\infty} \left(\sum_{i=1}^{k-1} u_k \Delta v_i \left|\alpha x_i + \beta y_i\right| + u_k v_k \left|\alpha x_k + \beta y_k\right|\right)^{p_k} \\ &\leq \sum_{k=1}^{\infty} \left(\sum_{i=1}^{k-1} u_k \Delta v_i \left|\alpha x_i\right| + u_k v_k \left|\alpha x_k\right| \right. \\ &\qquad \left. + \sum_{i=1}^{k-1} u_k \Delta v_i \left|\beta y_i\right| + u_k v_k \left|\beta y_k\right|\right)^{p_k} \\ &\leq \sum_{k=1}^{\infty} |\alpha|^{p_k} \left(\sum_{i=1}^{k-1} u_k \Delta v_i \left|x_i\right| + u_k v_k \left|x_k\right|\right)^{p_k} \\ &\quad + \sum_{k=1}^{\infty} |\beta|^{p_k} \left(\sum_{i=1}^{k-1} u_k \Delta v_i \left|y_i\right| + u_k v_k \left|y_k\right|\right)^{p_k} \\ &\leq \alpha\rho(x) + \beta\rho(y). \end{aligned} \tag{28}$$

For $x \in \ell_{\rho\Delta}(u, v, p)$, the modular ρ on $\ell_{\rho\Delta}(u, v, p)$ satisfies the following properties: □

(i) if $0 < a < 1$, then $a^M \rho(x/a) \leq \rho(x)$ and $\rho(ax) \leq a\rho(x)$,

(ii) if $a \geq 1$, then $\rho(x) \leq a^M \rho(x/a)$,

(iii) if $a \geq 1$, then $\rho(x) \leq a\rho(x) \leq \rho(ax)$.

Proof. It can be proved with standard techniques in a similar way as in [23]. □

Lemma 7. *For any $x \in \ell_{\rho\Delta}(u, v, p)$,*

(i) if $\|x\| < 1$, then $\rho(x) \leq \|x\|$,

(ii) if $\|x\| > 1$, then $\rho(x) \geq \|x\|$,

(iii) $\|x\| = 1 \Leftrightarrow \rho(x) = 1$,

(iv) if $\|x\| < 1$, then $\rho(x) < 1$,

(v) if $\|x\| > 1$, then $\rho(x) > 1$.

Proof. It can be proved with standard techniques in a similar way as in [23]. □

Lemma 8. *Let $\{x_n\}$ be a sequence in $\ell_{\rho\Delta}(u,v,p)$:*

(i) if $\lim_{n\to\infty}\|x_n\| = 1$, *then* $\lim_{n\to\infty}\rho(x_n) = 1$,

(ii) if $\lim_{n\to\infty}\rho(x_n) = 0$, *then* $\lim_{n\to\infty}\|x_n\| = 0$.

Proof. It can be proved with standard techniques in a similar way as in [23]. □

Lemma 9. *For any $L > 0$ and $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\left|\rho(u+v) - \rho(u)\right| < \varepsilon, \tag{29}$$

whenever $u, v \in \ell_{\rho\Delta}(u,v,p)$ with $\rho(u) \le L$ and $\rho(v) \le \delta$.

Proof. Since $p = (p_k)$ is bounded, it is easy to see that $\rho \in \Delta_2^s$. Hence, the lemma is obtained directly from Lemma 1. □

Lemma 10. *For any sequence $(x_n) \in \ell_{\rho\Delta}(u,v,p)$,*

$$\|x_n\| \longrightarrow 0 \Longleftrightarrow \rho(x_n) \longrightarrow 0. \tag{30}$$

Proof. Since $\rho \in \Delta_2^s$, the lemma is obtained directly from Lemma 3. □

Lemma 11. *For any $x \in \ell_{\rho\Delta}(u,v,p)$ and $\varepsilon \in (0,1)$, there exists $\delta \in (0,1)$ such that $\rho(x) \le 1-\varepsilon$ implies $\|x\| \le 1-\delta$.*

Proof. Since $\rho \in \Delta_2^s$, the lemma is obtained directly from Lemma 2. □

Now we will show that the $\ell_{\rho\Delta}(u,v,p)$ is a Banach space with respect to the Luxemburg norm

Theorem 12. *The space $\ell_{\rho\Delta}(u,v,p)$ is a Banach space with respect to the Luxemburg norm defined by*

$$\|x\| = \inf\left\{\lambda > 0 : \rho\left(\frac{x}{\lambda}\right) \le 1\right\}. \tag{31}$$

Proof. We will show that every Cauchy sequence in $\ell_{\rho\Delta}(u,v,p)$ is convergent according to the Luxemburg norm. Let (x_k^n) be a Cauchy sequence $\ell_{\rho\Delta}(u,v,p)$ and $\varepsilon \in (0,1)$. Thus, there exists $n_0(\varepsilon)$ such that $\|x^n - x^m\| < \varepsilon$ for all $m, n \ge n_0$. By the Lemma 8(i), we obtain

$$\rho\left(x^n - x^m\right) < \left\|x^n - x^m\right\| < \varepsilon, \tag{32}$$

for all $m, n \ge n_0(\varepsilon)$; that is,

$$\sum_{k=1}^{\infty}\left(\sum_{i=1}^{k-1} u_k \Delta v_i \left|x_n(i) - x_m(i)\right| + u_k v_k \left|x_n(k) - x_m(k)\right|\right)^{p_k} < \varepsilon. \tag{33}$$

For fixed k we get that

$$\left|x_n(i) - x_m(i)\right| < \varepsilon. \tag{34}$$

Hence, we obtain that the sequence $(x_n(i))$ is a Cauchy sequence in $\mathbb{R}$. Since the real number $\mathbb{R}$ is complete, $x_m(i) \to x(i)$ as $m \to \infty$. Therefore, for fixed k and

$$\begin{aligned}&\left(\sum_{i=1}^{k-1} u_k \Delta v_i \left|x_n(i) - x(i)\right| + u_k v_k \left|x_n(k) - x(k)\right|\right)^{p_k}\\ &\quad < \varepsilon \quad \forall n \ge n_0(\varepsilon).\end{aligned} \tag{35}$$

So, we obtain that for all $n \ge n_0(\varepsilon)$ and as m goes to infinity

$$\rho\left(x_n - x_m\right) \longrightarrow \rho\left(x_n - x\right). \tag{36}$$

So, for all $n \ge n_0(\varepsilon)$ from Lemma 8(i),

$$\rho\left(x_n - x\right) < \left\|x_n - x\right\| < \varepsilon. \tag{37}$$

It can be seen that, for all $n \ge n_0$, $x_n \to x$ and $(x_n - x) \in \ell_{\rho\Delta}(u,v,p)$.

From the linearity of the sequence space $\ell_{\rho\Delta}(u,v,p)$, we can write that

$$x = x_n - \left(x_n - x\right) \in \ell_{\rho\Delta}(u,v,p). \tag{38}$$

Hence, the sequence space $\ell_{\rho\Delta}(u,v,p)$ is a Banach space with respect to the Luxemburg norm. This completes the proof of the theorem. □

Lemma 13. *Let $x \in \ell_{\rho\Delta}(u,v,p)$ and $\{x_n\} \subseteq \ell_{\rho\Delta}(u,v,p)$. If $\rho(x_n) \to \rho(x)$ as $n \to \infty$ and $x_n(i) \to x(i)$ as $n \to \infty$ for all $i \in \mathbb{N}$, then $x_n \to x$ as $n \to \infty$.*

Now, we shall give the main theorems of this paper involving the geometric properties of the space $\ell_{\rho\Delta}(u,v,p)$.

Theorem 14. *The space $\ell_{\rho\Delta}(u,v,p)$ has the Kadec-Klee property.*

Proof. Let $x \in S(\ell_{\rho\Delta}(u,v,p))$ and $x_n(i) \subseteq \ell_{\rho\Delta}(u,v,p)$ such that $\|x_n(i)\| \to 1$ and $x_n(i) \xrightarrow{w} x(i)$ as $n \to \infty$. From Lemma 8(iii), we get $\rho(x) = 1$. So, from Lemma 9(i), it follows that $\rho(x_n) \to \rho(x)$ as $n \to \infty$. Since mapping $\pi_i : \ell_{\rho\Delta}(u,v,p) \to \mathbb{R}$ defined by $\pi_i(y) = y_i$ is a continuous linear functional on $\ell_{\rho\Delta}(u,v,p)$. It follows that $x_n(i) \to x(i)$ as $n \to \infty$ for all $i \in \mathbb{N}$. So from Lemma 13, $x_n \to x$ as $n \to \infty$. □

Theorem 15. *The space $\ell_{\rho\Delta}(u,v,p)$ is k-NUC for any integer $k \ge 2$ where $1 < p < \infty$.*

Proof. Let $\varepsilon > 0$ and $\{x_n\} \subseteq B(\ell_{\rho\Delta}(u,v,p))$ with $\operatorname{sep}(x_n) \ge \varepsilon$. For each $m \in \mathbb{N}$ let

$$x_n^m = \left(\underbrace{0,0,\ldots,0}_{m-1}, x_n(m), x_n(m+1), \ldots\right), \tag{39}$$

since for each $i \in \mathbb{N}(x_n(i))_{n=1}^{\infty}$ is bounded; by using the diagonal method, we have that, for each $m \in \mathbb{N}$, we can find that a subsequence (x_{n_j}) of (x_n) such that (x_{n_j}) converges for each $i \in \mathbb{N}$, $1 \le i \le m$. Therefore, there exists an increasing sequence of positive integer (t_m) such that $\operatorname{sep}((x_{n_j}^m)_{j>t_m}) \ge \varepsilon$. Hence, there is a sequence of positive integer $(r_m)_{m=1}^{\infty}$ with $r_1 < r_2 < r_3 < \cdots$ such that $\|x_{r_m}^m\| \ge \varepsilon/2$ for all $m \in \mathbb{N}$. Then, by Lemma 11, we may assume that there exists $\mu > 0$ such that

$$\rho\left(x_{r_m}^m\right) \ge \mu \quad \forall m \in \mathbb{N}. \tag{40}$$

Let $\alpha > 0$ and $1 < \alpha < \lim_{n\to\infty} \inf p_n$. For fixed integer $k \ge 2$, let $\varepsilon_1 = ((k^{\alpha-1}-1)/(k-1)k^{\alpha})(\mu/2)$. Then, by Lemma 10, there is a $\delta > 0$ such that

$$\left|\rho\left(u+v\right) - \rho\left(u\right)\right| < \varepsilon_1, \tag{41}$$

whenever $\rho(u) \le 1$ and $\rho(v) \le \delta$.

Since, by Lemma 8(i), $\rho(x_n) \le 1$ for all $n \in \mathbb{N}$, there exist positive integers m_i $(i = 1, 2, \dots, k-1)$ with $m_1 < m_2 < \cdots < m_{k-1}$ such that $\rho(x_i^{m_i}) \le \delta$ and $\alpha \le p_j$ for all $j \ge m_{k-1}$. Define $m_k = m_{k-1} + 1$. From (40), we have $\rho(x_{r_{m_k}}^{m_k}) \ge \mu$. Let $s_i = i$ for $1 \le i \le k-1$ and $s_k = r_{m_k}$. Then, in virtue of (40), (41), and convexity of function $f_i(u) = |u|^{p_i}$ $(i \in \mathbb{N})$, we have

$$\begin{aligned}
&\rho\left(\frac{x_{s_1} + x_{s_2} + \cdots + x_{s_k}}{k}\right) \\
&= \sum_{n=1}^{\infty}\left(\sum_{i=1}^{n-1} u_n \Delta v_i \left|\frac{x_{s_1}(i) + x_{s_2}(i) + \cdots + x_{s_k}(i)}{k}\right| + u_n v_n \left|\frac{x_{s_1}(n) + x_{s_2}(n) + \cdots + x_{s_k}(n)}{k}\right|\right)^{p_n} \\
&= \sum_{n=1}^{m_1}\left(\sum_{i=1}^{n-1} u_n \Delta v_i \left|\frac{x_{s_1}(i) + x_{s_2}(i) + \cdots + x_{s_k}(i)}{k}\right| + u_n v_n \left|\frac{x_{s_1}(n) + x_{s_2}(n) + \cdots + x_{s_k}(n)}{k}\right|\right)^{p_n} \\
&\quad + \sum_{n=m_1+1}^{\infty}\left(\sum_{i=1}^{n-1} u_n \Delta v_i \left|\frac{x_{s_1}(i) + x_{s_2}(i) + \cdots + x_{s_k}(i)}{k}\right| + u_n v_n \left|\frac{x_{s_1}(n) + x_{s_2}(n) + \cdots + x_{s_k}(n)}{k}\right|\right)^{p_n} \\
&\le \sum_{n=1}^{m_1}\frac{1}{k}\sum_{j=1}^{k}\left(\sum_{i=1}^{n-1} u_n \Delta v_i \left|x_{s_j}(i)\right| + u_n v_n \left|x_{s_j}(n)\right|\right)^{p_n} \\
&\quad + \sum_{n=m_1+1}^{\infty}\left(\sum_{i=1}^{n-1} u_n \Delta v_i \left|\frac{x_{s_2}(i) + x_{s_3}(i) + \cdots + x_{s_k}(i)}{k}\right| + u_n v_n \left|\frac{x_{s_2}(n) + x_{s_3}(n) + \cdots + x_{s_k}(n)}{k}\right|\right)^{p_n} \\
&\quad + \varepsilon_1 \\
&= \sum_{n=1}^{m_1}\frac{1}{k}\sum_{j=1}^{k}\left(\sum_{i=1}^{n-1} u_n \Delta v_i \left|x_{s_j}(i)\right| + u_n v_n \left|x_{s_j}(n)\right|\right)^{p_n} \\
&\quad + \sum_{n=m_1+1}^{m_2}\frac{1}{k}\sum_{j=2}^{k}\left(\sum_{i=1}^{n-1} u_n \Delta v_i \left|x_{s_j}(i)\right| + u_n v_n \left|x_{s_j}(n)\right|\right)^{p_n} \\
&\quad + \sum_{n=m_2+1}^{\infty}\left(\sum_{i=1}^{n-1} u_n \Delta v_i \left|\frac{x_{s_3}(i) + x_{s_4}(i) + \cdots + x_{s_k}(i)}{k}\right| + u_n v_n \left|\frac{x_{s_3}(n) + x_{s_4}(n) + \cdots + x_{s_k}(n)}{k}\right|\right)^{p_n} \\
&\quad + 2\varepsilon_1 \\
&\le \sum_{n=1}^{m_1}\frac{1}{k}\sum_{j=1}^{k}\left(\sum_{i=1}^{n-1} u_n \Delta v_i \left|x_{s_j}(i)\right| + u_n v_n \left|x_{s_j}(n)\right|\right)^{p_n} \\
&\quad + \sum_{n=m_1+1}^{m_2}\frac{1}{k}\sum_{j=2}^{k}\left(\sum_{i=1}^{n-1} u_n \Delta v_i \left|x_{s_j}(i)\right| + u_n v_n \left|x_{s_j}(n)\right|\right)^{p_n} \\
&\quad + \sum_{n=m_2+1}^{m_3}\frac{1}{k}\sum_{j=3}^{k}\left(\sum_{i=1}^{n-1} u_n \Delta v_i \left|x_{s_j}(i)\right| + u_n v_n \left|x_{s_j}(n)\right|\right)^{p_n} \\
&\quad + \cdots \\
&\quad + \sum_{n=m_{k-2}+1}^{m_{k-1}}\frac{1}{k}\sum_{j=k-1}^{k}\left(\sum_{i=1}^{n-1} u_n \Delta v_i \left|x_{s_j}(i)\right| + u_n v_n \left|x_{s_j}(n)\right|\right)^{p_n} \\
&\quad + \sum_{n=m_{k-1}+1}^{\infty}\left(\sum_{i=1}^{n-1} u_n \Delta v_i \left|\frac{x_{s_k}(i)}{k}\right| + u_n v_n \left|\frac{x_{s_k}(n)}{n}\right|\right)^{p_n} \\
&\quad + (k-1)\,\varepsilon_1 \\
&\le \frac{\rho\left(x_{s_1}\right) + \rho\left(x_{s_2}\right) + \cdots + \rho\left(x_{s_k}\right)}{k} \\
&\quad + \frac{1}{k}\sum_{n=1}^{m_k}\left(\sum_{i=1}^{n-1} u_n \Delta v_i \left|x_{s_j}(i)\right| + u_n v_n \left|x_{s_j}(n)\right|\right)^{p_n} \\
&\quad + \sum_{n=m_{k-1}+1}^{\infty}\left(\sum_{i=1}^{n-1} u_n \Delta v_i \left|\frac{x_{s_k}(i)}{k}\right| + u_n v_n \left|\frac{x_{s_k}(n)}{n}\right|\right)^{p_n} \\
&\quad + (k-1)\,\varepsilon_1 \\
&\le \frac{k-1}{k} + \frac{1}{k}\sum_{n=1}^{m_k}\left(\sum_{i=1}^{n-1} u_n \Delta v_i \left|x_{s_j}(i)\right| + u_n v_n \left|x_{s_j}(n)\right|\right)^{p_n} \\
&\quad + \frac{1}{k^{\alpha}}\sum_{n=m_k+1}^{\infty}\left(\sum_{i=1}^{n-1} u_n \Delta v_i \left|x_{s_j}(i)\right| + u_n v_n \left|x_{s_j}(n)\right|\right)^{p_n} \\
&\quad + (k-1)\,\varepsilon_1
\end{aligned}$$

$$\leq 1-\frac{1}{k}+\frac{1}{k}$$
$$\times\left[1-\sum_{n=m_k+1}^{\infty}\left(\sum_{i=1}^{n-1}u_n\Delta v_i\left|x_{s_j}(i)\right|+u_nv_n\left|x_{s_j}(n)\right|\right)^{p_n}\right]$$
$$+\frac{1}{k^{\alpha}}\sum_{n=m_k+1}^{\infty}\left(\sum_{i=1}^{n-1}u_n\Delta v_i\left|x_{s_j}(i)\right|+u_nv_n\left|x_{s_j}(n)\right|\right)^{p_n}$$
$$+(k-1)\varepsilon_1$$
$$=1-\frac{k^{\alpha}-1}{k^{\alpha}}\sum_{n=m_k+1}^{\infty}\left(\sum_{i=1}^{n-1}u_n\Delta v_i\left|x_{s_j}(i)\right|+u_nv_n\left|x_{s_j}(n)\right|\right)^{p_n}$$
$$+(k-1)\varepsilon_1$$
$$\leq 1+(k-1)\varepsilon_1-\frac{k^{\alpha-1}-1}{k^{\alpha}}\mu=1-\frac{k^{\alpha-1}-1}{k^{\alpha}}\frac{\mu}{2}. \tag{42}$$

By Lemma 13, there exists $\gamma>0$ such that

$$\left\|\frac{x_{s_1}+x_{s_2}+\cdots+x_{s_k}}{k}\right\|<1-\gamma. \tag{43}$$

Therefore, $\ell_{\rho\Delta}(u,v,p)$ is k-NUC. □

Theorem 16. *For any $1<p<\infty$, the space $\ell_{\rho\Delta}(u,v,p)$ has the uniform Opial property.*

Proof. Take any $\varepsilon>0$ and $x\in\ell_{\rho\Delta}(u,v,p)$ with $\|x\|\geq\varepsilon$. Let (x_n) be weakly null sequence in $S(\ell_{\rho\Delta}(u,v,p))$. By $\lim_{n\to\infty}\sup p_r<\infty$, that is, $\rho\in\delta_2^s$ by Lemma 13, there exists $\delta\in(0,1)$ independent of x such that $\rho(x)>\delta$. Also by $\rho\in\delta_2^s$ and Lemma 10 asserts that there exists $\delta_1\in(\delta,1)$ such that

$$\left|\rho(y+z)-\rho(y)\right|<\frac{\delta}{4}, \tag{44}$$

whenever $\rho(y)\leq 1$ and $\rho(z)\leq\delta_1$.

Choose $k_0\in\mathbb{N}$ such that

$$\sum_{k=k_0+1}^{\infty}\left(\sum_{i=1}^{k-1}u_k\Delta v_i\left|x_i\right|+u_kv_k\left|x_k\right|\right)^{p_k}<\frac{\delta_1}{4} \tag{45}$$

so, we have

$$\begin{aligned}\delta&<\sum_{k=1}^{k_0}\left(\sum_{i=1}^{k-1}u_k\Delta v_i\left|x_i\right|+u_kv_k\left|x_k\right|\right)^{p_k}\\&\quad+\sum_{k=k_0+1}^{\infty}\left(\sum_{i=1}^{k-1}u_k\Delta v_i\left|x_i\right|+u_kv_k\left|x_k\right|\right)^{p_k}\\&\leq\sum_{k=1}^{k_0}\left(\sum_{i=1}^{k-1}u_k\Delta v_i\left|x_i\right|+u_kv_k\left|x_k\right|\right)^{p_k}+\frac{\delta_1}{4}\end{aligned} \tag{46}$$

which implies that

$$\begin{aligned}\sum_{k=1}^{k_0}\left(\sum_{i=1}^{k-1}u_k\Delta v_i\left|x_i\right|+u_kv_k\left|x_k\right|\right)^{p_k}&>\delta-\frac{\delta_1}{4}\\&>\delta-\frac{\delta}{4}\\&=\frac{3\delta}{4}.\end{aligned} \tag{47}$$

Since $x_n\xrightarrow{w}0$, then there exists $n_0\in\mathbb{N}$ such that

$$\frac{3\delta}{4}\leq\sum_{k=1}^{k_0}\left(\sum_{i=1}^{k-1}u_k\Delta v_i\left|x_n(i)+x(i)\right|+u_kv_k\left|x_n(k)+x(k)\right|\right)^{p_k}, \tag{48}$$

for all $n>n_0$, since weak convergence implies coordinatewise convergence. We denote

$$\begin{aligned}x_{n|k_0}&=\left(x_n(1),x_n(2),\ldots,x_n(k_0),0,0\ldots\right)\\x_{n|\mathbb{N}-k_0}&=\left(0,0\ldots,0,x_n(k_0+1),x_n(k_0+2),\ldots\right).\end{aligned} \tag{49}$$

Again $x_n\xrightarrow{w}0$, and then there exists $n_1\in\mathbb{N}$ such that

$$\left\|x_{n|k_0}\right\|<1-\left(1-\frac{\delta}{4}\right)^M, \tag{50}$$

for all $n>n_1$.

Hence, by the triangle inequality of the norm, we get

$$\left\|x_{n|\mathbb{N}-k_0}\right\|>\left(1-\frac{\delta}{4}\right)^M. \tag{51}$$

It follows by the definition of norm that we have

$$\begin{aligned}1&<\rho\left(\frac{x_{n|\mathbb{N}-k_0}}{(1-\delta/4)^M}\right)\\&=\sum_{k=k_0+1}^{\infty}\left(\sum_{i=1}^{k-1}u_k\Delta v_i\left|\frac{x_n(i)}{(1-\delta/4)^M}\right|+u_kv_k\left|\frac{x_n(k)}{(1-\delta/4)^M}\right|\right)^{p_k}\\&\leq\frac{1}{(1-\delta/4)^M}\sum_{k=k_0+1}^{\infty}\left(\sum_{i=1}^{k-1}u_k\Delta v_i\left|x_n(i)\right|+u_kv_k\left|x_n(k)\right|\right)^{p_k},\end{aligned} \tag{52}$$

which implies that

$$\sum_{k=k_0+1}^{\infty}\left(\sum_{i=1}^{k-1}u_k\Delta v_i\left|x_n(i)\right|+u_kv_k\left|x_n(k)\right|\right)^{p_k}>\left(1-\frac{\delta}{4}\right)^{M^2}, \tag{53}$$

for all $n>n_1$.

By inequalities (44), (45), and (48), (53) yields for any $n > n_1$ that

$$\begin{aligned}\rho(x_n + x) &= \sum_{k=1}^{k_0}\left(\sum_{i=1}^{k-1} u_k \Delta v_i \left|x_n(i) + x(i)\right| + u_k v_k \left|x_n(k) + x(k)\right|\right)^{p_k} \\ &\quad + \sum_{k=k_0+1}^{\infty}\left(\sum_{i=1}^{k-1} u_k \Delta v_i \left|x_n(i) + x(i)\right| + u_k v_k \left|x_n(k) + x(k)\right|\right)^{p_k} \\ &\geq \frac{3\delta}{4} + \sum_{k=k_0+1}^{\infty}\left(\sum_{i=1}^{k-1} u_k \Delta v_i \left|x_n(i)\right| + u_k v_k \left|x_n(k)\right|\right)^{p_k} \\ &\geq \frac{3\delta}{4} + \left(1 - \frac{\delta}{4}\right)^{M^2} - \frac{\delta}{4}.\end{aligned} \tag{54}$$

Since $\rho \in \delta_2^s$ and by Lemma 4, there exists τ depending on δ only such that $\|x_n + x\| \geq 1 + \tau$, which implies that

$$\lim_{n\to\infty} \inf \left\|x_n + x\right\| \geq 1 + \tau. \tag{55}$$

Therefore, $\ell_{\rho\Delta}(u, v, p)$ has the uniform Opial property. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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