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Research Article

Weighted Estimates for Bilinear Operators

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We study the boundedness of weighted multilinear operators given by products of finite vectors of Calderón-Zygmund operators. We also investigate weighted estimates for bilinear operators related to Schrödinger operator.

1. Introduction

Bilinear (or multilinear) operators have attracted many researchers' attention, due to their relations closely connected to the Cauchy integral along with Lipschitz curves, Calderón commutators, and compensated compactness. In [1–3] and references therein, we can see an extensive study on the Hardy space estimate of bilinear operators. In [4], we can see the bilinear operators related to a Schrödinger operator $\mathscr{L}$ and estimates of them with respect to the Hardy type space associated with the Schrödinger operator $\mathscr{L}$ under some general conditions. In our paper, we study weighted estimates for bilinear operators which have the same expression as the operators in [4].

In this paper, N and K will denote fixed integers ≥ 2. Given a matrix of convolution Calderón-Zygmund kernels $\{K_i^j\}_{i=1,j=1}^{N\ \ K}$ on $\mathbb{R}^d$, we define $\{T_i^j\}_{i,j}$ as the associated Calderón-Zygmund operators. We denote by $L(f_1,\dots,f_K)$ the K-linear operator:

$$L(f_1,\dots,f_K)=\sum_{i=1}^{N}\left(T_i^1 f_1\right)\cdots\left(T_i^K f_K\right), \tag{1}$$

which originally is defined for smooth compactly supported functions $f_1,\dots,f_K$. For $p\leq 1$, $\omega\in A_1(\mathbb{R}^d)$, we denote by $H^p(\omega)$ the usual weighted Hardy space as defined in [5, 6], that is, the set of all distributions f on $\mathbb{R}^d$ for which the maximal function $\sup_{t>0}|\phi_t * f(x)|$ is in $L^p(\omega)$, where $\phi_t(x)=(1/t^n)\phi(x/t)$ and ϕ is smooth, nonzero, and compactly supported, and we also denote by $H^{p,\infty}(\omega)$ the weak H^p as defined in [7], that is, the set of all distributions f on $\mathbb{R}^d$ for which the maximal function $\sup_{t>0}|\phi_t * f(x)|$ is in weak $L^p(\omega)$. If $\omega(x)\equiv 1$, the weak Hardy space $H^{1,\infty}(\mathbb{R}^d)=H^{1,\infty}(\omega)$ was introduced by Feerman and Soria in [8]. $H^{p,\infty}(\mathbb{R}^d)$ first appeared in [9] (see also [10]).

Theorem 1. *Assume that $p_1,\dots,p_K>1$ are given and let $r=(p_1^{-1}+\cdots+p_K^{-1})^{-1}$ be their harmonic mean. We also assume that the harmonic mean of any proper subset of the p_j's is greater than 1. If, for all $(f_1,\dots,f_K)\in(C_0^\infty)^K$, the K-linear operator L satisfies*

$$\int L(f_1,\dots,f_K)\,dx=0. \tag{2}$$

Then, for $\omega\in A_1(\mathbb{R}^d)$, one has following conclusions.

(1) *If $r>1$, L maps $L^{p_1}(\omega)\times\cdots\times L^{p_K}(\omega)\to L^r(\omega)$.*

(2) *If $1\geq r>d/(d+1)$, L maps $L^{p_1}(\omega)\times\cdots\times L^{p_K}(\omega)\to H^r(\omega)$.*

(3) *If $r=d/(d+1)$, L maps $L^{p_1}(\omega)\times\cdots\times L^{p_K}(\omega)\to L^{r,\infty}(\omega)$.*

Let $\mathscr{L}=-\Delta+V$ be a Schrödinger operator on $\mathbb{R}^d$, $d\geq 3$, where $V\not\equiv 0$ is a fixed nonnegative potential. We will assume

that V belongs to reverse Hölder class $\mathrm{RH}_s(\mathbb{R}^d)$ for some $s \geq d/2$; that is, there exists $C = C(s, V) > 0$ such that

$$\left(\frac{1}{|B|}\int_B V(x)^s\,dx\right)^{1/s} \leq C\left(\frac{1}{|B|}\int_B V(x)\,dx\right), \tag{3}$$

for every ball $B \subset \mathbb{R}^d$. In what follows, $B(x, r)$ denotes the ball centered at x and of the radius r. Trivially, $\mathrm{RH}_q(\mathbb{R}^d) \subset \mathrm{RH}_p(\mathbb{R}^d)$ provided $1 < p \leq q < \infty$. It is well known that if $V \in \mathrm{RH}_q(\mathbb{R}^d)$ for some $q > 1$, then there exists $\varepsilon > 0$, which depends only on d and the constant C in (3), such that $V \in \mathrm{RH}_{q+\varepsilon}(\mathbb{R}^d)$ (see [11]). Throughout this paper, we always assume that $0 \not\equiv V \in \mathrm{RH}_{d/2}$. Thus, $V \in \mathrm{RH}_{q_0}$ for some $q_0 > d/2$.

Let $\{T_t^{\mathscr{L}}\}_{t>0}$ be the semigroup of linear operators generated by $\mathscr{L}$ and let $T_t^{\mathscr{L}}(x, y)$ be their kernels; that is,

$$\begin{aligned} T_t^{\mathscr{L}} f(x) &= e^{-t\mathscr{L}} f(x) \\ &= \int_{\mathbb{R}^d} T_t^{\mathscr{L}}(x, y) f(y)\,dy, \\ &\quad \text{for } t > 0,\ f \in L^2(\mathbb{R}^d). \end{aligned} \tag{4}$$

By the Trotter product formula (cf. [12]),

$$0 \leq T_t^{\mathscr{L}}(x, y) \leq H_t(x, y) = (4\pi t)^{-d/2} \exp\left(-\frac{|x-y|^2}{4t}\right). \tag{5}$$

The maximal function with respect to the semigroup $\{T_t^{\mathscr{L}}\}_{t>0}$ is defined by

$$\mathscr{T}^* f(x) = \sup_{t>0}\left|T_t^{\mathscr{L}} f(x)\right|. \tag{6}$$

The weighted Hardy-type space related to $\mathscr{L}$ is naturally defined by (see [13])

$$\begin{aligned} H_{\mathscr{L}}^1(\omega) &= \left\{f \in L^1(\omega) : \mathscr{T}^* f(x) \in L^1(\omega)\right\}, \\ &\quad \text{with } \|f\|_{H_{\mathscr{L}}^1(\omega)} = \|\mathscr{T}^* f\|_{L^1(\omega)}. \end{aligned} \tag{7}$$

Following [14], we define the auxiliary function $\rho(x, V) = \rho(x)$ by

$$\rho(x) = \rho(x, V) = \sup\left\{r > 0 : \frac{1}{r^{d-2}}\int_{B(x,r)} V(y)\,dy \leq 1\right\}. \tag{8}$$

The auxiliary function $\rho(x)$ plays an important role in studying the boundedness of singular integral operators related to the Schrödinger operator $\mathscr{L}$ as well as the atomic decomposition of $H_{\mathscr{L}}^1$ and $H_{\mathscr{L}}^1(\omega)$ (see [13–15]).

In our paper, we also consider the following bilinear operators:

$$T^{\pm}(f, g)(x) = (T_1 f)(x)(T_2 g)(x) \pm (T_2 f)(x)(T_1 g)(x), \tag{9}$$

where $f \in L^p(\omega)$, $g \in L^q(\omega)$ with $1 < p, q < \infty$ and $1/p + 1/q = 1$, T_i $(i = 1, 2)$ are Calderón-Zygmund operators related to $\mathscr{L}$ and satisfy the following two conditions.

(i) There exist parallel Calderón-Zygmund operators $\widetilde{T}_i$ related to the Laplacian Δ and a constant $\delta > 0$ such that

$$\left|T_i(x, y) - \widetilde{T}_i(x, y)\right| \leq \frac{C}{\rho(y)^{\delta}|x-y|^{d-\delta}}, \quad x \neq y, \tag{10}$$

where $T_i(x, y)$ and $\widetilde{T}_i(x, y)$ denote the kernels of T_i and $\widetilde{T}_i$, respectively.

(ii) One of the parallel bilinear operators

$$\begin{aligned} \widetilde{T}^{\pm}(f, g)(x) &= \left(\widetilde{T}_1 f\right)(x)\left(\widetilde{T}_2 g\right)(x) \\ &\quad \pm \left(\widetilde{T}_2 f\right)(x)\left(\widetilde{T}_1 g\right)(x) \end{aligned} \tag{11}$$

has the vanishing moment; that is, either $\widetilde{T}^+$ or $\widetilde{T}^-$ satisfies

$$\int_{\mathbb{R}^d} \widetilde{T}^{\pm}(f, g)(x)\,dx = 0 \quad \forall f, g \in C_c^{\infty}(\mathbb{R}^d). \tag{12}$$

We will show that either T^+ or T^- is bounded from $L^p(\omega) \times L^q(\omega)$ to $H_{\mathscr{L}}^1(\omega)$.

Theorem 2. *Suppose that the bilinear operators $T^{\pm}$ are defined as above. Let $1 < p, q < \infty$ and $1/p + 1/q = 1$. Then either T^+ or T^- (but not both), which corresponds to the parallel bilinear operator satisfying* (12), *maps $L^p(\omega) \times L^q(\omega)$ into $H_{\mathscr{L}}^1(\omega)$ and there exists a constant $C > 0$ such that*

$$\|T^{\pm}(f, g)\|_{H_{\mathscr{L}}^1(\omega)} \leq C\|f\|_{L^p(\omega)}\|g\|_{L^q(\omega)}. \tag{13}$$

This paper is organized as follows. In Section 2, we give some notation and preliminary estimates on $\rho(x)$ and the kernel $T_t(x, y)$ which have been proved in [14–16]. In Section 3, we prove Theorem 1, and in Section 4 we prove Theorem 2.

Throughout this paper, we will use C to denote a positive constant, which is not necessarily the same at each occurrence. By $A \sim B$, we mean that there exists a constant $C > 1$, such that $C^{-1} \leq A/B \leq C$. For a given ball B, we denote by B^* the concentric ball with twice radius, and $B^{**} = (B^*)^*$.

2. Preliminaries

Throughout this paper, we will denote $\omega(E) := \int_E \omega(x)dx$ for any set $E \subset \mathbb{R}^d$. For $1 \leq p \leq \infty$, denote by p' the adjoint number of p; that is, $1/p + 1/p' = 1$.

We review some needed background about Muckenhoupt weights. A weight ω is a nonnegative locally integrable function on $\mathbb{R}^d$. We say that $\omega \in A_p(\mathbb{R}^d)$, $1 < p < \infty$, if there exists a constant C such that

$$\left(\frac{1}{|B|}\int_B \omega\right)\left(\frac{1}{|B|}\int_B \omega^{1-p'}\right)^{p-1} \leq C, \tag{14}$$

for every ball $B \subset \mathbb{R}^d$. The class $A_1(\mathbb{R}^d)$ is defined replacing the above inequality by

$$M\omega(x) \leq C\omega(x) \quad \text{a.e. } x \in \mathbb{R}^d, \tag{15}$$

where M denotes the standard Hardy-Littlewood operator.

Let $0 < p < \infty$ and let ω be a locally integrable nonnegative function. We denote the weighted space $L^p(\mathbb{R}^d, \omega(x)dx)$ by $L^p(\omega)$ and set

$$\|f\|_{L^p(\omega)} = \left(\int_{\mathbb{R}^d} |f(x)|^p \omega(x)dx\right)^{1/p}. \tag{16}$$

We also denote the weak $L^p(\omega)$ by $L^{p,\infty}(\omega)$ and set

$$\|f\|_{L^{p,\infty}(\omega)} = \sup_{\lambda>0} \lambda \left[\omega\left(\left\{x \in \mathbb{R}^d : |f(x)| > \lambda\right\}\right)\right]^{1/p}. \tag{17}$$

We first sum up some properties of weights in the following results.

Lemma 3. *One has the following properties.*

(i) $A_p \subset A_q$ *for* $1 \le p \le q < \infty$.

(ii) *Let* $p \in (1, \infty)$. *Then* $\omega \in A_p$ *if and only if* $\omega^{1-p'} \in A_{p'}$.

(iii) *If* $\omega \in A_p, 1 \le p < \infty$, *then there exists* $\varepsilon > 0$ *such that* $\omega^{1+\varepsilon} \in A_p$.

(iv) *If* $\omega \in A_p, 1 \le p < \infty$, *then the measure* $\omega(x)dx$ *is doubling: precisely, for all* $\lambda > 1$ *and all cubes* Q *one has*

$$\omega(\lambda Q) \le \lambda^{dp} [\omega]_{A_p} \omega(Q), \tag{18}$$

(*where* $[\omega]_{A_p}$ *denotes the* A_p *Muckenhoupt characteristic constant and* λQ *denotes the cube with the same center as* Q *and side length* λ *times the side length of* Q).

(v) $A_\infty(\mathbb{R}^d) = \bigcup_{p\ge1} A_p(\mathbb{R}^d)$.

Proof. Properties are standard; see, for instance, [17–19]. □

Lemma 4. *Let* B *be a ball of* $\mathbb{R}^d$ *and let* E *be any measurable subset of* B. *Let* $\omega \in A_p, 1 \le p < \infty$. *Then, there exist constants* $C_1 > 0$, $C_2 > 0$, *and* $\delta > 0$ *such that*

$$C_1\left(\frac{|E|}{|B|}\right)^p \le \frac{\omega(E)}{\omega(B)} \le C_2\left(\frac{|E|}{|B|}\right)^\delta. \tag{19}$$

Proof. For the proof, we refer to [17]. □

Next we recall some basic properties of the auxiliary function $\rho(x)$. It is known that $0 < \rho(x) < \infty$ for any $x \in \mathbb{R}^d$ (cf. [14]). Therefore,

$$\mathbb{R}^d = \bigcup_{k=-\infty}^{\infty} \Omega_k, \tag{20}$$

where

$$\Omega_k = \left\{x \in \mathbb{R}^d : 2^{-k-1} \le \rho(x) < 2^{-k}\right\}. \tag{21}$$

Lemma 5 (see [15]). *There exist a constant* $N = N(V)$ *and a sequence of points* $\{x_{(k,\alpha)} \in \Omega_k : k, \alpha \in \mathbb{Z}\}$ *such that the family of critical balls* $\{B_{(k,\alpha)}\}$, *defined by* $B_{(k,\alpha)} = \{x \in \mathbb{R}^d : |x - x_{(k,\alpha)}| < \rho(x_{(k,\alpha)})\}$, *satisfies*

(i) $\bigcup_{(k,\alpha)} B_{(k,\alpha)} = \mathbb{R}^d$,

(ii) $\#\{(k', \alpha') : B^{**}_{(k',\alpha')} \cap B^{**}_{(k,\alpha)} \ne \emptyset\} \le N$ *for every* (k, α).

Furthermore, there exists a family of C^∞ *functions* $\{\psi_{(k,\alpha)}\}_{k,\alpha\in\mathbb{Z}}$ *such that*

(iii) $\operatorname{supp} \psi_{(k,\alpha)} \subseteq B^*_{(k,\alpha)}$,

(iv) $0 \le \psi_{(k,\alpha)}(x) \le 1$ *and* $\sum_{(k,\alpha)} \psi_{(k,\alpha)}(x) = 1$ *for all* $x \in \mathbb{R}^d$,

(v) $\|\nabla\psi_{(k,\alpha)}\|_{L^\infty} \le C/\rho(x_{(k,\alpha)})$.

Lemma 6 (see [14]). *There exist* $C > 0$ *and* $k_0 \ge 1$ *so that, for all* $x, y \in \mathbb{R}^d$,

$$\begin{aligned} C^{-1}\rho(x)\left(1 + \frac{|x-y|}{\rho(x)}\right)^{-k_0} \\ \le \rho(y) \le C\rho(x)\left(1 + \frac{|x-y|}{\rho(x)}\right)^{k_0/(k_0+1)}. \end{aligned} \tag{22}$$

In particular, $\rho(x) \sim \rho(y)$ *when* $y \in B(x, r)$ *and* $r \le C\rho(x)$.

To prove Theorem 2, we need the following estimates for the kernel $T_t^{\mathscr{L}}(x, y)$.

Lemma 7 (see [15, 16]). *For every* N, *there is a constant* $C_N > 0$ *such that*

$$\begin{aligned} 0 &\le T_t^{\mathscr{L}}(x, y) \\ &\le C_N t^{-(d/2)}\left(1 + \frac{\sqrt{t}}{\rho(x)} + \frac{\sqrt{t}}{\rho(y)}\right)^{-N} \exp\left(-\frac{|x-y|^2}{5t}\right). \end{aligned} \tag{23}$$

Also, there exists a constant $C > 0$ *such that*

$$\left|T_t^{\mathscr{L}}(x, y) - H_t(x, y)\right| \le \frac{C}{\rho(x)^\sigma |x-y|^{d-\sigma}}, \tag{24}$$

where $\sigma = 2 - d/q_0 > 0$.

3. The Proof of Theorem 1

We fix $p_1, \ldots, p_K > 1$ and we let r be their harmonic mean. For $r > 1$, we can get the result just by Hölder's inequality and the $L^p(\omega)$ boundedness of Calderón-Zygmund operators. Then we just need to study the case $r \le 1$. Fix a smooth compactly supported function ϕ in $\mathbb{R}^d$, an $x_0 \in \mathbb{R}^d$, and define $\phi_{t,x_0}(x) = (1/t^n)\phi((x - x_0)/t)$. Without loss of generality we may assume that ϕ is supported in $|x| \le 1$. We need to show that $\sup_{t>0}|\int \phi_{t,x_0} L(f_1, \ldots, f_K)dx|$ is in $L^r(\omega)$ when $r > n/(n+1)$ and in $L^{r,\infty}(\omega)$ when $r = n/(n+1)$. We also fix a smooth cutoff $\eta(x)$ such that $\eta \equiv 1$ on $|x| < 2$ and is supported in $|x| < 4$. We call for simplicity $\eta_0(x) =$

$\eta((x_0 - x)/t)$ and $\eta_1(x) = 1 - \eta_0(x)$. We now decompose $L(f_1, \ldots, f_K) = L_0 + L_1 + \cdots + L_{K+1}$, where

$$\begin{aligned} L_0 &= L(\eta_0 f_1, \eta_0 f_2, \ldots, \eta_0 f_K), \\ L_1 &= \sum_{j=1}^{K} L(f_1, \ldots, \eta_1 f_j, \ldots, f_K), \\ L_2 &= \sum_{\substack{1\le j,l\le K \\ j<l}} L(f_1, \ldots, \eta_1 f_j, \ldots, \eta_1 f_l, \ldots, f_K), \\ &\vdots \\ L_{K+1} &= (-1)^K L(\eta_1 f_1, \eta_1 f_2, \ldots, \eta_1 f_K). \end{aligned} \tag{25}$$

In each L_u above exactly u functions among the f_j's are multiplied by η_1 and the remaining are left intact.

For any fixed i, k and any x such that $|x - x_0| \le t$ we have

$$\begin{aligned} &\sup_{t>0} \left| T_i^k(\eta_1 f)(x) - T_i^k(\eta_1 f)(x_0) \right| \\ &\quad \le \sup_{t>0} \left| \int \left(K_i^k(x-y) - K_i^k(x_0 - y) \right) \eta_1(y) f(y)\, dy \right| \\ &\quad \le C \sup_{t>0} \int_{|y-x_0|\ge t} |x - x_0| \, |y - x_0|^{-n-1} |f(y)|\, dy \\ &\quad \le CM(|f|)(x_0), \end{aligned} \tag{26}$$

where $M(f)(x_0)$ is the Hardy-Littlewood maximal function of f at the point x_0. We denote the maximal truncated operator of T_i^j by $(T_i^j)^*$. We begin with the L_1:

$$\begin{aligned} L_1 = &\sum_{j=1}^{K} L(f_1, \ldots, (\eta_1 f_j)(x) - (\eta_1 f_j)(x_0), \ldots, f_K) \\ &+ \sum_{j=1}^{K} L(f_1, \ldots, (\eta_1 f_j)(x_0), \ldots, f_K). \end{aligned} \tag{27}$$

Then we have

$$\begin{aligned} I &= \sup_{t>0} \left| \int \phi_{t,x_0} L_1 dx \right| \\ &\le \sum_{j=1}^{K} \sum_{i=1}^{N} \sup_{t>0} \int \left| \phi_{t,x_0} \right| \prod_{\substack{1\le k\le K \\ k\ne j}} \left| T_i^k f_k \right| \\ &\qquad \times \left(\left| T_i^j (\eta_1 f_j)(x) - T_i^j (\eta_1 f_j)(x_0) \right| \right. \\ &\qquad \left. + \left| T_i^j (\eta_1 f_j)(x_0) \right| \right) dx \\ &\le C \sum_{j=1}^{K} \sum_{i=1}^{N} M\left(\prod_{\substack{1\le k\le K \\ k\ne j}} \left| T_i^k f_k \right| \right)(x_0) \\ &\quad \times \left(M(f_j)(x_0) + (T_i^j)^* f_j(x_0) \right). \end{aligned} \tag{28}$$

Define σ_j by $\sigma_j^{-1} + p_j^{-1} = r^{-1}$. By Hölder inequality and the boundedness of M and $(T_i^j)^*$, the $L^r(\omega)$ norm in x_0 of I is bounded by

$$\begin{aligned} &C \sum_{j=1}^{K} \sum_{i=1}^{N} \left(\left\| M(f_j) \right\|_{L^{p_j}(\omega)} + \left\| (T_i^j)^*(f_j) \right\|_{L^{p_j}(\omega)} \right) \\ &\quad \times \left\| M\left(\prod_{\substack{1\le k\le K \\ k\ne j}} \left| T_i^k f_k \right| \right) \right\|_{L^{\sigma_j}(\omega)} \\ &\le C \sum_{j=1}^{K} \sum_{i=1}^{N} \left\| f_j \right\|_{L^{p_j}(\omega)} \left\| \prod_{\substack{1\le k\le K \\ k\ne j}} \left| T_i^k f_k \right| \right\|_{L^{\sigma_j}(\omega)} \\ &\le C \sum_{j=1}^{K} \sum_{i=1}^{N} \left\| f_j \right\|_{L^{p_j}(\omega)} \prod_{\substack{1\le k\le K \\ k\ne j}} \left\| T_i^k f_k \right\|_{L^{p_k}(\omega)} \\ &\le C \sum_{j=1}^{K} \left\| f_j \right\|_{L^{p_j}(\omega)} \prod_{\substack{1\le k\le K \\ k\ne j}} \left\| f_k \right\|_{L^{p_k}(\omega)} \\ &= C \prod_{k=1}^{K} \left\| f_k \right\|_{L^{p_k}(\omega)}. \end{aligned} \tag{29}$$

We obtain that

$$\begin{aligned} &\left\| \sup_{t>0} \left| \int \phi_{t,x0} L_1 dx \right| \right\|_{L^r(\omega)} \le C \prod_{k-1}^{K} \left\| f_k \right\|_{L^{p_k}(\omega)}, \\ &\omega\left(\left\{ x_0 : \sup_{t>0} \left| \int \phi_{t,x_0} L_1 dx \right| > \lambda \right\} \right) \le C\lambda^{-r} \prod_{k=1}^{K} \left\| f_k \right\|_{L^{p_k}(\omega)}^r. \end{aligned} \tag{30}$$

For L_2, we write $L_2 = L_{21} + L_{22} + L_{23} + L_{24}$ where

$$\begin{aligned} L_{21} = -\sum_{\substack{1\le j,l\le K \\ j<l}} L\big(&f_1, \ldots, (\eta_1 f_j)(x) \\ &- (\eta_1 f_j)(x_0), \ldots, (\eta_1 f_l)(x) \\ &- (\eta_1 f_l)(x_0), \ldots, f_K\big), \\ L_{22} = -\sum_{\substack{1\le j,l\le K \\ j<l}} L\big(&f_1, \ldots, (\eta_1 f_j)(x_0), \ldots, (\eta_1 f_l)(x) \\ &- (\eta_1 f_l)(x_0), \ldots, f_K\big), \end{aligned}$$

$$L_{23} = -\sum_{\substack{1\le j,l\le K\\ j<l}} L\left(f_1,\ldots,\left(\eta_1 f_j\right)(x) - \left(\eta_1 f_j\right)(x_0),\ldots,\left(\eta_1 f_l\right)(x_0),\ldots,f_K\right),$$
$$L_{24} = -\sum_{\substack{1\le j,l\le K\\ j<l}} L\left(f_1,\ldots,\left(\eta_1 f_j\right)(x_0),\ldots,\left(\eta_1 f_l\right)(x_0),\ldots,f_K\right). \tag{31}$$

Using the same reason as before, we have the following estimate of L_{2u}, $u = 1,2,3,4$:

$$\sup_{t>0}\left|\int \phi_{t,x_0} L_{2u} dx\right| \le C \sum_{\substack{1\le j,l\le K\\ j<l}} \sum_{i=1}^{K} M\left(\prod_{\substack{1\le k\le K\\ k\ne j,l}} \left|T_i^k f_k\right|\right) \times (x_0)\left(\left(C_j f_j\right)(x_0)\left(C_l f_l\right)(x_0)\right), \tag{32}$$

where each $C_j f_j$ is either $M(f_j)$ or $(T_i^j)^* f_j$ and therefore $\|C_j f_j\|_{L^{p_j}(\omega)} \le C\|f_j\|_{L^{p_j}(\omega)}$. Define σ_{jl} by $\sigma_{jl}^{-1} + p_j^{-1} + p_l^{-1} = r^{-1}$. For each $u = 1,2,3,4$, Hölder inequality implies that

$$\begin{aligned}
&\left\|\sup_{t>0}\left|\int \phi_{t,x_0} L_{2u}\, dx\right|\right\|_{L^r(\omega)}\\
&\quad\le C \sum_{\substack{1\le j,l\le K\\ j<l}} \sum_{i=1}^{K} \left\|M\left(\prod_{\substack{1\le k\le K\\ k\ne j,l}} \left|T_i^k f_k\right|\right)\right\|_{L^{\sigma_{jl}}(\omega)} \times \left(\left\|C_j f_j\right\|_{L^{p_j}(\omega)}\left\|C_l f_l\right\|_{L^{p_l}(\omega)}\right)\\
&\quad\le C \sum_{\substack{1\le j,l\le K\\ j<l}} \sum_{i=1}^{K} \left\|\prod_{\substack{1\le k\le K\\ k\ne j,l}} \left|T_i^k f_k\right|\right\|_{L^{\sigma_{jl}}(\omega)} \times \left(\left\|f_j\right\|_{L^{p_j}(\omega)}\left\|f_l\right\|_{L^{p_l}(\omega)}\right)\\
&\quad\le C \sum_{\substack{1\le j,l\le K\\ j<l}} \sum_{i=1}^{K} \prod_{\substack{1\le k\le K\\ k\ne j,l}} \left\|T_i^k f_k\right\|_{L^{p_k}(\omega)} \times \left(\left\|f_j\right\|_{L^{p_j}(\omega)}\left\|f_l\right\|_{L^{p_l}(\omega)}\right)\\
&\quad\le C \sum_{\substack{1\le j,l\le K\\ j<l}} \prod_{\substack{1\le k\le K\\ k\ne j,l}} \left\|f_k\right\|_{L^{p_k}(\omega)} \times \left(\left\|f_j\right\|_{L^{p_j}(\omega)}\left\|f_l f_l\right\|_{L^{p_l}(\omega)}\right)\\
&\quad\le C \prod_{1\le k\le K} \left\|f_k\right\|_{L^{p_k}(\omega)}.
\end{aligned} \tag{33}$$

Then we get

$$\left\|\sup_{t>0}\left|\int \phi_{t,x0} L_1 dx\right|\right\|_{L^r(\omega)} \le C\prod_{k-1}^{K}\|f_k\|_{L^p k(\omega)},$$
$$\omega\left(\left\{x_0 : \sup_{t>0}\left|\int \phi_{t,x_0} L_2 dx\right| > \lambda\right\}\right) \le C\lambda^{-r}\prod_{k=1}^{K}\|f_k\|_{L^{p_k}(\omega)}^{r}. \tag{34}$$

For $L_3, L_4, \ldots, L_{K+1}$, by the similar method, we get the desired conclusions. Now we turn to the term L_0, and we use two inequalities which have been proved in [3] as follows:

$$\sup_{t>0}\left|\int \phi_{t,x_0} L_0\, dx\right| \le C\prod_{k=1}^{K}\left(M\left(|f_k|^{s_k}\right)(x_0)\right)^{1/s_k} \quad \text{when } r > \frac{d}{(d+1)}, \tag{35}$$

$$\sup_{t>0}\left|\int \phi_{t,x_0} L_0 dx\right| \le C\prod_{k=1}^{K}\left(M\left(|f_k|^{p_k}\right)(x_0)\right)^{1/p_k} \quad \text{when } r = \frac{d}{(d+1)}, \tag{36}$$

where $1 < s_j < p_j$.

Then, for $r > d/(d+1)$, we get

$$\begin{aligned}
&\left\|\sup_{t>0}\left|\int \phi_{t,x_0} L_0 dx\right|\right\|_{L^r(\omega)}\\
&\quad\le C\left\|\prod_{k=1}^{K}\left(M\left(|f_k|^{s_k}\right)(x_0)\right)^{1/s_k}\right\|_{L^r(\omega)}\\
&\quad\le C\prod_{k=1}^{K}\left\|\left(M\left(|f_k|^{s_k}\right)(x_0)\right)^{1/s_k}\right\|_{L^{p_k}(\omega)}\\
&\quad\le C\prod_{k=1}^{K}\|f_k\|_{L^{p_k}(\omega)},
\end{aligned} \tag{37}$$

where we use $p_k/s_k > 1$, and we obtain conclusion (2) of Theorem 1.

For conclusion (3), let $\varepsilon_0 = \lambda/C$, $\varepsilon_K = 1$, and $\varepsilon_1, \ldots, \varepsilon_{K-1} > 0$ be arbitrary; then, by using (36) and the weak (1, 1) result for the Hardy-Littlewood maximal function, we can get

$$\begin{aligned}
&\omega\left(\left\{x_0 : \sup_{t>0}\left|\int \phi_{t,x_0} L_0 dx\right| > \lambda\right\}\right) \\
&\quad \le \sum_{j=1}^{K} \omega\left(\left\{x_0 : M\left(\left|f_j\right|^{p_j}\right)(x_0) > \left(\frac{\varepsilon_{j-1}}{\varepsilon_j}\right)^{p_j}\right\}\right) \\
&\quad \le C\sum_{j=1}^{K}\left(\frac{\varepsilon_{j-1}}{\varepsilon_j}\right)^{-p_j} \int \left|f_j\right|^{p_j} \omega(x)\, dx,
\end{aligned} \tag{38}$$

and the expression minimizes when all the terms that appear in the sum are equal.

Finally, we get the weak type estimate:

$$\omega\left(\left\{x_0 : \sup_{t>0}\left|\int \phi_{t,x_0} L_0 dx\right| > \lambda\right\}\right) \le C\lambda^{-r}\prod_{j=1}^{K}\left\|f_j\right\|_{L^{p_j}(\omega)}^{r}, \tag{39}$$

and we complete the proof.

4. The Proof of Theorem 2

We give the proof for the bilinear operator T^- only and the proof for T^+ is similar. Let $f \in L^p(\omega)$ and $g \in L^q(\omega)$. Assume that the parallel Calderón-Zygmund operators $\widetilde{T}_i$ $(i = 1, 2)$ and the parallel bilinear operator $\widetilde{T}^-$ satisfy (10) and (12). We split $T^-(f, g)$ into three parts:

$$T^-(f,g)(x) = T_1^-(f,g)(x) + T_2^-(f,g)(x) + \widetilde{T}^-(f,g)(x), \tag{40}$$

where

$$\begin{aligned}
T_1^-(f,g) &= \left(T_1 f - \widetilde{T}_1 f\right)\left(T_2 g\right) - \left(T_2 f - \widetilde{T}_2 f\right)\left(T_1 g\right), \\
T_2^-(f,g) &= \left(\widetilde{T}_1 f\right)\left(T_2 g - \widetilde{T}_2 g\right) - \left(\widetilde{T}_2 f\right)\left(T_1 g - \widetilde{T}_1 g\right).
\end{aligned} \tag{41}$$

It follows from Theorem 1 that $\widetilde{T}^- \in H^1(\omega)$ and

$$\left\|\widetilde{T}^-(f,g)\right\|_{H^1(\omega)} \le C\|f\|_{L^p(\omega)}\|g\|_{L^q(\omega)}. \tag{42}$$

Since $H^1(\omega) \subset H^1_{\mathscr{L}}(\omega)$, we have $\widetilde{T}^-(f, g)$ and

$$\left\|\widetilde{T}^-(f,g)\right\|_{H^1_{\mathscr{L}}(\omega)} \le C\|f\|_{L^p(\omega)}\|g\|_{L^q(\omega)}. \tag{43}$$

It suffices to show $T_1^-(f, g) + T_2^-(f, g) \in H^1_{\mathscr{L}}(\omega)$; that is,

$$\begin{aligned}
&\mathscr{T}^*\left(T_1^-(f,g) + T_2^-(f,g)\right)(x) \\
&\quad = \sup_{t>0}\left|T_t^{\mathscr{L}}\left(T_1^-(f,g) + T_2^-(f,g)\right)(x)\right| \in L^1(\omega).
\end{aligned} \tag{44}$$

For simplicity we write $F(f, g)(x) = T_1^-(f, g) + T_2^-(f, g)$. Choose a sequence of points $\{x_{(k,\alpha)}\}$ and a family of functions $\{\psi_{(k,\alpha)}\}$ as in Lemma 5. Then

$$\begin{aligned}
&\left\|T_1^-(f,g) + T_2^-(f,g)\right\|_{H^1_{\mathscr{L}}(\omega)} \\
&\quad \le \sum_{(k,\alpha)}\left(\int_{(B^{**}_{(k,\alpha)})^c} + \int_{B^{**}_{(k,\alpha)}}\right) \\
&\qquad \times \sup_{t>0}\left|T_t^{\mathscr{L}}\left(\psi_{(k,\alpha)} F(f,g)\right)(x)\right| \omega(x)\, dx \\
&\quad = \sum_{(k,\alpha)} I_{(k,\alpha)} + \sum_{(k,\alpha)} J_{(k,\alpha)}.
\end{aligned} \tag{45}$$

For $I_{(k,\alpha)}$, by using (23), we get

$$\begin{aligned}
&I_{(k,\alpha)} \\
&\quad = \int_{(B^{**}_{(k,\alpha)})^c} \sup_{t>0}\left|\int_{\mathbb{R}^d} T_t^{\mathscr{L}}(x,y)\, \psi_{(k,\alpha)}(y)\, F(f,g)(y)\, dy\right| \\
&\qquad \times \omega(x)\, dx \\
&\quad \le C_N \int_{(B^{**}_{(k,\alpha)})^c} \sup_{t>0} \int_{\mathbb{R}^d} \rho(y)^N t^{-((d+N)/2)} e^{-(|x-y|^2/5t)} \\
&\qquad \times \left|\psi_{(k,\alpha)}(y)\, F(f,g)(y)\right| dy\, \omega(x)\, dx \\
&\quad \le C_N \int_{(B^{**}_{(k,\alpha)})^c} \int_{\mathbb{R}^d} \frac{\rho(x)^N}{|x-y|^{d+N}} \\
&\qquad \times \left|\psi_{(k,\alpha)}(y)\, F(f,g)(y)\right| dy\, \omega(x)\, dx,
\end{aligned} \tag{46}$$

and for $x \notin B^{**}_{k,\alpha}$ and $y \in \operatorname{supp} \psi_{(k,\alpha)} \subset B^*_{(k,\alpha)}$, Lemma 6 yields $\rho(y) \sim \rho(x_{(k,\alpha)})$ and $|x - y| \ge (1/2)|x - x_{(k,\alpha)}|$; then we have

$$\begin{aligned}
I_{(k,\alpha)} &\le C \int_{B^*_{(k,\alpha)}} \left|\psi_{(k,\alpha)}(y)\, F(f,g)(y)\right| \\
&\qquad \times \left(\int_{(B^{**}_{(k,\alpha)})^c} \frac{\rho\left(x_{(k,\alpha)}\right)^N}{\left|x - x_{(k,\alpha)}\right|^{d+N}} \omega(x)\, dx\right) dy.
\end{aligned} \tag{47}$$

For the inner integral we have

$$\begin{aligned}
&\int_{(B^{**}_{(k,\alpha)})^c} \frac{\rho\left(x_{(k,\alpha)}\right)^N}{\left|x - x_{(k,\alpha)}\right|^{d+N}} \omega(x)\, dx \\
&\quad = \int_{|x - x_{(k,\alpha)}| > 4\rho(x_{(k,\alpha)})} \frac{\rho\left(x_{(k,\alpha)}\right)^N}{\left|x - x_{(k,\alpha)}\right|^{d+N}} \omega(x)\, dx \\
&\quad = \sum_{j=2}^{\infty} \int_{|x - x_{(k,\alpha)}| \sim 2^j \rho(x_{(k,\alpha)})} \frac{\rho\left(x_{(k,\alpha)}\right)^N}{\left|x - x_{(k,\alpha)}\right|^{d+N}} \omega(x)\, dx
\end{aligned}$$

$$\begin{aligned}&\le \sum_{j=2}^{\infty}\frac{\rho\left(x_{(k,\alpha)}\right)^{N}}{\left(2^{j}\rho\left(x_{(k,\alpha)}\right)\right)^{d+N}}\omega\left(B\left(x_{(k,\alpha)},2^{j+1}\rho\left(x_{(k,\alpha)}\right)\right)\right)\\&\le C\frac{\omega\left(B^{*}_{(k,\alpha)}\right)}{\left|B^{*}_{(k,\alpha)}\right|}\le C\omega\left(y\right),\end{aligned}\tag{48}$$

where $y\in B^{*}_{(k,\alpha)}$. Then we obtain

$$\begin{aligned}I_{(k,\alpha)}&\le C\int_{B^{*}_{(k,\alpha)}}\left|\psi_{(k,\alpha)}\left(y\right)F\left(f,g\right)\left(y\right)\right|\omega\left(y\right)dy\\&\le C\left\|\psi_{(k,\alpha)}F\left(f,g\right)\right\|_{L^{1}(\omega)},\end{aligned}\tag{49}$$

and hence

$$\begin{aligned}\sum_{(k,\alpha)}I_{(k,\alpha)}&\le C\sum_{(k,\alpha)}\left\|\psi_{(k,\alpha)}F(f,g)\right\|_{L^{1}(\omega)}\\&\le C\left\|F\left(f,g\right)\right\|_{L^{1}(\omega)}\le C\left\|f\right\|_{L^{p}(\omega)}\left\|g\right\|_{L^{q}(\omega)}.\end{aligned}\tag{50}$$

To estimate $\sum_{(k,\alpha)}J_{(k,\alpha)}$, we write

$$\begin{aligned}&\sum_{(k,\alpha)}J_{(k,\alpha)}\\&\quad\le\sum_{(k,\alpha)}\int_{B^{**}_{(k,\alpha)}}\sup_{t\ge\rho(x_{(k,\alpha)})^{2}}\left|T^{\mathscr{L}}_{t}\left(\psi_{(k,\alpha)}F\left(f,g\right)\right)\left(x\right)\right|\\&\qquad\times\omega\left(x\right)dx\\&\quad+\sum_{(k,\alpha)}\int_{B^{**}_{(k,\alpha)}}\sup_{0<t<\rho(x_{(k,\alpha)})^{2}}\left|T^{\mathscr{L}}_{t}\left(\psi_{(k,\alpha)}F\left(f,g\right)\right)\left(x\right)\right|\\&\qquad\times\omega\left(x\right)dx\\&\quad=J_{1}+J_{2}.\end{aligned}\tag{51}$$

For J_1, we apply (23) and Lemma 6 to obtain

$$\begin{aligned}&\int_{B^{**}_{(k,\alpha)}}\sup_{t\ge\rho(x_{(k,\alpha)})^{2}}\left|T^{\mathscr{L}}_{t}\left(\psi_{(k,\alpha)}F\left(f,g\right)\right)\left(x\right)\right|\omega\left(x\right)dx\\&\quad\le C_{N}\int_{B^{**}_{(k,\alpha)}}\sup_{t\ge\rho(x_{(k,\alpha)})^{2}}\int_{\mathbb{R}^{d}}\rho\left(y\right)^{N}t^{-((d+N)/2)}e^{-(|x-y|^{2}/5t)}\\&\qquad\times\left|\psi_{(k,\alpha)}\left(y\right)F\left(f,g\right)\left(y\right)\right|dy\omega\left(x\right)dx\\&\quad\le C_{N}\int_{B^{**}_{(k,\alpha)}}\sup_{t\ge\rho(x_{(k,\alpha)})^{2}}\int_{\mathbb{R}^{d}}\rho\left(x_{(k,\alpha)}\right)^{N}t^{-((d+N)/2)}\\&\qquad\times\left|\psi_{(k,\alpha)}\left(y\right)F\left(f,g\right)\left(y\right)\right|dy\omega\left(x\right)dx\\&\quad\le C_{N}\int_{B^{*}_{(k,\alpha)}}\left|\psi_{(k,\alpha)}\left(y\right)F\left(f,g\right)\left(y\right)\right|\\&\qquad\times\left(\frac{1}{\rho\left(x_{(k,\alpha)}\right)^{d}}\int_{B^{**}_{(k,\alpha)}}\omega\left(x\right)dx\right)dy\\&\quad\le C\int_{B^{*}_{(k,\alpha)}}\left|\psi_{(k,\alpha)}\left(y\right)F\left(f,g\right)\left(y\right)\right|\omega\left(y\right)dy\\&\quad\le C\left\|\psi_{(k,\alpha)}F(f,g)\right\|_{L^{1}(\omega)},\end{aligned}\tag{52}$$

and hence

$$J_{1}\le C\sum_{(k,\alpha)}\left\|\psi_{(k,\alpha)}F(f,g)\right\|_{L^{1}(\omega)}\le C\left\|f\right\|_{L^{p}(\omega)}\left\|g\right\|_{L^{q}(\omega)}.\tag{53}$$

To estimate J_2, we denote

$$\begin{aligned}&J_{2}\\&\quad\le\sum_{(k,\alpha)}\int_{B^{**}_{(k,\alpha)}}\sup_{0<t<\rho(x_{(k,\alpha)})^{2}}\left|\left(T^{\mathscr{L}}_{t}-H_{t}\right)\left(\psi_{(k,\alpha)}F\left(f,g\right)\right)\left(x\right)\right|\\&\qquad\times\omega\left(x\right)dx\\&\quad+\sum_{(k,\alpha)}\int_{B^{**}_{(k,\alpha)}}\sup_{0<t<\rho(x_{(k,\alpha)})^{2}}\left|H_{t}\left(\psi_{(k,\alpha)}\left(T^{-}_{1}\left(f,g\right)\right.\right.\right.\\&\qquad\left.\left.\left.+T^{-}_{2}\left(f,g\right)\right)\left(x\right)\right)\right|\\&\qquad\times\omega\left(x\right)dx\\&\quad=J_{21}+J_{22}.\end{aligned}\tag{54}$$

For $x\in B^{**}_{k,\alpha}$ and $y\in\operatorname{supp}\psi_{(k,\alpha)}\subset B^{*}_{(k,\alpha)}$, by Lemma 6 we have $|x-y|<6\rho(x_{(k,\alpha)})\sim\rho(x)$; then the estimate (24) gives

$$\begin{aligned}&\int_{B^{**}_{(k,\alpha)}}\sup_{0<t<\rho(x_{(k,\alpha)})^{2}}\left|\left(T^{\mathscr{L}}_{t}-H_{t}\right)\left(\psi_{(k,\alpha)}F\left(f,g\right)\right)\left(x\right)\right|\omega\left(x\right)dx\\&\quad\le C\int_{B^{**}_{(k,\alpha)}}\int_{B^{*}_{(k,\alpha)}}\frac{1}{\rho(x)^{\sigma}\left|x-y\right|^{d-\sigma}}\\&\qquad\times\left|\psi_{(k,\alpha)}\left(y\right)F\left(f,g\right)\left(y\right)\right|dy\omega\left(x\right)dx\\&\quad\le C\int_{B^{*}_{(k,\alpha)}}\left|\psi_{(k,\alpha)}\left(y\right)F\left(f,g\right)\left(y\right)\right|\\&\qquad\times\left(\int_{B^{**}_{(k,\alpha)}}\frac{\omega\left(x\right)dx}{\rho\left(x_{(k,\alpha)}\right)^{\sigma}\left|x-y\right|^{d-\sigma}}\right)dy.\end{aligned}\tag{55}$$

For the inner integral, by properties of weight, we have

$$\begin{aligned}
&\int_{B^{**}_{(k,\alpha)}} \frac{\omega(x)}{\rho(x_{(k,\alpha)})^{\sigma}|x-y|^{d-\sigma}}dx \\
&\quad\le \int_{|x-y|<8\rho(x_{(k,\alpha)})} \frac{\omega(x)}{\rho(x_{(k,\alpha)})^{\sigma}|x-y|^{d-\sigma}}dx \\
&\quad\le \sum_{j=-2}^{\infty}\int_{|x-y|\sim 2^{-j}\rho(x_{(k,\alpha)})} \frac{\omega(x)}{\rho(x_{(k,\alpha)})^{\sigma}|x-y|^{d-\sigma}}dx \\
&\quad\le C\sum_{j=-2}^{\infty}\frac{1}{(2^{\sigma})^{j}}\frac{\omega\left(B\left(y,2^{-j}\rho(x_{(k,\alpha)})\right)\right)}{\left(2^{-j}\rho(x_{(k,\alpha)})\right)^{d}} \le C\omega(y),
\end{aligned} \tag{56}$$

which yield

$$J_{21} \le C\sum_{(k,\alpha)}\|\psi_{(k,\alpha)}F(f,g)\|_{L^1(\omega)} \le C\|f\|_{L^p(\omega)}\|g\|_{L^q(\omega)}. \tag{57}$$

Now let us estimate J_{22}. For $t>0$, let $\Gamma_0=\{y\in\mathbb{R}^d : |x-y|<\sqrt{t}\}$ and $\Gamma_n=\{y\in\mathbb{R}^d : 2^{n-1}\sqrt{t}\le|x-y|<2^n\sqrt{t}\}$, $n\in\mathbb{N}$. Then

$$\begin{aligned}
&H_t\left(\psi_{(k,\alpha)}T^{-}_m(f,g)\right)(x) \\
&\quad= \sum_{n=0}^{\infty}H_t\left(\psi_{(k,\alpha)}T^{-}_m(f,g)\chi_{\Gamma_n}\right)(x), \quad m=1,2,
\end{aligned} \tag{58}$$

where χ_{Γ_n} denotes the characteristic function of the set Γ_n. Let $\eta(y)\in C_0^{\infty}(\mathbb{R}^d)$ with $0\le\eta\le1$ satisfy $\eta(y)=1$ for $|y|<2$ and $\eta(y)=0$ for $|y|>4$. Set $\eta_0^n(y)=\eta((x-y)/2^n\sqrt{t})$ and $\eta_1^n(y)=1-\eta_0^n(y)$, $n\in\mathbb{N}\cup\{0\}$. We split the operator $T^{-}_m(f,g)$ into four parts:

$$\begin{aligned}
T^{-}_m(f,g) &= T^{-}_m(f,\eta_1^n g)+T^{-}_m(\eta_1^n f,g) \\
&\quad - T^{-}_m(\eta_1^n f,\eta_1^n g)+T^{-}_m(\eta_0^n f,\eta_0^n g), \quad m=1,2.
\end{aligned} \tag{59}$$

We first estimate

$$\int_{B^{**}_{(k,\alpha)}}\sup_{0<t<\rho(x_{(k,\alpha)})^2}\left|H_t\left(\psi_{(k,\alpha)}T^{-}_1(f,\eta_1^n g)\chi_{\Gamma_n}\right)(x)\right|\omega(x)\,dx. \tag{60}$$

Write

$$T^{-}_1(f,\eta_1^n g)=\left(U_1^1 f\right)U_1^2(\eta_1^n g)-\left(U_2^1 f\right)U_2^2(\eta_1^n g), \tag{61}$$

where

$$U_1^1=T_1-\widetilde{T}_1,\quad U_2^1=T_2-\widetilde{T}_2,\quad U_1^2=T_2,\quad U_2^2=T_1. \tag{62}$$

Then we have

$$\begin{aligned}
&\left|H_t\left(\psi_{(k,\alpha)}T^{-}_1(f,\eta_1^n g)\chi_{\Gamma_n}\right)(x)\right| \\
&\quad\le \sum_{l=1}^{2}\int_{\Gamma_n}H_t(x-y)\left|U_l^1 f(y)\right| \\
&\qquad\times\left|U_l^2(\eta_1^n g)(y)-U_l^2(\eta_1^n g)(x)\right|dy \\
&\qquad+\sum_{l=1}^{2}\int_{\Gamma_n}H_t(x-y)\left|U_l^1 f(y)\right|\left|U_l^2(\eta_1^n g)(x)\right|dy \\
&\quad\le C2^{nd}e^{-2^{2n-4}}\sum_{l=1}^{2}\frac{1}{\left(2^n\sqrt{t}\right)^d} \\
&\qquad\times\int_{|x-y|<2^n\sqrt{t}}\left|U_l^1 f(y)\right| \\
&\qquad\qquad\times\left|U_l^2(\eta_1^n g)(y)-U_l^2(\eta_1^n g)(x)\right|dy \\
&\qquad+C2^{nd}e^{-2^{2n-4}}\sum_{l=1}^{2}\frac{1}{\left(2^n\sqrt{t}\right)^d} \\
&\qquad\times\int_{|x-y|<2^n\sqrt{t}}\left|U_l^1 f(y)\right|\left|U_l^2(\eta_1^n g)(x)\right|dy,
\end{aligned} \tag{63}$$

where we have used (5) in the last inequality. Since U_l^j $(j,l=1,2)$ are Calderón-Zygmund operators, their kernels satisfy the standard kernel estimate for some $\delta>0$. For $|x-y|\le 2^n\sqrt{t}$,

$$\begin{aligned}
&\left|U_l^2(\eta_1^n g)(y)-U_l^2(\eta_1^n g)(x)\right| \\
&\quad\le\int_{|x-z|\ge 2^{n+1}\sqrt{t}}\frac{|x-y|^{\delta}}{|x-z|^{d+\delta}}\left|(\eta_1^n g)(z)\right|dz \le CM(\eta_1^n g)(x),
\end{aligned} \tag{64}$$

where M is the Hardy-Littlewood maximal operator. Thus,

$$\begin{aligned}
&\left|H_t\left(\psi_{(k,\alpha)}T^{-}_1(f,\eta_1^n g)\chi_{\Gamma_n}\right)(x)\right| \\
&\quad\le C2^{nd}e^{-2^{2n-4}}\sum_{l=1}^{2}\left(M\left(U_l^1 f\right)(x)M(\eta_1^n g)(x)\right. \\
&\qquad\left.+M\left(U_l^1 f\right)(x)\left|U_l^2(\eta_1^n g)(x)\right|\right),
\end{aligned} \tag{65}$$

and hence

$$\begin{aligned}
&\sum_{(k,\alpha)}\int_{B^{**}_{(k,\alpha)}}\sup_{0<t<\rho\left(x_{(k,\alpha)}\right)^2}\left|H_t\left(\psi_{(k,\alpha)}T_1^-\left(f,\eta_1^n g\right)\chi_{\Gamma_n}\right)(x)\right| \\
&\qquad\times\omega(x)\,dx \\
&\le C2^{nd}e^{-2^{2n-4}}\int_{\mathbb{R}^d}\sum_{l=1}^{2}M\left(U_l^1 f\right)(x) \\
&\qquad\times\left(M\left(\eta_1^n g\right)(x)+\left|U_l^2\left(\eta_1^n g\right)(x)\right|\right) \\
&\qquad\times\omega(x)\,dx \\
&\le C2^{nd}e^{-2^{2n-4}}\sum_{l=1}^{2}\left\|M(U_l^1 f)(x)\right\|_{L^p(\omega)} \\
&\qquad\times\left(\|M(g)\|_{L^q(\omega)}+\left\|U_l^2 g\right\|_{L^q(\omega)}\right) \\
&\le C2^{nd}e^{-2^{2n-4}}\|f\|_{L^p(\omega)}\|g\|_{L^q(\omega)}.
\end{aligned}\tag{66}$$

A similar argument shows that

$$\begin{aligned}
&\sum_{(k,\alpha)}\int_{B^{**}_{(k,\alpha)}}\sup_{0<t<\rho(x_{(k,\alpha)})^2}\left|H_t\left(\psi_{(k,\alpha)}T_1^-\left(\eta_1^n f,g\right)\chi_{\Gamma_n}\right)(x)\right|\omega(x)\,dx \\
&\qquad\le C2^{nd}e^{-2^{2n-4}}\|f\|_{L^p(\omega)}\|g\|_{L^q(\omega)}, \\
&\sum_{(k,\alpha)}\int_{B^{**}_{(k,\alpha)}}\sup_{0<t<\rho(x_{(k,\alpha)})^2}\left|H_t\left(\psi_{(k,\alpha)}T_1^-\left(\eta_1^n f,\eta_1^n g\right)\chi_{\Gamma_n}\right)(x)\right| \\
&\qquad\times\omega(x)\,dx \\
&\qquad\le C2^{nd}e^{-2^{2n-4}}\|f\|_{L^p(\omega)}\|g\|_{L^q(\omega)}.
\end{aligned}\tag{67}$$

Finally we consider the term $T_1^-(\eta_0^n f,\eta_0^n g)$. By using (5) and (10) we have

$$\begin{aligned}
&\sup_{0<t<\rho(x_{(k,\alpha)})^2}\left|H_t\left(\psi_{(k,\alpha)}T_1^-\left(\eta_0^n f,\eta_0^n g\right)\chi_{\Gamma_n}\right)(x)\right| \\
&\le\sum_{l=1}^{2}\sup_{0<t<\rho(x_{(k,\alpha)})^2}\int_{\Gamma_n}H_t(x-y)\left|U_l^1\left(\eta_0^n f\right)(y)\right| \\
&\qquad\times\left|\psi_{(k,\alpha)}(y)\right|\left|U_l^2\left(\eta_0^n g\right)(y)\right|dy \\
&\le\sup_{0<t<\rho(x_{(k,\alpha)})^2}\int_{\Gamma_n}H_t(x-y)\left(\int_{\mathbb{R}^d}\left|T_1(y,z)-\widetilde{T}_1(y,z)\right|\right. \\
&\qquad\left.\times\left|\left(\eta_0^n f\right)(z)\right|dz\right) \\
&\qquad\times\left|T_2\left(\eta_0^n g\right)(y)\right|dy \\
&\quad+\sup_{0<t<\rho(x_{(k,\alpha)})^2}\int_{\Gamma_n}H_t(x-y) \\
&\qquad\times\left(\int_{\mathbb{R}^d}\left|T_2(y,z)-\widetilde{T}_2(y,z)\right|\right. \\
&\qquad\left.\times\left|\left(\eta_0^n f\right)(z)\right|dz\right)\left|T_1\left(\eta_0^n g\right)(y)\right|dy \\
&\le Ce^{-2^{2n-4}}\sup_{0<t<\rho(x_{(k,\alpha)})^2}t^{-d/2}\left\|\eta_0^n g\right\|_{L^{p_1}(\mathbb{R}^d)} \\
&\qquad\times\left\{\int_{\Gamma_n}\left(\int_{\mathbb{R}^d}\frac{\left|\left(\eta_0^n f\right)(z)\right|}{\rho(z)^\delta|y-z|^{d-\delta}}dz\right)^{p_1'}dy\right\}^{1/p_1'},
\end{aligned}\tag{68}$$

where $1/p_1'+1/p_1=1$. For $z\in\operatorname{supp}\eta_0^n=\{y\in\mathbb{R}^d:|x-y|\le 2^{n+2}\sqrt{t}\}$, $x\in B^{**}_{(k,\alpha)}$, and $s<\rho(x_{(k,\alpha)})^2$, we have $|z-x_{(k,\alpha)}|\le 2^{n+3}\rho(x_{(k,\alpha)})$. It follows from Lemma 6 that $\rho(z)^{-\delta}\le C2^{n\delta k_0}\rho(x_{(k,\alpha)})^{-\delta}$ for a fixed constant $k_0>0$. Hence, a well-known result for fractional integrals gives

$$\begin{aligned}
&\sup_{0<t<\rho(x_{(k,\alpha)})^2}\left|H_t\left(\psi_{(k,\alpha)}T_1^-\left(\eta_0^n f,\eta_0^n g\right)\chi_{\Gamma_n}\right)(x)\right| \\
&\le C2^{n\delta k_0}e^{-2^{2n-4}}\sup_{0<t<\rho(x_{(k,\alpha)})^2}t^{-d/2}\rho\left(x_{(k,\alpha)}\right)^{-\delta} \\
&\qquad\times\left\|\left|\eta_0^n f\right|*\frac{1}{|\cdot|^{d-\delta}}\right\|_{L^{p_1'}(\mathbb{R}^d)}\left\|\eta_0^n g\right\|_{L^{p_1}(\mathbb{R}^d)} \\
&\le C2^{n\delta k_0}e^{-2^{2n-4}}\sup_{0<t<\rho(x_{(k,\alpha)})^2}\frac{\left(2^n\sqrt{t}\right)^{d+\delta}}{\sqrt{t^d}\rho\left(x_{(k,\alpha)}\right)^\delta}\frac{1}{\left(2^n\sqrt{t}\right)^{d+\delta}} \\
&\qquad\times\left\|\eta_0^n f\right\|_{L^{q_1}(\mathbb{R}^d)}\left\|\eta_0^n g\right\|_{L^{p_1}(\mathbb{R}^d)} \\
&\le C2^{n(d+\delta+\delta k_0)}e^{-2^{2n-4}}M\left(|f|^{q_1}\right)^{1/q_1}(x)\,M\left(|g|^{p_1}\right)^{1/p_1}(x),
\end{aligned}\tag{69}$$

where $1/p_1'=1/q_1-\delta/d$. Since $1/p_1+1/q_1=1+\delta/d>1/p+1/q$, we are always able to choose p_1 and q_1 such that $1<p_1<q$ and $1<q_1<p$. Then we get

$$\begin{aligned}
&\sum_{(k,\alpha)}\int_{B^{**}_{(k,\alpha)}}\sup_{0<t<\rho(x_{(k,\alpha)})^2}\left|H_t\left(\psi_{(k,\alpha)}T_1^-\left(\eta_0^n f,\eta_0^n g\right)\chi_{\Gamma_n}\right)(x)\right| \\
&\qquad\times\omega(x)\,dx \\
&\le C2^{n(d+\delta+\delta k_0)}e^{-2^{2n-4}} \\
&\qquad\times\int_{\mathbb{R}^d}M\left(|f|^{q_1}\right)^{1/q_1}(x)\,M\left(|g|^{p_1}\right)^{1/p_1}(x)\,\omega(x)\,dx \\
&\le C2^{n(d+\delta+\delta k_0)}e^{-2^{2n-4}}
\end{aligned}$$

$$\times\left(\int_{\mathbb{R}^d} M\left(|f|^{q_1}\right)^{p/q_1}(x)\omega(x)dx\right)^{1/p}$$
$$\times\left(\int_{\mathbb{R}^d} M\left(|g|^{p_1}\right)^{q/p_1}(x)\omega(x)dx\right)^{1/q}$$
$$\leq C2^{n(d+\delta+\delta k_0)}e^{-2^{2n-4}}\|f\|_{L^p(\omega)}\|g\|_{L^q(\omega)}. \tag{70}$$

Therefore, we have

$$\begin{aligned}
&\sum_{(k,\alpha)}\int_{B^{**}_{(k,\alpha)}}\sup_{0<t<\rho(x_{(k,\alpha)})^2}\left|H_t\left(\psi_{(k,\alpha)}T_1^-(f,g)\right)(x)\right|\omega(x)\,dx\\
&\leq \sum_{(k,\alpha)}\int_{B^{**}_{(k,\alpha)}}\sup_{0<t<\rho(x_{(k,\alpha)})^2}\sum_{n=0}^{\infty}\left|H_t\left(\psi_{(k,\alpha)}T_1^-(f,g)\chi_{\Gamma_n}\right)(x)\right|\\
&\qquad\times\omega(x)\,dx\\
&\leq C\|f\|_{L^p(\omega)}\|g\|_{L^q(\omega)}\sum_{n=0}^{\infty}2^{n(d+\delta+\delta k_0)}e^{-2^{2n-4}}\\
&\leq C\|f\|_{L^p(\omega)}\|g\|_{L^q(\omega)}.
\end{aligned} \tag{71}$$

Similarly, we get

$$\begin{aligned}
&\sum_{(k,\alpha)}\int_{B^{**}_{(k,\alpha)}}\sup_{0<t<\rho(x_{(k,\alpha)})^2}\left|H_t\left(\psi_{(k,\alpha)}T_2^-(f,g)\right)(x)\right|\omega(x)\,dx\\
&\qquad\leq C\|f\|_{L^p(\omega)}\|g\|_{L^q(\omega)}.
\end{aligned} \tag{72}$$

Then we obtain

$$J_{22}\leq C\|f\|_{L^p(\omega)}\|g\|_{L^q(\omega)}, \tag{73}$$

and we complete the proof of Theorem 2.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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