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Research Article

Solvability of a Quadratic Integral Equation of Fredholm Type with Supremum in Hölder Spaces

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Using the classical Schauder fixed point theorem, we prove the existence of solutions of a quadratic integral equation of Fredholm type with supremum in the space of functions satisfying the Hölder condition.

1. Introduction

Quadratic integral equations arise naturally in applications of real-world problems. For example, problems in the theory of radiative transfer in the theory of neutron transport and in the kinetic theory of gases lead to the quadratic equation

$$x(t) = 1 + tx(t)\int_0^1 \frac{\Phi(s)}{t+s}x(s)\,ds, \tag{1}$$

where Φ is a continuous function defined on the interval $[0,1]$ (see [1–4]).

Equations of this type have been studied by several authors [5–11].

The aim of this paper is to investigate the existence of solutions of the following quadratic integral equation of Fredholm type with supremum:

$$x(t) = p(t) + x(t)\int_0^1 k(t,\tau)\left\{\max_{\eta\in[0,r(\tau)]}|x(\eta)|\right\}d\tau, \quad t\in[0,1]. \tag{2}$$

Differential and integral equations with supremum are adequate models of real-world problems, in which the present state depends significantly on this maximum value on a past time interval [12]. Equations of such kind have been studied in some papers appearing in the literature (see [13–20]).

Our solutions are placed in the space of functions satisfying the Hölder condition. A sufficient condition for the relative compactness in these spaces and the classical Schauder fixed point theorem are the main tools used in our study.

2. Preliminaries

Our starting point in this section is to introduce the space of functions satisfying the Hölder condition and some properties in this space. These properties appear in [21].

Let $[a,b]$ be a closed interval in $\mathbb{R}$; by $C[a,b]$ we denote the space of the continuous functions on $[a,b]$ equipped with the norm $\|x\|_\infty = \sup\{|x(t)| : t\in[a,b]\}$.

For fixed $0<\alpha\leqslant 1$, by $H_\alpha[a,b]$, we will denote the space of the real functions x defined on $[a,b]$ and satisfying the Hölder condition, that is, those functions x for which there exists a constant H_x^α such that

$$|x(t)-x(s)| \leqslant H_x^\alpha |t-s|^\alpha, \tag{3}$$

for all $t,s\in[a,b]$.

It is easily seen that $H_\alpha[a,b]$ forms a linear subspace of $C[a,b]$.

In what follows, for $x \in H_\alpha[a,b]$, by H_x^α, we will denote the least possible constant for which inequality (3) is satisfied. More "precisely," we put

$$H_x^\alpha = \sup\left\{\frac{|x(t)-x(s)|}{|t-s|^\alpha} : t,s \in [a,b],\ t \neq s\right\}. \quad (4)$$

The space $H_\alpha[a,b]$ with $0 < \alpha \leqslant 1$ can be normed under the following norm:

$$\|x\|_\alpha = |x(a)| + \sup\left\{\frac{|x(t)-x(s)|}{|t-s|^\alpha} : t,s \in [a,b],\ t \neq s\right\}, \quad (5)$$

for $x \in H_\alpha[a,b]$.

It is proved in [21] that $(H_\alpha[a,b], \|\cdot\|_\alpha)$ with $0 < \alpha \leqslant 1$ is a Banach space.

Now, we recollect some results about the spaces $H_\alpha[a,b]$ with $0 < \alpha \leqslant 1$ which appear in [21].

Lemma 1. *For $x \in H_\alpha[a,b]$ with $0 < \alpha \leqslant 1$ the following inequality is satisfied:*

$$\|x\|_\infty \leqslant \max\left(1,(b-a)^\alpha\right)\|x\|_\alpha. \quad (6)$$

Lemma 2. *For $0 < \alpha < \gamma \leqslant 1$, one has*

$$H_\gamma[a,b] \subset H_\alpha[a,b] \subset C[a,b]. \quad (7)$$

Moreover, for $x \in H_\gamma[a,b]$ the following inequality holds:

$$\|x\|_\alpha \leqslant \max\left(1,(b-a)^{\gamma-\alpha}\right)\|x\|_\gamma. \quad (8)$$

The following sufficient condition for relative compactness in the spaces $H_\alpha[a,b]$ with $0 < \alpha \leqslant 1$ appears in Example 6 of [21].

Theorem 3. *Suppose that $0 < \alpha < \beta \leqslant 1$ and that A is a bounded subset in $H_\beta[a,b]$ (this means that there exists a constant $M > 0$ such that $|x(t)-x(s)| \leqslant M|t-s|^\beta$ for any $x \in A$ and for any $t,s \in [a,b]$). Then A is a relatively compact subset of $H_\alpha[a,b]$.*

Remark 4. Suppose that $0 < \alpha < \beta \leqslant 1$ and by B_r^β we denote the ball centered at θ and radius r in the space $H_\beta[a,b]$; that is, $B_r^\beta = \{x \in H_\beta[a,b] : \|x\|_\beta \leqslant r\}$. Then B_r^β is compact in the space $H_\alpha[a,b]$.

Proof. In fact, by Theorem 3, since B_r^β is a bounded subset in $H_\beta[a,b]$, B_r^β is a relatively compact subset of $H_\alpha[a,b]$.

Suppose that $(x_n) \subset B_r^\beta$ and $x_n \xrightarrow{\|\cdot\|_\alpha} x$ with $x \in H_\alpha[a,b]$. This means that for $\varepsilon > 0$ we can find $n_0 \in \mathbb{N}$ such that

$$\|x_n - x\|_\alpha \leqslant \varepsilon \quad \text{for any } n \geqslant n_0, \quad (9)$$

or, equivalently

$$|x_n(a) - x(a)| + \sup\left\{\frac{|x_n(t)-x(t)-(x_n(s)-x(s))|}{|t-s|^\alpha} : t,s \in [a,b],\ t \neq s\right\} \leqslant \varepsilon \quad \text{for any } n \geqslant n_0. \quad (10)$$

This implies that $x_n(a) \to x(a)$.

Moreover, if in (10) we put $s = a$, then we get

$$\sup\left\{\frac{|x_n(t)-x(t)-(x_n(a)-x(a))|}{|t-a|^\alpha} : t \in [a,b],\ t \neq a\right\} < \varepsilon \quad \text{for any } n \geqslant n_0. \quad (11)$$

The last inequality implies that

$$|x_n(t)-x(t)-(x_n(a)-x(a))| < \varepsilon|t-a|^\alpha \leqslant \varepsilon(b-a)^\alpha \quad \text{for any } n \geqslant n_0 \text{ and for any } t \in [a,b]. \quad (12)$$

Therefore, for any $n \geqslant n_0$ and any $t \in [a,b]$ and taking into account (10) and (12), we have

$$\begin{aligned}|x_n(t)-x(t)| &\leqslant |(x_n(t)-x(t))-(x_n(a)-x(a))| + |x_n(a)-x(a)| < \varepsilon(b-a)^\alpha + \varepsilon \\ &= \varepsilon\left(1+(b-a)^\alpha\right).\end{aligned} \quad (13)$$

Consequently,

$$\|x_n - x\|_\infty \longrightarrow 0. \quad (14)$$

Next, we will prove that $x \in B_r^\beta$.

In fact, since $(x_n) \subset B_r^\beta \subset H_\beta[a,b]$, we have that

$$\frac{|x_n(t)-x_n(s)|}{|t-s|^\beta} \leqslant r \quad \text{for any } t,s \in [a,b] \text{ with } t \neq s, \quad (15)$$

and, accordingly,

$$|x_n(t)-x_n(s)| \leqslant r|t-s|^\beta \quad \text{for any } t,s \in [a,b]. \quad (16)$$

Letting $n \to \infty$ in the above inequality and taking into account (14), we deduce that

$$|x(t)-x(s)| \leqslant r|t-s|^\beta \quad \text{for any } t,s \in [a,b]. \quad (17)$$

Hence, we get

$$\frac{|x(t)-x(s)|}{|t-s|^\beta} \leqslant r \quad \text{for any } t,s \in [a,b], \quad (18)$$

and this means that $x \in B_r^\beta$.

This proves that B_r^β is a closed subset of $H_\alpha[a,b]$. Thus, B_r^β is a compact subset of $H_\alpha[a,b]$.

This finishes the proof. □

Next, we recall some results appearing in [18].

In what follows, we consider $[a,b] = [0,1]$.

Lemma 5. *Let $r : [0,1] \to [0,1]$ be a continuous and nondecreasing function and $x \in C[0,1]$. Let Gx be the function defined by*

$$(Gx)(t) = \max_{\tau\in[0,r(t)]} |x(\tau)| \quad \text{for } t \in [0,1]. \tag{19}$$

Then $Gx \in C[0,1]$.

Under the above assumption we have the following.

Lemma 6. *Let (x_n) be a sequence in $C[0,1]$ and $x \in C[0,1]$ such that $x_n \to x$ in $C[0,1]$. Then*

$$\|Gx_n - Gx\|_\infty \leqslant \|x_n - x\|_\infty. \tag{20}$$

Finally, we recall Schauder's fixed point theorem.

Theorem 7. *Let Q be a nonempty, convex, and compact subset of a Banach space $(X, \|\cdot\|)$ and let $T : Q \to Q$ be a continuity mapping. Then T has at least one fixed point in Q.*

3. Main Result

In this section we will investigate the solvability of the integral equation (2) in the Hölder spaces.

We will formulate the following assumptions:

(i) $p \in H_\beta[0,1]$ $(0 < \beta \leqslant 1)$;

(ii) $k : [0,1] \times [0,1] \to \mathbb{R}$ is a continuous function such that it satisfies the Hölder condition with exponent β with respect to the first variable; that is, there exists a constant $k_\beta > 0$ such that

$$|k(t,\varrho) - k(s,\varrho)| \leqslant k_\beta |t-s|^\beta, \tag{21}$$

for any $t, s, \varrho \in [0,1]$;

(iii) $r : [0,1] \to [0,1]$ is a continuous and nondecreasing function;

(iv) the following inequality is satisfied:

$$\|p\|_\beta (2K + k_\beta) < \frac{1}{4}, \tag{22}$$

where the constant K is defined by

$$K = \sup\left\{\int_0^1 |k(t,\tau)|\, d\tau : t \in [0,1]\right\} \tag{23}$$

and whose existence is guaranteed by virtue of (ii).

Theorem 8. *Under assumptions (i)–(iv), (2) has at least one solution belonging to the space $H_\alpha[0,1]$, where α is arbitrarily fixed number satisfying $0 < \alpha < \beta$.*

Proof. Let us consider the operator F defined on the space $H_\beta[0,1]$ by

$$(Fx)(t) = p(t) + x(t)\int_0^1 k(t,\tau)\left\{\max_{\eta\in[0,r(\tau)]} |x(\eta)|\right\} d\tau, \tag{24}$$

for $t \in [0,1]$.

First, we will prove that F transforms the space $H_\beta[0,1]$ into itself.

In fact, we take an arbitrary function $x \in H_\beta[0,1]$ and $t, s \in [0,1]$ such that $t \neq s$. Then, in view of assumptions (i) and (ii), we get

$$\begin{aligned}
&\frac{|(Fx)(t) - (Fx)(s)|}{|t-s|^\beta} \\
&= \left| p(t) + x(t)\int_0^1 k(t,\tau)\left\{\max_{\eta\in[0,r(\tau)]} |x(\eta)|\right\} d\tau \right. \\
&\qquad \left. - p(s) - x(s)\int_0^1 k(s,\tau)\left\{\max_{\eta\in[0,r(\tau)]} |x(\eta)|\right\} d\tau \right| \\
&\quad \times \left(|t-s|^\beta\right)^{-1} \\
&\leqslant \frac{|p(t) - p(s)|}{|t-s|^\beta} \\
&\quad + \left| x(t)\int_0^1 k(t,\tau)\left\{\max_{\eta\in[0,r(\tau)]} |x(\eta)|\right\} d\tau - x(s) \right. \\
&\qquad \left. \times \int_0^1 k(t,\tau)\left\{\max_{\eta\in[0,r(\tau)]} |x(\eta)|\right\} d\tau \right| \\
&\quad \times \left(|t-s|^\beta\right)^{-1} \\
&\quad + \left| x(s)\int_0^1 k(t,\tau)\left\{\max_{\eta\in[0,r(\tau)]} |x(\eta)|\right\} d\tau \right. \\
&\qquad \left. - x(s)\int_0^1 k(s,\tau)\left\{\max_{\eta\in[0,r(\tau)]} |x(\eta)|\right\} d\tau \right| \\
&\quad \times \left(|t-s|^\beta\right)^{-1} \\
&\leqslant \frac{|p(t) - p(s)|}{|t-s|^\beta} + \frac{|x(t) - x(s)|}{|t-s|^\beta} \\
&\quad \cdot \int_0^1 |k(t,\tau)| \left\{\max_{\eta\in[0,r(\tau)]} |x(\eta)|\right\} d\tau \\
&\quad + \frac{|x(s)| \int_0^1 |k(t,\tau) - k(s,\tau)| \left\{\max_{\eta\in[0,r(\tau)]} |x(\eta)|\right\} d\tau}{|t-s|^\beta} \\
&\leqslant \frac{|p(t) - p(s)|}{|t-s|^\beta} + \frac{|x(t) - x(s)|}{|t-s|^\beta}\|x\|_\infty \\
&\quad \times \int_0^1 |k(t,\tau)|\, d\tau \\
&\quad + \frac{\|x\|_\infty \|x\|_\infty \int_0^1 |k(t,\tau) - k(s,\tau)|\, d\tau}{|t-s|^\beta} \\
&\leqslant \frac{|p(t) - p(s)|}{|t-s|^\beta} + K\|x\|_\infty
\end{aligned}$$

$$\times \frac{|x(t)-x(s)|}{|t-s|^{\beta}} + \frac{\|x\|_{\infty}^{2}\int_{0}^{1} k_{\beta}|t-s|^{\beta}d\tau}{|t-s|^{\beta}}$$
$$\leqslant H_{p}^{\beta} + K\|x\|_{\infty}H_{x}^{\beta} + k_{\beta}\|x\|_{\infty}^{2}. \tag{25}$$

By Lemma 1, $\|x\|_{\infty} \leqslant \|x\|_{\beta}$, and, since $H_{x}^{\beta} \leqslant \|x\|_{\beta}$, from the last inequality it follows that

$$\begin{aligned}\frac{|(Fx)(t)-(Fx)(s)|}{|t-s|^{\beta}} &\leqslant H_{p}^{\beta} + K\|x\|_{\beta}^{2} + k_{\beta}\|x\|_{\beta}^{2}\\ &= H_{p}^{\beta} + \left(K+k_{\beta}\right)\|x\|_{\beta}^{2}.\end{aligned} \tag{26}$$

Therefore,

$$\begin{aligned}&\|Fx\|_{\beta}\\ &= |(Fx)(0)| + \sup\left\{\frac{|(Fx)(t)-(Fx)(s)|}{|t-s|^{\beta}} : t,s\in[0,1],\ t\neq s\right\}\\ &\leqslant |(Fx)(0)| + H_{p}^{\beta} + \left(K+k_{\beta}\right)\|x\|_{\beta}^{2}\\ &\leqslant \left|p(0)\right| + |x(0)|\int_{0}^{1}|k(0,\tau)|\left\{\max_{\eta\in[0,r(\tau)]}\left|x(\eta)\right|\right\}d\tau\\ &\quad + H_{p}^{\beta} + \left(K+k_{\beta}\right)\|x\|_{\beta}^{2}\\ &\leqslant \|p\|_{\beta} + \|x\|_{\infty}\cdot\|x\|_{\infty}\int_{0}^{1}|k(0,\tau)|\,d\tau\\ &\quad + \left(K+k_{\beta}\right)\|x\|_{\beta}^{2} \leqslant \|p\|_{\beta} + K\|x\|_{\infty}^{2} + \left(K+k_{\beta}\right)\|x\|_{\beta}^{2}\\ &\leqslant \|p\|_{\beta} + K\|x\|_{\beta}^{2} + \left(K+k_{\beta}\right)\|x\|_{\beta}^{2}\\ &= \|p\|_{\beta} + \left(2K+k_{\beta}\right)\|x\|_{\beta}^{2} < \infty,\end{aligned} \tag{27}$$

where we have used the fact that $\|p\|_{\beta} = |p(0)| + H_{p}^{\beta}$.

This shows that the operator F transforms $H_{\beta}[0,1]$ into itself.

On the other hand, the inequality

$$\|p\|_{\beta} + \left(2K+k_{\beta}\right)r^{2} \leqslant r \tag{28}$$

is satisfied by the number

$$\begin{aligned}r_{1} &= \frac{1-\sqrt{1-4\|p\|_{\beta}\left(2K+k_{\beta}\right)}}{2\left(2K+k_{\beta}\right)},\\ r_{2} &= \frac{1+\sqrt{1-4\|p\|_{\beta}\left(2K+k_{\beta}\right)}}{2\left(2K+k_{\beta}\right)},\end{aligned} \tag{29}$$

which are positive by virtue of (iv), and, consequently, from (27) we infer that F transforms the ball $B_{r_0}^{\beta} = \{x\in H_{\beta}[0,1] : \|x\|_{\beta}\leqslant r_0\}$ into itself, for any $r_0\in[r_1,r_2]$.

Therefore, $F : B_{r_0}^{\beta} \to B_{r_0}^{\beta}$, where $r_1\leqslant r_0\leqslant r_2$.

By Theorem 3 and Remark 4, $B_{r_0}^{\beta}$ is a compact subset in the space $H_{\alpha}[0,1]$ for any $0<\alpha<\beta\leqslant 1$.

Next, we will prove that the operator F is continuous on $B_{r_0}^{\beta}$, where in $B_{r_0}^{\beta}$ we consider the induced norm by $\|\cdot\|_{\alpha}$, where $0<\alpha<\beta\leqslant 1$.

To this end fix $x\in B_{r_0}^{\beta}$ and $\varepsilon>0$. Suppose that $y\in B_{r_0}^{\beta}$ and $\|x-y\|_{\alpha}\leqslant\delta$, where $\delta\leqslant\varepsilon/2(2K+3k_{\beta})r_0$.

Then, for any $t,s\in[0,1]$ with $t\neq s$, we have

$$\begin{aligned}&\frac{\left|\left[(Fx)(t)-(Fy)(t)\right]-\left[(Fx)(s)-(Fy)(s)\right]\right|}{|t-s|^{\alpha}}\\ &= \left|\left[x(t)\int_{0}^{1}k(t,\tau)\left\{\max_{\eta\in[0,r(\tau)]}\left|x(\eta)\right|\right\}d\tau\right.\right.\\ &\qquad \left.- y(t)\int_{0}^{1}k(t,\tau)\left\{\max_{\eta\in[0,r(\tau)]}\left|y(\eta)\right|\right\}d\tau\right]\\ &\quad -\left[x(s)\int_{0}^{1}k(s,\tau)\left\{\max_{\eta\in[0,r(\tau)]}\left|x(\eta)\right|\right\}d\tau\right.\\ &\qquad \left.\left.- y(s)\int_{0}^{1}k(s,\tau)\left\{\max_{\eta\in[0,r(\tau)]}\left|y(\eta)\right|\right\}d\tau\right]\right|\\ &\quad \times\left(|t-s|^{\alpha}\right)^{-1}\\ &= \left|\left[x(t)\int_{0}^{1}k(t,\tau)\left\{\max_{\eta\in[0,r(\tau)]}\left|x(\eta)\right|\right\}d\tau\right.\right.\\ &\qquad \left.- y(t)\int_{0}^{1}k(t,\tau)\left\{\max_{\eta\in[0,r(\tau)]}\left|x(\eta)\right|\right\}d\tau\right]\\ &\quad +\left[y(t)\int_{0}^{1}k(t,\tau)\left\{\max_{\eta\in[0,r(\tau)]}\left|x(\eta)\right|\right\}d\tau\right.\\ &\qquad \left.- y(t)\int_{0}^{1}k(t,\tau)\left\{\max_{\eta\in[0,r(\tau)]}\left|y(\eta)\right|\right\}d\tau\right]\\ &\quad -\left[x(s)\int_{0}^{1}k(s,\tau)\left\{\max_{\eta\in[0,r(\tau)]}\left|x(\eta)\right|\right\}d\tau\right.\\ &\qquad \left.- y(s)\int_{0}^{1}k(s,\tau)\left\{\max_{\eta\in[0,r(\tau)]}\left|x(\eta)\right|\right\}d\tau\right]\\ &\quad -\left[y(s)\int_{0}^{1}k(s,\tau)\left\{\max_{\eta\in[0,r(\tau)]}\left|x(\eta)\right|\right\}d\tau\right.\\ &\qquad \left.\left.- y(s)\int_{0}^{1}k(s,\tau)\left\{\max_{\eta\in[0,r(\tau)]}\left|y(\eta)\right|\right\}d\tau\right]\right|\\ &\quad \times\left(|t-s|^{\alpha}\right)^{-1}\\ &= \left|\left[x(t)-y(t)\right]\int_{0}^{1}k(t,\tau)\left\{\max_{\eta\in[0,r(\tau)]}\left|x(\eta)\right|\right\}d\tau\right.\\ &\quad + y(t)\int_{0}^{1}k(t,\tau)\left\{\max_{\eta\in[0,r(\tau)]}\left|x(\eta)\right| - \max_{\eta\in[0,r(\tau)]}\left|y(\eta)\right|\right\}d\tau\end{aligned}$$

$$
\begin{aligned}
&- [x(s) - y(s)] \int_0^1 k(s,\tau) \left\{ \max_{\eta\in[0,r(\tau)]} |x(\eta)| \right\} d\tau \\
&- y(s) \int_0^1 k(s,\tau) \left\{ \max_{\eta\in[0,r(\tau)]} |x(\eta)| - \max_{\eta\in[0,r(\tau)]} |y(\eta)| \right\} d\tau \Bigg| \\
&\cdot \frac{1}{|t-s|^\alpha} \\
\leqslant &\Bigg\{ \left| [x(t) - y(t)] - [x(s) - y(s)] \right| \\
&\cdot \left| \int_0^1 k(t,\tau) \left\{ \max_{\eta\in[0,r(\tau)]} |x(\eta)| \right\} d\tau \right| + |x(s) - y(s)| \\
&\cdot \left| \int_0^1 (k(t,\tau) - k(s,\tau)) \left\{ \max_{\eta\in[0,r(\tau)]} |x(\eta)| \right\} d\tau \right| \\
&+ \Bigg| y(t) \int_0^1 k(t,\tau) \Bigg\{ \max_{\eta\in[0,r(\tau)]} |x(\eta)| \\
&\qquad - \max_{\eta\in[0,r(\tau)]} |y(\eta)| \Bigg\} d\tau - y(s) \\
&\quad \times \int_0^1 k(s,\tau) \left\{ \max_{\eta\in[0,r(\tau)]} |x(\eta)| - \max_{\eta\in[0,r(\tau)]} |y(\eta)| \right\} d\tau \Bigg| \Bigg\} \\
&\cdot \frac{1}{|t-s|^\alpha} \\
\leqslant &\frac{\left| [x(t) - y(t)] - [x(s) - y(s)] \right|}{|t-s|^\alpha} \\
&\cdot \|x\|_\infty \int_0^1 |k(t,\tau)| \, d\tau \\
&+ \left[\left| [x(s) - y(s)] - [x(0) - y(0)] \right| \right. \\
&\quad \left. + |x(0) - y(0)| \right] \cdot \|x\|_\infty \\
&\times \int_0^1 \frac{|k(t,\tau) - k(s,\tau)|}{|t-s|^\alpha} d\tau \\
&+ \Bigg| y(t) \int_0^1 k(t,\tau) \left\{ \max_{\eta\in[0,r(\tau)]} |x(\eta)| - \max_{\eta\in[0,r(\tau)]} |y(\eta)| \right\} d\tau \\
&\quad - y(s) \int_0^1 k(t,\tau) \left\{ \max_{\eta\in[0,r(\tau)]} |x(\eta)| - \max_{\eta\in[0,r(\tau)]} |y(\eta)| \right\} d\tau \Bigg| \\
&\cdot \frac{1}{|t-s|^\alpha} \\
&+ \Bigg| y(s) \int_0^1 k(t,\tau) \left\{ \max_{\eta\in[0,r(\tau)]} |x(\eta)| - \max_{\eta\in[0,r(\tau)]} |y(\eta)| \right\} d\tau \\
&\quad - y(s) \int_0^1 k(s,\tau) \left\{ \max_{\eta\in[0,r(\tau)]} |x(\eta)| - \max_{\eta\in[0,r(\tau)]} |y(\eta)| \right\} d\tau \Bigg| \\
&\cdot \frac{1}{|t-s|^\alpha} \\
\leqslant &\|x - y\|_\alpha \|x\|_\infty K \\
&+ \sup_{t,s\in[0,1]} \left| [x(t) - y(t)] - [x(s) - y(s)] \right| \\
&\cdot \|x\|_\infty \cdot \int_0^1 \frac{k_\beta |t-s|^\beta}{|t-s|^\alpha} d\tau + |x(0) - y(0)| \cdot \|x\|_\infty \\
&\times \int_0^1 \frac{k_\beta |t-s|^\beta}{|t-s|^\alpha} d\tau + \frac{|y(t) - y(s)|}{|t-s|^\alpha} \\
&\times \int_0^1 |k(t,\tau)| \left| \left\{ \max_{\eta\in[0,r(\tau)]} |x(\eta)| - \max_{\eta\in[0,r(\tau)]} |y(\eta)| \right\} \right| d\tau \\
&+ |y(s)| \int_0^1 \frac{|k(t,\tau) - k(s,\tau)|}{|t-s|^\alpha} \\
&\qquad \times \left| \left\{ \max_{\eta\in[0,r(\tau)]} |x(\eta)| - \max_{\eta\in[0,r(\tau)]} |y(\eta)| \right\} \right| d\tau \\
\leqslant &K \|x\|_\infty \|x - y\|_\alpha \\
&+ \sup_{t,s\in[0,1]} \left\{ \frac{\left| [x(t) - y(t)] - [x(s) - y(s)] \right|}{|t-s|^\alpha} \cdot |t-s|^\alpha \right\} \\
&\cdot \|x\|_\infty k_\beta |t-s|^{\beta-\alpha} + k_\beta \|x\|_\infty |t-s|^{\beta-\alpha} |x(0) - y(0)| \\
&+ H_y^\alpha K \|x - y\|_\infty + \|y\|_\infty \|x - y\|_\infty \int_0^1 \frac{k_\beta |t-s|^\beta}{|t-s|^\alpha} d\tau \\
\leqslant &K \|x\|_\beta \|x - y\|_\alpha + k_\beta \|x\|_\beta \|x - y\|_\alpha \\
&+ k_\beta \|x\|_\beta \|x - y\|_\alpha + H_y^\alpha K \|x - y\|_\alpha \\
&+ k_\beta \|y\|_\beta \|x - y\|_\alpha \\
\leqslant &\left(K r_0 + 3 k_\beta r_0 \right) \|x - y\|_\alpha \\
&+ K \|y\|_\alpha \|x - y\|_\alpha \leqslant \left(K r_0 + 3 k_\beta r_0 \right) \\
&\times \|x - y\|_\alpha + K \|y\|_\beta \|x - y\|_\alpha \\
\leqslant &\left(2 K r_0 + 3 k_\beta r_0 \right) \|x - y\|_\alpha \\
\leqslant &\left(2 K r_0 + 3 k_\beta r_0 \right) \delta < \frac{\varepsilon}{2}.
\end{aligned}
\tag{30}
$$

On the other hand, we have

$$
\begin{aligned}
&|(Fx)(0) - (Fy)(0)| \\
&= \Bigg| x(0) \int_0^1 k(0,\tau) \left\{ \max_{\eta\in[0,r(\tau)]} |x(\eta)| \right\} d\tau \\
&\qquad - y(0) \int_0^1 k(0,\tau) \left\{ \max_{\eta\in[0,r(\tau)]} |y(\eta)| \right\} d\tau \Bigg|
\end{aligned}
$$

$$\begin{aligned}
&\leqslant \left| x(0) \int_0^1 k(0,\tau) \left\{ \max_{\eta\in[0,r(\tau)]} |x(\eta)| \right\} d\tau \right. \\
&\qquad \left. - x(0) \int_0^1 k(0,\tau) \left\{ \max_{\eta\in[0,r(\tau)]} |y(\eta)| \right\} d\tau \right| \\
&\quad + \left| x(0) \int_0^1 k(0,\tau) \left\{ \max_{\eta\in[0,r(\tau)]} |y(\eta)| \right\} d\tau \right. \\
&\qquad \left. - y(0) \int_0^1 k(0,\tau) \left\{ \max_{\eta\in[0,r(\tau)]} |y(\eta)| \right\} d\tau \right| \\
&\leqslant \left| x(0) \int_0^1 k(0,\tau) \left\{ \max_{\eta\in[0,r(\tau)]} |x(\eta)| - \max_{\eta\in[0,r(\tau)]} |y(\eta)| \right\} d\tau \right| \\
&\quad + \left| [x(0) - y(0)] \int_0^1 k(0,\tau) \left\{ \max_{\eta\in[0,r(\tau)]} |y(\eta)| \right\} d\tau \right| \\
&\leqslant \|x\|_\infty K \|x-y\|_\infty + \|x-y\|_\alpha \|y\|_\infty K \\
&\leqslant \|x\|_\beta K \|x-y\|_\alpha + \|x-y\|_\alpha \|y\|_\beta K \\
&\leqslant 2Kr_0 \|x-y\|_\alpha < 2Kr_0\delta < \frac{\varepsilon}{2}.
\end{aligned} \tag{31}$$

From (30) and (31), we infer that

$$\begin{aligned}
&\|Fx - Fy\|_\alpha \\
&= |(Fx)(0) - (Fy)(0)| \\
&\quad + \sup \left\{ \frac{|[(Fx)(t) - (Fy)(t)] - [(Fx)(s) - (Fy)(s)]|}{|t-s|^\alpha} : \right. \\
&\qquad \left. t,s \in [0,1], t \neq s \right\} \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned} \tag{32}$$

This shows that the operator F is continuous at the point $x \in B_{r_0}^\beta$, where x is arbitrary.

Finally, applying Theorem 7, we deduce that desired result.

Finally, we illustrate our results by presenting an example.

Example 9. Let us consider the following quadratic integral equation:

$$\begin{aligned}
x(t) &= \sqrt{qt+r} + x(t) \\
&\quad \times \int_0^1 \sqrt[3]{mt^2+\tau} \left\{ \max_{\eta\in[0,\tau/(\tau+1)]} |x(\eta)| \right\} d\tau,
\end{aligned} \tag{33}$$

where $t \in [0,1]$ and q, r, and m are positive constants.

Notice that (33) is a particular case of (2), where $p(t) = \sqrt{qt+r}$, $k(t,\tau) = \sqrt[3]{mt^2+\tau}$ and $r(\tau) = \tau/(\tau+1)$.

It is easily seen that

$$|p(t) - p(s)| \leqslant \sqrt{q}|t-s|^{1/2}, \quad \text{for any } t,s \in [0,1], \tag{34}$$

and, consequently, $p(t) = \sqrt{qt+r} \in H_{1/2}[0,1]$.

Moreover, using the inequality proved in [11], we have

$$\begin{aligned}
|k(t,\tau) - k(s,\tau)| &= \left| \sqrt[3]{mt^2+\tau} - \sqrt[3]{ms^2+\tau} \right| \\
&\leqslant \sqrt[3]{m}|t-s|^{2/3},
\end{aligned} \tag{35}$$

for any $t, s, \tau \in [0,1]$. Therefore, assumption (ii) of Theorem 8 is satisfied, since for any $t, s, \tau \in [0,1]$

$$\begin{aligned}
|k(t,\tau) - k(s,\tau)| &\leqslant \sqrt[3]{m}|t-s|^{2/3} \\
&= \sqrt[3]{m}|t-s|^{1/2}|t-s|^{1/6} \leqslant \sqrt[3]{m}|t-s|^{1/2},
\end{aligned} \tag{36}$$

where $k_\beta = \sqrt[3]{m}$.

Moreover, we have that

$$\begin{aligned}
\|p\|_{1/2} &= |p(0)| + \sup \left\{ \frac{|p(t) - p(s)|}{|t-s|^{1/2}} : t,s \in [0,1], t \neq s \right\} \\
&= \sqrt{r} + \sqrt{q}.
\end{aligned} \tag{37}$$

Since $r(\tau) = \tau/(\tau+1)$, the function r is continuous and increasing on $[0,1]$ and $0 \leqslant r(\tau) \leqslant 1/2$ for all $\tau \in [0,1]$. Thus, assumption (iii) of Theorem 8 is satisfied.

In our case, the constant K is given by

$$\begin{aligned}
K &= \sup \left\{ \int_0^1 |k(t,\tau)| \, d\tau : t \in [0,1] \right\} \\
&= \sup \left\{ \int_0^1 \sqrt[3]{mt^2+\tau} d\tau : t \in [0,1] \right\} \\
&= \sup \left\{ \frac{3}{4} \left[\sqrt[3]{(mt^2+1)^4} - \sqrt[3]{(mt^2)^4} \right] : t \in [0,1] \right\} \\
&= \frac{3}{4} \left[\sqrt[3]{(m+1)^4} - \sqrt[3]{m^4} \right].
\end{aligned} \tag{38}$$

In our case, the inequality appearing in assumption (iv) of Theorem 8 takes the form

$$\begin{aligned}
&\|p\|_{1/2} \left(2K + k_\beta\right) \\
&\quad = \left(\sqrt{r} + \sqrt{q}\right) \left(\frac{3}{2} \left[\sqrt[3]{(m+1)^4} - \sqrt[3]{m^4} \right] + \sqrt[3]{m} \right) \\
&\quad < \frac{1}{4}.
\end{aligned} \tag{39}$$

It is easily seen that the above inequality is satisfied when, for example, $q = r = 1/216$ and $m = 1/128$.

Therefore, using Theorem 8, we infer that (33) for $q = r = 1/216$ and $m = 1/128$ has at least one solution in the space $H_\alpha[0,1]$ with $0 < \alpha < 1/2$. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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