Hindawi Publishing Corporation
Journal of Function Spaces
Volume 2014, Article ID 727341, 7 pages
http://dx.doi.org/10.1155/2014/727341

Research Article

Wick Analysis for Bernoulli Noise Functionals

Caishi Wang and Jihong Zhang

School of Mathematics and Statistics, Northwest Normal University, Lanzhou, Gansu 730070, China

Correspondence should be addressed to Caishi Wang; cswangnwnu@163.com

Received 4 December 2013; Revised 10 February 2014; Accepted 10 February 2014; Published 12 March 2014

Academic Editor: Dachun Yang

Copyright © 2014 C. Wang and J. Zhang. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

A Gel'fand triple $S(\Omega) \subset L^2(\Omega) \subset S^*(\Omega)$ is constructed of functionals of Z, where $Z = (Z_n)_{n\in\mathbb{N}}$ is an appropriate Bernoulli noise on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. Characterizations are given to both $S(\Omega)$ and $S^*(\Omega)$. It is also shown that a Wick-type product can be defined on $S^*(\Omega)$ and moreover $S^*(\Omega)$ forms a commutative algebra with the product. Finally, a transform named S-transform is defined on $S^*(\Omega)$ and its continuity as well as other properties are examined.

1. Introduction

In the constructive quantum field theory, there have been encountered infinite quantities, which originated (from a mathematical point of view) from the product of generalized functions (see [1, 2]). To obtain useful information out of these infinite quantities, Wick [3] first introduced the now so-called Wick renormalization, which was actually a formal manipulation. Hida and Ikeda [4] first rigorously defined the Wick renormalization on functionals of Brownian motion and called it Wick product instead. Meyer and Yan [5] extended the Wick product to cover generalized functionals of Gaussian white noises (also known as Hida distributions). Now the Wick product is widely used as a tool in stochastic differential equations, stochastic partial differential equations, stochastic quantization [2], and many other fields.

Bernoulli noise functionals (also known as functionals of Bernoulli noises) have attracted much attention in recent years. In 2001 Émery [6] discussed the chaotic representation property of a class of discrete-time stochastic processes including Bernoulli noises. In 2008 Privault [7] surveyed his discrete-time chaotic calculus. In 2010 Nourdin [8] considered Rademacher functionals (a special case of Bernoulli noise functionals) by using Stein's method. The same year, Wang et al. [9] introduced a notion of quantum Bernoulli noises and defined corresponding quantum stochastic integrals, which are actually about operator processes acting on Bernoulli noise functionals. Recently Wang et al. [10] have presented an alternative approach to Privault's discrete-time chaotic calculus. There are other works devoted to development of a theory of quantum Bernoulli noises (see, for instance, [11, 12]).

As can be seen, most of the existing work concerning the Wick product has been done within generalized functionals of Gaussian white noises. In this paper, we aim to construct generalized functionals of Bernoulli noises and develop a Wick-type calculus on them.

The paper is organized as follows. In Section 2, we fix some necessary notation and recall main notions and facts about Bernoulli noises for our later use.

Sections 3, 4, and 5 are our main work. In Section 3, we first introduce a weight function on the finite power set of $\mathbb{N}$, where $\mathbb{N}$ denotes the set of all nonnegative integers. And then, by using the weight function and the full Wiener integral operator $\mathbb{J}$, we construct a nuclear space $S(\Omega)$ of functionals of Z, where $Z = (Z_n)_{n\in\mathbb{N}}$ is the Bernoulli noise fixed in Section 2. With $S(\Omega)$ as the testing functional space, we get the generalized functional space $S^*(\Omega)$ by taking the dual. We also give characterizations to both $S(\Omega)$ and $S^*(\Omega)$. In Section 4, we define the Wick product on the generalized functional space $S^*(\Omega)$ and prove that $S^*(\Omega)$ forms a commutative algebra with the Wick product. Finally, in Section 5, we introduce a transform, called S-transform, on the generalized functional space $S^*(\Omega)$ by using the coherent states. We examine its continuity as well as other properties.

2. Preliminaries

Throughout this paper, $\mathbb{N}$ designates the set of all nonnegative integers. We denote by Γ the finite power set of $\mathbb{N}$, namely,

$$\Gamma = \{\sigma \mid \sigma \subset \mathbb{N}, \sharp\sigma < \infty\}, \tag{1}$$

where $\sharp\sigma$ means the cardinality of σ as a set. We use $l^2(\Gamma)$ to mean the usual Hilbert space of square summable real-valued functions on Γ.

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space and $Z = (Z_n)_{n\in\mathbb{N}}$ an independent sequence of real-valued random variables on $(\Omega, \mathscr{F}, \mathbb{P})$ satisfying that

$$\mathbb{P}\{Z_n = \varepsilon_n\} = \theta_n, \quad \mathbb{P}\left\{Z_n = \frac{-1}{\varepsilon_n}\right\} = 1 - \theta_n, \quad n \in \mathbb{N} \tag{2}$$

with $\varepsilon_n = \sqrt{(1-\theta_n)/\theta_n}$, $0 < \theta_n < 1$, and $\mathscr{F} = \sigma(Z_n; n \in \mathbb{N})$, namely, Z generates $\mathscr{F}$. We note that such a sequence of random variables does exist. In fact, Z is a discrete-time Bernoulli stochastic process and if we put

$$S_n = \sum_{k=1}^{n} Z_k, \quad n \in \mathbb{N}, \tag{3}$$

then $(S_n)_{n\in\mathbb{N}}$ is a martingale. Hence we may call Z a Bernoulli noise.

In the following, we set $\mathscr{F}_{-1} = \{\emptyset, \Omega\}$, $\mathscr{F}_n = \sigma(Z_k; 0 \le k \le n)$, $n \in \mathbb{N}$. In this way, the sequence $(\mathscr{F}_n)_{n\ge -1}$ forms a filtration of σ-fields over $(\Omega, \mathscr{F}, \mathbb{P})$. As usual, we use

$$L^2(\Omega) \equiv L^2(\Omega, \mathscr{F}, \mathbb{P}) \tag{4}$$

to mean the usual space of square integrable real-valued random variables on $(\Omega, \mathscr{F}, \mathbb{P})$. We denote by $\|\cdot\|$ the norm of $L^2(\Omega)$.

The next lemma [9] shows that Z has the chaotic representation property.

Lemma 1. *The family $\{Z_\sigma \mid \sigma \in \Gamma\}$ forms a countable orthonormal basis of $L^2(\Omega)$, where*

$$Z_\sigma = \prod_{j\in\sigma} Z_j, \quad \sigma \ne \emptyset, \ \sigma \in \Gamma, \tag{5}$$

and $Z_\emptyset = 1$.

Lemma 2. *There exists a unique isometric isomorphism $\mathbb{J} : l^2(\Gamma) \mapsto L^2(\Omega)$ such that*

$$\mathbb{J}(f) = \sum_{\sigma\in\Gamma} f(\sigma) Z_\sigma, \quad f \in l^2(\Gamma), \tag{6}$$

where the series on the right-hand side converges in the norm $\|\cdot\|$.

The isometric isomorphism $\mathbb{J}$ is known as the full Wiener integral operator [10].

Lemma 3. *For each $k \ge 0$, there exists a bounded operator ∂_k on $L^2(\Omega)$ such that*

$$\partial_k Z_\sigma = \mathbf{1}_\sigma(k) Z_{\sigma\setminus k}, \quad \sigma \in \Gamma, \tag{7}$$

where $\sigma \setminus k = \sigma \setminus \{k\}$ and $\mathbf{1}_\sigma(k)$ the indicator of σ as a subset of $\mathbb{N}$.

Lemma 4. *Let $k \ge 0$; then ∂_k^*, the adjoint operator of ∂_k, has following property:*

$$\partial_k^* Z_\sigma = [1 - \mathbf{1}_\sigma(k)] Z_{\sigma\cup k}, \quad \sigma \in \Gamma, \tag{8}$$

where $\sigma \cup k = \sigma \cup \{k\}$.

The operator ∂_k and its adjoint operator ∂_k^* are referred to as the annihilation operator and creation operator [10], respectively.

3. The Framework of Bernoulli Noise Analysis

In the present section, we mainly construct a Gel'fand triple of functionals of the Bernoulli noise Z, which serves as the framework where we work.

To start with, we first introduce a weight function on Γ, which will play a key role in stating and proving our main results.

Proposition 5. *Define an $\mathbb{N}$-valued function $\sigma \mapsto \lambda_\sigma$ on Γ as*

$$\lambda_\sigma = \prod_{k\in\sigma}(k+1), \quad \sigma \ne \emptyset, \ \sigma \in \Gamma \tag{9}$$

and $\lambda_\emptyset = 1$. Then for each $p > 1$, it holds that

$$\sum_{\sigma\in\Gamma} \lambda_\sigma^{-p} < \infty. \tag{10}$$

Proof. For $n \in \mathbb{N}$, let Δ_n be the subset of $\mathbb{N}^n$ defined by

$$\Delta_n = \left\{(k_1, k_2, \ldots, k_n) \in \mathbb{N}^n \mid k_i \ne k_j, \text{for } i \ne j, 1 \le i, j \le n\right\}. \tag{11}$$

Writing $\vec{k} = (k_1, k_2, \ldots, k_n)$, we obtain

$$\begin{aligned}
&\sum_{\sigma\in\Gamma} \lambda_\sigma^{-p} \\
&\quad = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\vec{k}\in\Delta_n} (k_1+1)^{-p}(k_2+1)^{-p}\cdots(k_n+1)^{-p} \\
&\quad \le 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\vec{k}\in\mathbb{N}^n} (k_1+1)^{-p}(k_2+1)^{-p}\cdots(k_n+1)^{-p} \\
&\quad = \exp\left[\sum_{k\in\mathbb{N}} (k+1)^{-p}\right].
\end{aligned} \tag{12}$$

This, together with the fact $\sum_{k\in\mathbb{N}}(k+1)^{-p} < \infty$, gives (10). □

Remark 6. For $\sigma \in \Gamma$, we can verify that $\lambda_\sigma \ge (\sharp\sigma)!$, where $\sharp\sigma$ denotes the cardinality of σ.

For $p \ge 0$, let $\mathscr{S}_p(\Gamma)$ be the collection of real-valued functions f on Γ satisfying

$$\sum_{\sigma\in\Gamma} \lambda_\sigma^{2p} |f(\sigma)|^2 < \infty. \tag{13}$$

Then $\mathcal{S}_p(\Gamma)$ forms a Hilbert space with the usual linear operations and the inner product $\langle\cdot,\cdot\rangle_p$ defined by

$$\langle f,g\rangle_p=\sum_{\sigma\in\Gamma}\lambda_\sigma^{2p}f(\sigma)g(\sigma),\quad f,g\in\mathcal{S}_p(\Gamma). \tag{14}$$

We use $|\cdot|_p$ to denote the norm generated by $\langle\cdot,\cdot\rangle_p$. Note that $\mathcal{S}_p(\Gamma)\subset l^2(\Gamma)$ for any $p\geq 0$ and $\mathcal{S}_0(\Gamma)=l^2(\Gamma)$. In the sequel, we set

$$\mathcal{S}_+(\Gamma)=\bigcap_{p=0}^{\infty}\mathcal{S}_p(\Gamma). \tag{15}$$

Recall that the full Wiener integral operator $\mathbb{J}:l^2(\Gamma)\mapsto L^2(\Omega)$ is an isometric isomorphism. This, together with the fact $\mathcal{S}_p(\Gamma)\subset l^2(\Gamma)$, $p\geq 0$, suggests the next definition.

Definition 7. For $p\geq 0$, one defines $S_p(\Omega)$ as the image of $\mathcal{S}_p(\Gamma)$ under $\mathbb{J}$, namely,

$$S_p(\Omega)=\left\{\varphi\in L^2(\Omega)\mid\varphi=\mathbb{J}(f),f\in\mathcal{S}_p(\Gamma)\right\} \tag{16}$$

and endows it with the inner product $\langle\langle\cdot,\cdot\rangle\rangle_p$ given by

$$\langle\langle\xi,\eta\rangle\rangle_p=\left\langle\mathbb{J}^{-1}(\xi),\mathbb{J}^{-1}(\eta)\right\rangle_p,\quad\xi,\eta\in S_p(\Omega). \tag{17}$$

One uses $\|\cdot\|_p$ to denote the norm generated by $\langle\langle\xi,\eta\rangle\rangle_p$.

Clearly, for each $p\geq 0$, $S_p(\Omega)$ is a Hilbert space and the restriction of $\mathbb{J}$ to $\mathcal{S}_p(\Gamma)$ is a Hilbert isomorphism from $\mathcal{S}_p(\Gamma)$ to $S_p(\Omega)$. Additionally, if $0\leq p<q$, then one has

$$S_q(\Omega)\subset S_p(\Omega)\subset S_0(\Omega)=L^2(\Omega). \tag{18}$$

Proposition 8. *For $p\geq 0$, the system $\{\lambda_\sigma^{-p}Z_\sigma\}_{\sigma\in\Gamma}$ forms an orthonormal basis of the Hilbert space $S_p(\Omega)$ and moreover,*

$$\|\xi\|_p^2=\sum_{\sigma\in\Gamma}\lambda_\sigma^{2p}|\mathbb{E}(Z_\sigma\xi)|^2,\quad\xi\in S_p(\Omega). \tag{19}$$

Proof. It can be verified that $\{\lambda_\sigma^{-p}\pi_\sigma\}_{\sigma\in\Gamma}$ forms an orthonormal basis of $\mathcal{S}_p(\Gamma)$, where

$$\pi_\sigma(\tau)=\begin{cases}1, & \tau=\sigma,\ \tau\in\Gamma;\\ 0, & \tau\neq\sigma,\ \tau\in\Gamma.\end{cases} \tag{20}$$

Clearly $\lambda_\sigma^{-p}Z_\sigma=\mathbb{J}(\lambda_\sigma^{-p}\pi_\sigma)$, $\sigma\in\Gamma$. Thus $\{\lambda_\sigma^{-p}Z_\sigma\}_{\sigma\in\Gamma}$ forms an orthonormal basis of $S_p(\Omega)$. Now for $\xi\in S_p(\Omega)$, we have

$$\langle\langle Z_\sigma,\xi\rangle\rangle_p=\lambda_\sigma^{2p}\mathbb{E}(Z_\sigma\xi),\quad\sigma\in\Gamma. \tag{21}$$

This together with the well-known Parseval's equality yields (19). □

Definition 9. One sets

$$S(\Omega)=\bigcap_{p=0}^{\infty}S_p(\Omega) \tag{22}$$

and endows it with the topology generated by the norm sequence $\{\|\cdot\|_p\}_{p\geq 0}$. One calls $S(\Omega)$ the testing functional space and its elements testing functionals.

Note that, for each $0\leq p\leq q$, $S_p(\Omega)$ is just the completion of $S(\Omega)$ with respect to norm $\|\cdot\|_p$. Thus, with inner product norms $\|\cdot\|_p$, $p\geq 0$, $S(\Omega)$ forms a countably-Hilbert space. It is also worthwhile to note that $\|\cdot\|_p\leq\|\cdot\|_q$ on $S(\Omega)$ whenever $0\leq p\leq q$ and, in particular, the inclusion mapping from $S(\Omega)$ to $L^2(\Omega)$ is continuous.

Proposition 10. *The testing functional space $S(\Omega)$ is a nuclear space, namely, for any $p\geq 0$, there exists $q\geq p$ such that the inclusion mapping $i_{pq}:\xi\mapsto\xi$ from $S_q(\Omega)$ to $S_p(\Omega)$ is a Hilbert-Schmidt operator.*

Proof. Let $p\geq 0$. Then there exists $q>p$ such that $2(q-p)>1$. By Proposition 8, $\{\lambda_\sigma^{-q}Z_\sigma\}_{\sigma\in\Gamma}$ forms an orthonormal basis of $S_q(\Omega)$. Thus, we have

$$\|i_{pq}\|_{\mathrm{HS}}^2=\sum_{\sigma\in\Gamma}\left\|i_{pq}(\lambda_\sigma^{-q}Z_\sigma)\right\|_p^2=\sum_{\sigma\in\Gamma}\lambda_\sigma^{-2(q-p)}<\infty, \tag{23}$$

where $\|\cdot\|_{\mathrm{HS}}$ denotes the Hilbert-Schmidt norm of an operator. Therefore the inclusion mapping $i_{pq}:\xi\mapsto\xi$ from $S_q(\Omega)$ to $S_p(\Omega)$ is a Hilbert-Schmidt operator. □

Proposition 11. *Let $\xi\in L^2(\Omega)$ and $f\in l^2(\Gamma)$ with $\xi=\mathbb{J}(f)$. Then $\xi\in S(\Omega)$ if and only if $f\in\mathcal{S}_+(\Gamma)$. In that case*

$$\xi=\sum_{\sigma\in\Gamma}f(\sigma)Z_\sigma. \tag{24}$$

Proof. The first part is easy to verify. We need only to show that for each $p\geq 0$, the series in (24) converges to ξ in norm $\|\cdot\|_p$. To do so, we take $p\geq 0$. Since $\{\lambda_\sigma^{-p}Z_\sigma\}_{\sigma\in\Gamma}$ forms an orthonormal basis of $S_p(\Omega)$, we have

$$\xi=\sum_{\sigma\in\Gamma}\langle\langle\lambda_\sigma^{-p}Z_\sigma,\xi\rangle\rangle_p\lambda_\sigma^{-p}Z_\sigma \tag{25}$$

in norm $\|\cdot\|_p$. Note that the series on the right-hand side also converges to ξ in the norm of $L^2(\Omega)$ since $\|\cdot\|_{L^2(\Omega)}=\|\cdot\|\leq\|\cdot\|_p$. On the other hand, because of $\xi=\mathbb{J}(f)$, we have

$$\xi=\sum_{\sigma\in\Gamma}f(\sigma)Z_\sigma\ \text{in the norm of}\ L^2(\Omega). \tag{26}$$

Thus $f(\sigma)=\langle\langle\lambda_\sigma^{-p}Z_\sigma,\xi\rangle\rangle_p\lambda^{-p}$, $\sigma\in\Gamma$, which together with (25) implies

$$\xi=\sum_{\sigma\in\Gamma}f(\sigma)Z_\sigma\ \text{in norm}\ \|\cdot\|_p. \tag{27}$$

□

Definition 12. One defines $S^*(\Omega)$ as the dual of $S(\Omega)$ and endows it with the strong topology. One calls $S^*(\Omega)$ the generalized functional space and its elements generalized functionals.

Now, by identifying $L^2(\Omega)$ with its dual, we come to a real Gel'fand triple

$$S(\Omega)\subset L^2(\Omega)\subset S^*(\Omega). \tag{28}$$

This justifies naming $S^*(\Omega)$ the generalized functional space. By convention we may call the above triple the framework of Bernoulli noise analysis.

Remark 13. For $p \geq 0$, let $S_p^*(\Omega)$ be the dual of $S_p(\Omega)$. Then $S_0^*(\Omega) \subset \cdots \subset S_p^*(\Omega) \subset S_{p+1}^*(\Omega) \subset \cdots \subset S^*(\Omega)$ and moreover

$$S^*(\Omega) = \bigcup_{p=0}^{\infty} S_p^*(\Omega). \tag{29}$$

Thus $S^*(\Omega)$ can also be endowed with the inductive limit topology given by space sequence $\{S_p^*(\Omega)\}_{p\geq 1}$, which, however, coincides with the strong topology because $S(\Omega)$ is a countably-Hilbert space.

For $p \geq 0$, we define $\mathcal{S}_{-p}(\Gamma)$ as the vector space of real-valued functions F on Γ satisfying

$$\sum_{\sigma\in\Gamma} \lambda_\sigma^{-2p} |F(\sigma)|^2 < \infty, \tag{30}$$

and endow it with the inner product $\langle \cdot, \cdot \rangle_{-p}$ given by

$$\langle F, G \rangle_{-p} = \sum_{\sigma\in\Gamma} \lambda_\sigma^{-2p} F(\sigma) G(\sigma), \quad F, G \in \mathcal{S}_{-p}(\Gamma). \tag{31}$$

Clearly $\mathcal{S}_{-p}(\Gamma)$ forms a Hilbert space. We set

$$\mathcal{S}_-(\Gamma) = \bigcup_{p=0}^{\infty} \mathcal{S}_{-p}(\Gamma). \tag{32}$$

Note that for $f \in \mathcal{S}_+(\Gamma)$, $F \in \mathcal{S}_-(\Gamma)$, the series $\sum_{\sigma\in\Gamma} F(\sigma) f(\sigma)$ always converges absolutely.

Proposition 14. *For each $\Phi \in S^*(\Omega)$, there exists a unique $F \in \mathcal{S}_-(\Gamma)$ such that*

$$\Phi(\xi) = \sum_{\sigma\in\Gamma} F(\sigma) f(\sigma), \quad \xi \in S(\Omega), \tag{33}$$

where $f = \mathbb{J}^{-1}(\xi) \in \mathcal{S}_+(\Gamma)$. Conversely, given $F \in \mathcal{S}_-(\Gamma)$, a generalized functional $\Phi \in S^*(\Omega)$ is defined by (33).

Proof. The proof of the second part is trivial. We are now turning to the proof of the first part. Let $\Phi \in S^*(\Omega)$. In view of Remark 13, there exists some $p \geq 0$ such that $\Phi \in S_p^*(\Omega)$. According to the Riesz representation theorem, there exists a unique $\zeta \in S_p(\Omega)$ such that

$$\Phi(\xi) = \langle\langle \zeta, \xi \rangle\rangle_p, \quad \xi \in S_p(\Omega). \tag{34}$$

Note that $\{Z_\sigma\}_{\sigma\in\Gamma} \subset S_p(\Omega)$. Thus we can define a function F on Γ as $F(\sigma) = \Phi(Z_\sigma)$, $\sigma \in \Gamma$. Direct computation gives that

$$\sum_{\sigma\in\Gamma} \lambda_\sigma^{-2p} |F(\sigma)|^2 = \sum_{\sigma\in\Gamma} \left| \langle\langle \zeta, \lambda_\sigma^{-p} Z_\sigma \rangle\rangle_p \right|^2 = \|\zeta\|_p^2 < \infty, \tag{35}$$

which shows that $F \in \mathcal{S}_{-p}(\Gamma)$, in particular $F \in \mathcal{S}_-(\Gamma)$. For $\xi \in S(\Omega)$, using the fact that

$$\xi = \sum_{\sigma\in\Gamma} f(\sigma) Z_\sigma \text{ in the topology of } S(\Omega) \tag{36}$$

and taking the continuity of Φ into account, we have

$$\Phi(\xi) = \sum_{\sigma\in\Gamma} f(\sigma) \Phi(Z_\sigma) = \sum_{\sigma\in\Gamma} f(\sigma) F(\sigma) = \sum_{\sigma\in\Gamma} F(\sigma) f(\sigma). \tag{37}$$

Finally we show that such an $F \in \mathcal{S}_-(\Gamma)$ is unique. In fact, if $G \in \mathcal{S}_-(\Gamma)$ satisfies

$$\Phi(\xi) = \sum_{\sigma\in\Gamma} G(\sigma) f(\sigma), \quad \xi \in S(\Omega), \tag{38}$$

with $f = \mathbb{J}^{-1}(\xi) \in \mathcal{S}_+(\Gamma)$, then for each $\sigma \in \Gamma$, by using $Z_\sigma = \mathbb{J}(\pi_\sigma)$, we get $\Phi(Z_\sigma) = G(\sigma)$. Thus $F = G$. □

Definition 15. For $\Phi \in S^*(\Omega)$, one calls the function $F \in \mathcal{S}_-(\Gamma)$ given in Proposition 14 the Guichardet representation of Φ.

4. Wick Product

In this section, we define the Wick product on the generalized functional space $S^*(\Omega)$. We prove that $S^*(\Omega)$ forms a commutative algebra with this product. Some other properties are also shown of the Wick product.

Definition 16. For $F, G \in \mathcal{S}_-(\Gamma)$, one defines a function $F \diamond G$ on Γ as

$$F \diamond G(\sigma) = \sum_{\tau\subset\sigma} F(\tau) G(\sigma \setminus \tau), \quad \sigma \in \Gamma \tag{39}$$

and calls it the Wick product of F and G.

Proposition 17. *Let $p \geq 0$ and $F, G \in \mathcal{S}_{-p}(\Gamma)$. Then $F \diamond G \in \mathcal{S}_{-q}(\Gamma)$ whenever $q > p + 1/2$.*

Proof. For $q > p + 1/2$, it follows from Proposition 5 that

$$\sum_{\sigma\in\Gamma} \lambda_\sigma^{-2(q-p)} < \infty. \tag{40}$$

Thus

$$\begin{aligned}
&\sum_{\sigma\in\Gamma} \lambda_\sigma^{-2q} |F \diamond G(\sigma)|^2 \\
&\quad = \sum_{\sigma\in\Gamma} \lambda_\sigma^{-2q} \left| \sum_{\tau\subset\sigma} F(\tau) G(\sigma \setminus \tau) \right|^2 \\
&\quad = \sum_{\sigma\in\Gamma} \lambda_\sigma^{-2q} \left| \sum_{\tau\subset\sigma} \lambda_\tau^{-p} F(\tau) \lambda_{\sigma\setminus\tau}^{-p} G(\sigma \setminus \tau) \right|^2 \lambda_\sigma^{2p} \\
&\quad \leq \sum_{\sigma\in\Gamma} \lambda_\sigma^{-2(q-p)} \sum_{\tau\subset\sigma} \lambda_\tau^{-2p} |F(\tau)|^2 \sum_{\tau\subset\sigma} \lambda_\tau^{-2p} |G(\tau)|^2 \\
&\quad \leq \|F\|_{-p}^2 \|G\|_{-p}^2 \sum_{\sigma\in\Gamma} \lambda_\sigma^{-2(q-p)} \\
&\quad < \infty,
\end{aligned} \tag{41}$$

which shows that $F \diamond G \in \mathcal{S}_{-q}(\Gamma)$. □

From Proposition 17, we easily see that $F \diamond G$ remains in $\mathcal{S}_-(\Gamma)$ for all $F, G \in \mathcal{S}_-(\Gamma)$; namely, $\mathcal{S}_-(\Gamma)$ is closed under the Wick product.

Note that the generalized functional space $S^*(\Omega)$ can be characterized in terms of $\mathcal{S}_-(\Gamma)$. Thus we naturally come to the following definition.

Definition 18. For generalized functionals $\Phi, \Psi \in S^*(\Omega)$, their Wick product $\Phi \diamond \Psi$ is defined as

$$\Phi \diamond \Psi(\xi) = \sum_{\sigma \in \Gamma} \left(\sum_{\tau \subset \sigma} F(\tau) G(\sigma \setminus \tau) \right) f(\sigma), \quad \xi \in S(\Omega), \tag{42}$$

where $f = \mathbb{J}^{-1}(\xi)$ and $F, G \in \mathcal{S}_-(\Gamma)$ the Guichardet representations of Φ and Ψ, respectively.

Proposition 19. *Let $p \geq 0$ and $\Phi, \Psi \in S_p^*(\Omega)$. Then $\Phi \diamond \Psi \in S_q^*(\Omega)$ whenever $q > p + 1/2$. In particular, $\Phi \diamond \Psi \in S^*(\Omega)$ for all $\Phi, \Psi \in S^*(\Omega)$.*

Proof. Let $F, G \in \mathcal{S}_-(\Gamma)$ be the Guichardet representations of Φ and Ψ, respectively. Then, by Definition 18, we find that $\Phi \diamond \Psi$ just has $F \diamond G$ as its Guichardet representation. Thus by using Proposition 17 we know the claim is true. □

Proposition 20. *Let $\Phi, \Psi \in S^*(\Omega)$. Then*

$$\Phi \diamond \Psi = \Psi \diamond \Phi. \tag{43}$$

Proof. Let $F, G \in \mathcal{S}_-(\Gamma)$ be the Guichardet representations of Φ and Ψ, respectively. Then, by Definition 18, $\Phi \diamond \Psi$ and $\Psi \diamond \Phi$ have $F \diamond G$ and $G \diamond F$ as their Guichardet representations, respectively. Clearly $F \diamond G = G \diamond F$. Thus $\Phi \diamond \Psi = \Psi \diamond \Phi$. □

Proposition 21. *Let $\Phi_1, \Phi_2, \Phi_3 \in S^*(\Omega)$. Then*

$$(\Phi_1 \diamond \Phi_2) \diamond \Phi_3 = \Phi_1 \diamond (\Phi_2 \diamond \Phi_3). \tag{44}$$

Proof. Let $F_i \in \mathcal{S}_-(\Gamma)$ be the Guichardet representation of Φ_i, where $i = 1, 2, 3$. Then, for each $\sigma \in \Gamma$, we can show that

$$\begin{aligned} (F_1 \diamond F_2) \diamond F_3(\sigma) &= \sum_{\vec{\tau} \in \Xi(\sigma)} F_1(\tau_1) F_2(\tau_2) F_3(\tau_3) \\ &= F_1 \diamond (F_2 \diamond F_3)(\sigma), \end{aligned} \tag{45}$$

where $\Xi(\sigma) = \{\vec{\tau} = (\tau_1, \tau_2, \tau_3) \mid (\tau_1, \tau_2, \tau_3) \text{ is a partition of } \sigma\}$. Thus

$$(F_1 \diamond F_2) \diamond F_3 = F_1 \diamond (F_2 \diamond F_3), \tag{46}$$

which implies $(\Phi_1 \diamond \Phi_2) \diamond \Phi_3 = \Phi_1 \diamond (\Phi_2 \diamond \Phi_3)$. □

The following two propositions can be proved similarly and their proofs are omitted.

Proposition 22. *Let $\Phi_1, \Phi_2, \Phi_3 \in S^*(\Omega)$. Then*

$$\Phi_1 \diamond (\Phi_2 + \Phi_3) = \Phi_1 \diamond \Phi_2 + \Phi_1 \diamond \Phi_3. \tag{47}$$

Proposition 23. *Let $\Phi_1, \Phi_2 \in S^*(\Omega)$ and $\alpha \in \mathbb{R}$, the real numbers. Then*

$$\Phi_1 \diamond (\alpha \Phi_2) = \alpha (\Phi_1 \diamond \Phi_2). \tag{48}$$

Remark 24. It follows from the above propositions that $S^*(\Omega)$ forms a commutative algebra with the Wick product.

Let $k \in \mathbb{N}$. Then for $\Phi \in S^*(\Omega)$ with $F \in \mathcal{S}_-(\Gamma)$ being its Guichardet representation, there exists an element, denoted by $\partial_k \Phi$, in $S^*(\Omega)$ such that $\partial_k \Phi$ has the function $\sigma \mapsto F(\sigma \cup k)(1 - \mathbf{1}_\sigma(k))$ being its Guichardet representation. In this way, an operator $\partial_k : S^*(\Omega) \to S^*(\Omega)$ is naturally defined.

The following proposition offers a link between the Wick product and the operator ∂_k.

Proposition 25. *Let $k \in \mathbb{N}$ and $\Phi, \Psi \in S^*(\Omega)$. Then*

$$\partial_k (\Phi \diamond \Psi) = \partial_k \Phi \diamond \Psi + \Phi \diamond \partial_k \Psi. \tag{49}$$

Obviously, for $f \in \mathcal{S}_+(\Gamma)$ and $F \in \mathcal{S}_-(\Gamma)$, their Wick product $f \diamond F$ makes sense and belongs to $\mathcal{S}_-(\Gamma)$. This leads us to the following definition.

Definition 26. For $\phi \in S(\Omega)$ and $\Phi \in S^*(\Omega)$, one defines their Wick product $\phi \diamond \Phi$ as

$$\phi \diamond \Phi(\xi) = \sum_{\sigma \in \Gamma} g_\phi \diamond F(\sigma) f(\sigma), \quad \xi \in S(\Omega), \tag{50}$$

where $f = \mathbb{J}^{-1}(\xi)$, $g_\phi = \mathbb{J}^{-1}(\phi)$ and F the Guichardet representation of Φ.

Proposition 27. *Let $p \geq 0$, $\phi \in S_p(\Omega)$ and $\Phi \in S_p^*(\Omega)$. Then $\phi \diamond \Phi \in S_p^*(\Omega)$.*

Proof. Let $g_\phi = \mathbb{J}^{-1}(\phi)$ and F the Guichardet representation of Φ. Then $g_\phi \in \mathcal{S}_p(\Gamma)$ and $F \in \mathcal{S}_{-p}(\Gamma)$. Thus

$$\sum_{\sigma \in \Gamma} \lambda_\sigma^{-2p} \left| g_\phi \diamond F(\sigma) \right|^2 = \sum_{\sigma \in \Gamma} \lambda_\sigma^{-2p} \left| \sum_{\tau \subset \sigma} g_\phi(\tau) F(\sigma \setminus \tau) \right|^2 < \infty, \tag{51}$$

which implies $g_\phi \diamond F \in \mathcal{S}_{-p}(\Gamma)$. On the other hand, $\phi \diamond \Phi$ has $g_\phi \diamond F$ as its Guichardet representation. Therefore $\phi \diamond \Phi \in S_p^*(\Omega)$. □

5. S-Transform

It is known that the S-transform on generalized functionals of Gaussian white noises plays an important role in Hida's white noise analysis [13]. In the last section, we define the S-transform on our generalized functional space $S^*(\Omega)$ and examine its fundamental properties.

Let $\mathcal{S}_+(\mathbb{N}) = \bigcap_{p \geq 0} \mathcal{S}_p(\mathbb{N})$, where

$$\mathcal{S}_p(\mathbb{N}) = \left\{ x \in l^2(\mathbb{N}) \mid \sum_{k \in \mathbb{N}} (k+1)^{2p} |x(k)|^2 < \infty \right\}. \tag{52}$$

Clearly $\mathcal{S}_+(\mathbb{N})$ is a dense linear subspace of $l^2(\mathbb{N})$. Additionally $\mathcal{S}_+(\mathbb{N})$ forms a countably-Hilbert space with the sequence $\langle\cdot,\cdot\rangle_{\mathcal{S}_p(\mathbb{N})}$, $p \geq 0$, of inner products given by

$$\langle x, y\rangle_{\mathcal{S}_p(\mathbb{N})} = \sum_{k\in\mathbb{N}} (k+1)^{2p} x(k)\, y(k), \quad x, y \in \mathcal{S}_+(\mathbb{N}). \tag{53}$$

For $x \in l^2(\mathbb{N})$, the coherent state ϕ_x is defined as

$$\phi_x = \prod_{k=0}^{\infty} (1 + x(k) Z_k), \tag{54}$$

which belongs to $L^2(\Omega)$ (see [11] for details). It can be shown that $\phi_x \in S(\Omega)$ whenever $x \in \mathcal{S}_+(\mathbb{N})$ and moreover $\{\phi_x \mid x \in \mathcal{S}_+(\mathbb{N})\}$ is total in $S(\Omega)$.

In the following, we denote by $\langle\langle\cdot,\cdot\rangle\rangle$ the canonical bilinear form on $S^*(\Omega) \times S(\Omega)$ given by

$$\langle\langle\Phi, \xi\rangle\rangle = \Phi(\xi), \quad \Phi \in S^*(\Omega), \quad \xi \in S(\Omega). \tag{55}$$

Definition 28. For $\Phi \in S^*(\Omega)$, one defines its S-transform $S\Phi$ as a function on $\mathcal{S}_+(\mathbb{N})$ given by

$$S\Phi(x) = \langle\langle\Phi, \phi_x\rangle\rangle, \quad x \in \mathcal{S}_+(\mathbb{N}). \tag{56}$$

Obviously, a generalized functional $\Phi \in S^*(\Omega)$ is completely determined by its S-transform $S\Phi$ since $\{\phi_x \mid x \in \mathcal{S}_+(\mathbb{N})\}$ is total in $S(\Omega)$.

Remark 29. It can be proved that the coherent mapping $x \mapsto \phi_x$ is a continuous mapping from $\mathcal{S}_+(\mathbb{N})$ to $S(\Omega)$. Thus, for a generalized functional $\Phi \in S^*(\Omega)$, its S-transform $S\Phi$ is actually a continuous function on $\mathcal{S}_+(\mathbb{N})$.

The next proposition offers a norm estimate to the coherent state ϕ_x.

Proposition 30. *Let $x \in \mathcal{S}_+(\mathbb{N})$. Then for each $p \geq 0$, it holds that*

$$\|\phi_x\|_p \leq \exp\left(\frac{1}{2}\|x\|^2_{\mathcal{S}_p(\mathbb{N})}\right), \tag{57}$$

where $\|\phi_x\|_p = \|\phi_x\|_{S_p(\Omega)}$ and $\|x\|^2_{\mathcal{S}_p(\mathbb{N})} = \sum_{k\in\mathbb{N}}(k+1)^{2p}|x(k)|^2$.

Proof. Define a function $\mathcal{E}_x$ on Γ as $\mathcal{E}_x(\emptyset) = 1$ and

$$\mathcal{E}_x(\sigma) = \prod_{k\in\sigma} x(k), \quad \sigma \neq \emptyset, \ \sigma \in \Gamma. \tag{58}$$

Then it is easy to show that $\mathcal{E}_x \in \mathcal{S}_+(\Gamma)$ and moreover $\phi_x = \mathbb{J}(\mathcal{E}_x)$, where $\mathbb{J}$ is the full Wiener integral operator (see Section 2). Thus, by using the following expression:

$$\|\phi_x\|^2_p = |\mathcal{E}_x|^2_p = \sum_{\sigma\in\Gamma} \lambda_\sigma^{2p} |\mathcal{E}_x(\sigma)|^2, \tag{59}$$

we have

$$\begin{aligned}\|\phi_x\|^2_p &= \sum_{\sigma\in\Gamma}\prod_{k\in\sigma}(k+1)^{2p}|x(k)|^2 = \sum_{n=0}^{\infty}\sum_{\sharp\sigma=n}\prod_{k\in\sigma}(k+1)^{2p}|x(k)|^2 \\ &\leq \sum_{n=0}^{\infty}\frac{1}{n!}\left(\sum_{k=0}^{\infty}(k+1)^{2p}|x(k)|^2\right)^n = \exp\left(\|x\|^2_{\mathcal{S}_p(\mathbb{N})}\right),\end{aligned} \tag{60}$$

which gives (57). □

Note that the left-hand side of (57) can be strictly less than the right-hand side [11]. Using Proposition 30, we can get a growth estimate of the S-transform of a generalized functional.

Proposition 31. *Let $\Phi \in S^*(\Omega)$. Then there exist constants $M \geq 0$ and $p \geq 0$ such that*

$$|S\Phi(x)| \leq M \exp\left(\frac{1}{2}\|x\|^2_{\mathcal{S}_p(\mathbb{N})}\right). \tag{61}$$

Proof. Since $S^*(\Omega) = \bigcup_{p\geq 0} S^*_p(\Omega)$, there exists some $p \geq 0$ such that $\Phi \in S^*_p(\Omega)$. Take $M = \|\Phi\|_{S^*_p(\Omega)}$. Then, by using Proposition 30, we get

$$\begin{aligned}|S\Phi(x)| &= |\langle\langle\Phi, \phi_x\rangle\rangle| \leq \|\Phi\|_{S^*_p(\Omega)}\|\phi_x\|_p \\ &\leq M \exp\left(\frac{1}{2}\|x\|^2_{\mathcal{S}_p(\mathbb{N})}\right).\end{aligned} \tag{62}$$

□

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgment

This paper is supported by National Natural Science Foundation of China (no. 11061032) and Natural Science Foundation of Gansu Province (Grant no. 0710RJZA106).

References

[1] M. Hitsuda, "Formula for Brownian partial derivatives," *Publication of Faculty of Integrated Arts and Sciences, Hiroshima University, Series III*, vol. 4, pp. 1–15, 1978.

[2] Y. Hu and G. Kallianpur, "Exponential integrability and application to stochastic quantization," *Applied Mathematics and Optimization*, vol. 37, no. 3, pp. 295–353, 1998.

[3] G. C. Wick, "The evaluation of the collision matrix," *Physical Review*, vol. 80, pp. 268–272, 1950.

[4] T. Hida and N. Ikeda, "Analysis on Hilbert space with reproducing kernel arising from multiple Wiener integral," in *Proceedings of the 5th Berkeley Symposium on Mathematical Statistics and Probability (Berkeley, Calif., 1965/66). Vol. II: Contributions to Probability Theory, Part 1*, pp. 117–143, University California Press, Berkeley, Calif, USA, 1967.

[5] P.-A. Meyer and J. A. Yan, "Les "fonctions caractéristiques" des distributions sur l'espace de Wiener," in *Séminaire de Probabilités, XXV*, vol. 1485 of *Lecture Notes in Mathematics*, pp. 61–78, Springer, Berlin, Germany, 1991.

[6] M. Émery, "A discrete approach to the chaotic representation property," in *Séminaire de Probabilités, XXXV*, vol. 1755 of *Lecture Notes in Mathematics*, pp. 123–138, Springer, Berlin, Germany, 2001.

[7] N. Privault, "Stochastic analysis of Bernoulli processes," *Probability Surveys*, vol. 5, pp. 435–483, 2008.

[8] I. Nourdin, G. Peccati, and G. Reinert, "Stein's method and stochastic analysis of Rademacher functionals," *Electronic Journal of Probability*, vol. 15, p. no. 55, 1703–1742, 2010.

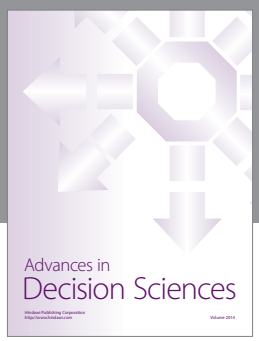
[9] C. Wang, H. Chai, and Y. Lu, "Discrete-time quantum Bernoulli noises," *Journal of Mathematical Physics*, vol. 51, no. 5, Article ID 053528, 8 pages, 2010.

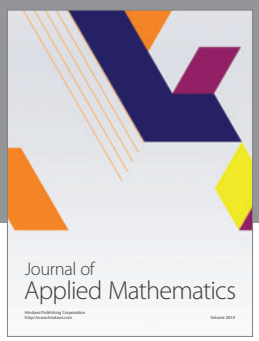
[10] C. Wang, Y. Lu, and H. Chai, "An alternative approach to Privault's discrete-time chaotic calculus," *Journal of Mathematical Analysis and Applications*, vol. 373, no. 2, pp. 643–654, 2011.

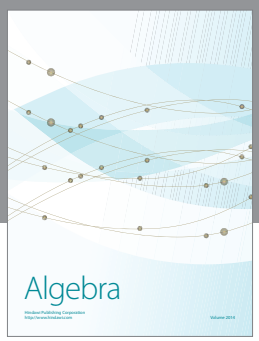
[11] C. Wang and Q. Han, "Coherent states in Bernoulli noise functionals," *Bulletin of the Australian Mathematical Society*, vol. 84, no. 1, pp. 116–126, 2011.

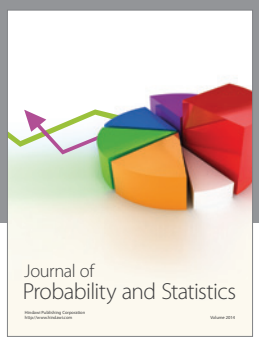
[12] C. S. Wang and J. H. Zhang, "Localization of quantumn Bernoulli noises," *Journal of Mathematical Physics*, vol. 54, no. 10, Article ID 103502, 2013.

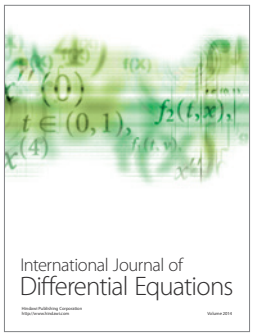
[13] H. Holden, B. Øksendal, J. Ubøe, and T. Zhang, *Stochastic Partial Differential Equations: A Modeling, White Noise Functional Approach*, Universitext, Springer, New York, NY, USA, 2nd edition, 2010.

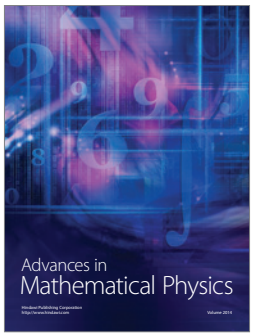
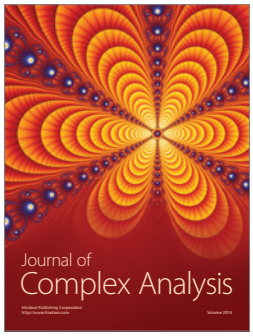
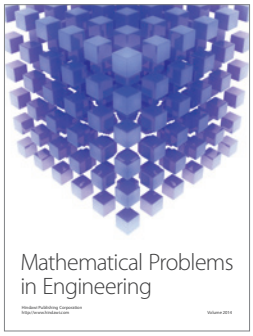
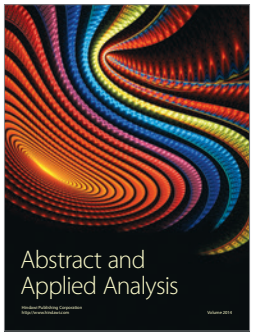
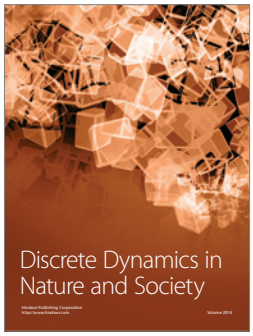
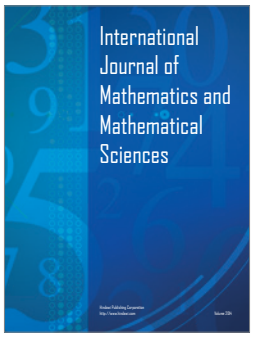
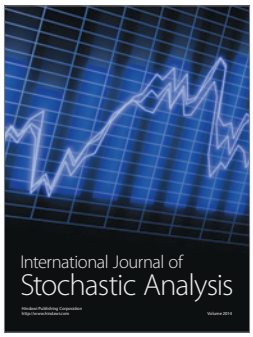










Hindawi

Submit your manuscripts at
http://www.hindawi.com