Hindawi Publishing Corporation
Journal of Function Spaces
Volume 2014, Article ID 918796, 8 pages
http://dx.doi.org/10.1155/2014/918796

Research Article

A System of Differential Set-Valued Variational Inequalities in Finite Dimensional Spaces

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Received 29 May 2013; Revised 5 November 2013; Accepted 19 November 2013; Published 4 february 2014

Academic Editor: John R. Akeroyd

A system of differential set-valued variational inequalities is introduced and studied in finite dimensional Euclidean spaces. An existence theorem of weak solutions for the system of differential set-valued variational inequalities in the sense of Carathéodory is proved under some suitable conditions. Furthermore, a convergence result on Euler time-dependent procedure for solving the system of differential set-valued variational inequalities is also given.

1. Introduction

For a set-valued mapping $F : R^n \rightrightarrows R^n$ and a nonempty closed convex set K in R^n, the VI(K, F), is to find $u \in K$ and $u^* \in F(u)$ such that $\langle u^*, u' - u\rangle \geq 0$ for all $u' \in K$. Let SOL(K, F) denote the solution set of this problem. We write $\dot{x} := dx/dt$ for the time-derivative of a function $x(t)$. In this paper, we consider the following system of differential set-valued variational inequalities:

$$\begin{gathered}
\dot{x}(t) = f(t, x(t)) + B_1(t, x(t))\, u(t) + B_2(t, x(t))\, v(t), \\
\left\langle G_1(t, x(t)) + F_1(u(t)), u' - u(t)\right\rangle \geq 0, \quad \forall u' \in K, \\
\left\langle G_2(t, x(t)) + F_2\left(v'(t)\right), v' - v(t)\right\rangle \geq 0, \quad \forall v' \in K, \\
x(0) = x_0,
\end{gathered} \tag{1}$$

where $\Omega \equiv [0, T] \times R^m$, $f : \Omega \to R^m$, $B_i : \Omega \to R^{m\times n}$, $G_i : \Omega \to R^n$, and $F_i : R^n \rightrightarrows R^n$ $(i = 1, 2)$ are given mappings.

In [1], Pang and Stewart introduced a class of differential variational inequalities in finite dimensional Euclidean spaces. For some related results, we refer to [2–17]. Recently, the differential variational inequalities have been used in cellular biology (see [18]). In [18], the authors needed two or more variational inequalities to formulate the switching between the metabolic models. Sometimes it is convenient to apply the differential vector variational inequalities in [19] to show the fermentation dynamics. However, when we study the fermentation model (20) in [18], we find that the system (1) in this paper can help us a lot.

In this paper, we establish an existence theorem of weak solutions for the system (1) in the sense of Carathéodory under some suitable conditions. Furthermore, we give a convergence result on Euler time-dependent procedure for solving the system (1).

2. Preliminaries

In this section, we will introduce some basic notations and preliminary results.

In the rest of this paper, we will use the following assumptions (A) and (B).

(A) f, B_1, B_2, G_1, and G_2 are Lipschitz continuous functions on Ω with Lipschitz constants L_f, L_{B_1}, L_{B_2}, L_{G_1}, and L_{G_2}, respectively.

(B) B_1 is bounded on Ω with $\sigma_{B_1} \equiv \sup_{(t,x)\in\Omega}\|B_1(t, x)\| < \infty$; B_2 is bounded on Ω with $\sigma_{B_2} \equiv \sup_{(t,x)\in\Omega}\|B_2(t, x)\| < \infty$.

Definition 1. A set-valued map $F : R^n \rightrightarrows R^n$ is said to be

(i) monotone on a convex set $K \subset R^n$ if for each pair of points $x, y \in K$, and for all $x^* \in F(x)$ and $y^* \in F(y)$, $\langle x^* - y^*, x - y\rangle \geq 0$;

(ii) pseudo monotone on a convex set $K \subset R^n$ if for each pair of points $x, y \in K$, and for all $x^* \in F(x)$ and $y^* \in F(y)$, $\langle y^*, x - y\rangle \geq 0$ implies that $\langle x^*, \ x - y\rangle \geq 0$.

Definition 2. A function $f : \Omega \rightarrow R^n$ (resp., $B : \Omega \rightarrow R^{n\times m}$) is said to be Lipschitz continuous if there exists a constant $L_f > 0$ (resp., $L_B > 0$) such that, for any $(t_1, x), \ (t_2, y) \in \Omega$,

$$\|f(t_1, x) - f(t_2, y)\| \leq L_f(|t_1 - t_2| + \|x - y\|)$$
$$(\text{resp.}, \|B(t_1, x) - B(t_2, y)\| \leq L_B(|t_1 - t_2| + \|x - y\|)). \quad (2)$$

Definition 3. Let X, Y be topological spaces and let $F : X \rightrightarrows Y$ be a set-valued mapping with nonempty values. One says that F is upper semicontinuous at $x_0 \in X$ if and only if, for any neighborhood $\mathcal{N}(F(x_0))$ of $F(x_0)$, there exists a neighborhood $\mathcal{N}(x_0)$ of x_0 such that

$$F(x) \subset \mathcal{N}(F(x_0)), \quad \forall x \in \mathcal{N}(x_0). \quad (3)$$

Lemma 4 (see [1]). *Let $\mathbb{F} : \Omega \rightrightarrows R^m$ be an upper semicontinuous set-valued map with nonempty closed convex values. Suppose that there exists a scalar $\rho_{\mathbb{F}} > 0$ satisfying*

$$\sup\{\|y\| : y \in \mathbb{F}(t, x)\} \leq \rho_{\mathbb{F}}(1 + \|x\|), \quad \forall (t, x) \in \Omega. \quad (4)$$

For every $x^0 \in R^n$, the DI : $\dot{x} \in \mathbb{F}(t, x), x(0) = x^0$ has a weak solution in the sense of Carathéodory.

Lemma 5 (see [1]). *Let $h : \Omega \times R^m \rightarrow R^n$ be a continuous function and let $U : \Omega \rightrightarrows R^m$ be a closed set-valued map such that, for some constant $\eta_U > 0$,*

$$\sup_{u\in U(t,x)} \|u\| \leq \eta_U(1 + \|x\|), \quad \forall (t, x) \in \Omega. \quad (5)$$

Let $v : [0, T] \rightarrow R^n$ be a measurable function and let $x : [0, T] \rightarrow R^n$ be a continuous function satisfying $v(t) \in h(t, x(t), U(t, x(t)))$ for almost all $t \in [0, T]$. There exists a measurable function $u : [0, T] \rightarrow R^m$ such that $u(t) \in U(t, x(t))$ and $v(t) = h(t, x(t), u(t))$ for almost all $t \in [0, T]$.

Lemma 6 (see [20]). *Let $\widehat{m}$ denote the Lebesgue measure on R^n and let $f : R^n \rightarrow R^m$ be a measurable function. Let L be a measurable set in R^n with $\widehat{m}(L) < \infty$. Then, for any $\varepsilon > 0$, there exists a compact set $K \subseteq L$ with $\widehat{m}(L \setminus K) < \varepsilon$ such that the restriction of f to K is continuous.*

Definition 7 (see [21]). An acyclic set is a set whose homology is the same as the homology of the space consisting of just one point. An acyclic map is an upper semicontinuous set-valued map which has compact acyclic values.

In [21], we can find that every homeomorphic image of a compact convex set is an acyclic set.

Lemma 8 (see [1]). *Every acyclic set-valued map $F : X \rightarrow X$ on a compact convex set X has a fixed point: $x \in F(x)$ for some $x \in X$.*

3. Main Results

In this section, we obtain existence theorem for weak solutions of the differential set-valued variational inequality in the sense of Carathéodory. Furthermore, we establish a convergence result for solving differential set-valued variational inequality.

Theorem 9. *Assume that (f, B_1, B_2, G_1, G_2) satisfy conditions (A) and (B) and $F_i : R^n \rightrightarrows R^n$ $(i = 1, 2)$ are upper semicontinuous with nonempty and compact values such that $q_i + F_i$ $(i = 1, 2)$ are pseudo monotone on R^n for each $q_i \in G_i(\Omega)$ $(i = 1, 2)$. If K is a bounded, closed, and convex subset of R^n, then initial-value system* (1) *has a weak solution.*

Proof. From the proofs of Lemmas 3.2, 3.3, and 3.4 and Theorem 3.1 in [19], it is easy to see that the assumption "F is pseudo monotone on R^n" in there should be replaced by the assumption "$q + F$ is pseudo monotone on R^n for each $q \in G(\Omega)$." Since K is a bounded, closed, and convex subset of R^n, it follows from Lemma 3.3 in [19] that $\text{SOL}(K, q_i + F_i)$ $(i = 1, 2)$ are nonempty and bounded. Let $u = (u_1, u_2)$, where $u_i \in \text{SOL}(K, q_i + F_i)$, $(i = 1, 2)$. Then it follows that u is bounded on R^{2n}. Moreover, Lemma 3.4 in [19] shows that $\text{SOL}(K, q_i + F_i)$ $(i = 1, 2)$ are closed and convex for all $q_i \in G_i(\Omega)$. Therefore, $\text{SOL}(K, q_1 + F_2) \times \text{SOL}(K, q_2 + F_2)$ is closed and convex. Let

$$\mathbb{F}(t, x) \equiv \{f(t, x) + B_1(t, x)u_1 + B_2(t, x)u_2 : u_i \in \text{SOL}(K, G_i(t, x) + F_i)\}. \quad (6)$$

We can prove in a similar way as Lemma 6.3 in [1] that $\mathbb{F}$ has linear growth and it is upper semicontinuous on Ω. Now it follows from Lemmas 4 and 5 that system (1) has a weak solution. This completes the proof. □

Remark 10. If $F_i : R^n \rightrightarrows R^n$ $(i = 1, 2)$ are monotone, then it is easy to see that $q_i + F_i$ $(i = 1, 2)$ are pseudo monotone on R^n for each $q_i \in G_i(\Omega)$ $(i = 1, 2)$.

Lemma 11. *Let $G : \Omega \times R^m \rightarrow R^n$ be a continuous function, $F : L^2[0, T] \rightrightarrows L^2[0, T]$ a set-valued function, and $u(t) \in K$ with $u \in L^2[0, T]$. Suppose there exists $u^* \in F(u)$ such that, for any continuous function $\tilde{u} : [0, T] \rightarrow K$, one has*

$$\int_0^T \langle G(t, x(t)) + u^*(t), \tilde{u}(t) - u(t)\rangle\, dt \geq 0. \quad (7)$$

Then, for almost all $t \in [0, T]$, $u(t) \in SOL(K, G(t, x(t)) + F(\cdot))$.

Proof. We assume that the contrary holds. Then there exists a set $E \subset [0, T]$ with $\widehat{m}(E) > 0$ (where $\widehat{m}(E)$ denotes the Lebesgue measure of E such that, for all $t \in E$, $u(t) \notin \text{SOL}(K, G(t, x(t)) + F(\cdot))$. By Lemma 6, we know that there exists a closed subset E_1 of E with $\widehat{m}(E_1) > 0$ such that $u(t)$

and $u^*(t)$ are continuous on E_1, where $u^*(t) \in F(u(t))$. Then there exists a closed subset E_2 of E_1 with $\widehat{m}(E_2) > 0$ and $v_0 \in K$ such that

$$\langle G(t, x(t)) + u^*(t), v_0 - u(t) \rangle < 0, \tag{8}$$

and so

$$\int_{E_2} \langle G(t, x(t)) + u^*(t), v_0 - u(t) \rangle \, dt < 0. \tag{9}$$

Let

$$u_0(t) = \begin{cases} v_0, & t \in E_2, \\ u(t), & t \in [0, T] \setminus E_2. \end{cases} \tag{10}$$

We know that $u_0(t) \in K$ is an integrable function on $[0, T]$. Since the space of continuous functions $C([0, T]; R^m)$ is dense in $L^1([0, T]; R^m)$, we can approximate $u_0(t) \in L^1([0, T]; R^m)$ by continuous functions $\overline{u}(t) \in K$ and obtain that there exists a continuous function $\overline{u}(t)$ such that

$$\int_0^T \langle G(t, x(t)) + u^*(t), \overline{u}(t) - u(t) \rangle \, dt < 0, \tag{11}$$

which contradicts (7). This completes the proof. □

Remark 12. If $u(t)$ is an integrable function satisfying, for almost all $t \in [0, T]$,

$$u(t) \in \mathrm{SOL}(K, G(t, x(t), \cdot) + F(\cdot)), \tag{12}$$

then the integral inequality (7) must hold for any continuous $\widehat{u} : [0, T] \to K$.

Now we begin to design a computational method for solving DVI (1). With $x^{h,0} := x^0$, we compute

$$\begin{gathered} \{x^{h,1}, x^{h,2}, \ldots, x^{h,N_h+1}\} \subset R^n, \\ \{u^{h,1}, u^{h,2}, \ldots, u^{h,N_h+1}\} \subset K, \\ \{v^{h,1}, v^{h,2}, \ldots, v^{h,N_h+1}\} \subset K, \end{gathered} \tag{13}$$

by the recursion, for $i = 0, 1, \ldots, N_h$, where $N_h = (T/h) - 1$,

$$\begin{aligned} x^{h,i+1} = {} & x^{h,i} + h\left[f\left(t_{h,i+1}, \theta x^{h,i} + (1-\theta) x^{h,i+1}\right) \right. \\ & \left. + B_1\left(t_{h,i}, x^{h,i}\right) u^{h,i+1} + B_2\left(t_{h,i}, x^{h,i}\right) v^{h,i+1} \right], \\ u^{h,i+1} & \in \mathrm{SOL}\left(K, G_1\left(t_{h,i+1}, x^{h,i+1}\right) + F_1\right), \\ v^{h,i+1} & \in \mathrm{SOL}\left(K, G_2\left(t_{h,i+1}, x^{h,i+1}\right) + F_2\right); \end{aligned} \tag{14}$$

that is,

$$\begin{aligned} x^{h,i+1} = {} & x^{h,i} + h\left[f\left(t_{h,i+1}, \theta x^{h,i} + (1-\theta) x^{h,i+1}\right) \right. \\ & \left. + B_1\left(t_{h,i}, x^{h,i}\right) u^{h,i+1} + B_2\left(t_{h,i}, x^{h,i}\right) v^{h,i+1} \right], \\ & \left\langle G_1\left(t_{h,i+1}, x^{h,i+1}\right) + F_1\left(u^{h,i+1}\right), u' - u^{h,i+1} \right\rangle \geq 0, \quad \forall u' \in K, \\ & \left\langle G_2\left(t_{h,i+1}, x^{h,i+1}\right) + F_2\left(v^{h,i+1}\right), v' - v^{h,i+1} \right\rangle \geq 0, \quad \forall v' \in K. \end{aligned} \tag{15}$$

Lemma 13. *Let (f, B_1, B_2, G_1, G_2) satisfy conditions (A) and (B). Then there exists an $h_0 > 0$ such that, for any $h \in (0, h_0]$, $(x^{ref}, u, v) \in R^{n+m+m}$ with $\theta \in [0, 1]$ and t, t_{ref} in $[0, T]$, there exists a unique vector x_{uv} satisfying*

$$\begin{aligned} x_{uv} - x^{ref} = {} & h\left[f\left(t, \theta x^{ref} + (1-\theta) x_{uv}\right) + B_1\left(t_{ref}, x^{ref}\right) u \right. \\ & \left. + B_2\left(t_{ref}, x^{ref}\right) v \right]. \end{aligned} \tag{16}$$

Moreover, for any $u, v, u', v' \in R^m$, one has

$$\begin{aligned} \left\| x_{uv} - x_{u'v'} \right\| & \leq \frac{h\sigma_{B_1} \left\| u - u' \right\| + h\sigma_{B_2} \left\| v - v' \right\|}{1 - h(1-\theta) L_f}, \\ \left\| x_{uv} - x^{ref} \right\| & \leq \frac{\rho_f \left(1 + \left\| x^{ref} \right\|\right) + \sigma_{B_1} \|u\| + \sigma_{B_2} \|v\|}{1 - h(1-\theta) \rho_f}. \end{aligned} \tag{17}$$

Proof. It suffices to choose h_0 satisfying

$$0 < h_0 < \min\left\{ \frac{1}{(1-\theta) L_f}, \frac{1}{(1-\theta) \rho_f} \right\}. \tag{18}$$

The right-hand side is taken to be ∞ if $\theta = 1$. Under this choice, consider any tuple $(h, x^{\mathrm{ref}}, u, v, t, t_{\mathrm{ref}})$ as specified. Let

$$\begin{aligned} \mathcal{F}(x) = {} & hf\left(t, \theta x^{\mathrm{ref}} + (1-\theta) x\right) + hB_1\left(t_{\mathrm{ref}}, x^{\mathrm{ref}}\right) u \\ & + hB_2\left(t_{\mathrm{ref}}, x^{\mathrm{ref}}\right) v + x^{\mathrm{ref}}. \end{aligned} \tag{19}$$

Then

$$\begin{aligned} & \left\| \mathcal{F}(x_1) - \mathcal{F}(x_2) \right\| \\ & \quad = \left\| hf\left(t, \theta x^{\mathrm{ref}} + (1-\theta) x_1\right) - hf\left(t, \theta x^{\mathrm{ref}} + (1-\theta) x_2\right) \right\| \\ & \quad \leq hL_f (1-\theta) \left\| x_1 - x_2 \right\|, \end{aligned} \tag{20}$$

with $0 < hL_f(1-\theta) < 1$. This shows that the map $\mathcal{F}$ is contractive and so there exists a unique vector x_{uv} such that

$$\begin{aligned} x_{uv} - x^{\mathrm{ref}} = {} & h\left[f\left(t, \theta x^{\mathrm{ref}} + (1-\theta) x_{uv}\right) + B_1\left(t_{\mathrm{ref}}, x^{\mathrm{ref}}\right) u \right. \\ & \left. + B_2\left(t_{\mathrm{ref}}, x^{\mathrm{ref}}\right) v \right]. \end{aligned} \tag{21}$$

It implies that, for any $(u_1, v_1), (u_2, v_2) \in R^{m \times m}$, there exist $x_{u_1 v_1}$ and $x_{u_2 v_2}$ such that

$$\begin{aligned} x_{u_1 v_1} - x^{\mathrm{ref}} = {} & h\left[f\left(t, \theta x^{\mathrm{ref}} + (1-\theta) x_{u_1 v_1}\right) + B_1\left(t_{\mathrm{ref}}, x^{\mathrm{ref}}\right) u_1 \right. \\ & \left. + B_2\left(t_{\mathrm{ref}}, x^{\mathrm{ref}}\right) v_1 \right], \\ x_{u_2, v_2} - x^{\mathrm{ref}} = {} & h\left[f\left(t, \theta x^{\mathrm{ref}} + (1-\theta) x_{u_2, v_2}\right) + B_1\left(t_{\mathrm{ref}}, x^{\mathrm{ref}}\right) u_2 \right. \\ & \left. + B_2\left(t_{\mathrm{ref}}, x^{\mathrm{ref}}\right) v_2 \right]. \end{aligned} \tag{22}$$

By (22), we have

$$\begin{aligned}&\left\|x_{u_1v_1}-x_{u_2,v_2}\right\|\\&\quad\le hL_f(1-\theta)\left\|x_{u_1v_1}-x_{u_2,v_2}\right\|+h\sigma_{B_1}\|u_1-u_2\|\\&\qquad+h\sigma_{B_2}\|v_1-v_2\|\end{aligned}\tag{23}$$

and so

$$\left\|x_{u_1v_1}-x_{u_2,v_2}\right\|\le\frac{h\sigma_{B_1}\|u_1-u_2\|+h\sigma_{B_2}\|v_1-v_2\|}{1-hL_f(1-\theta)}.\tag{24}$$

Now the Lipschitz continuity of f implies that there exists ρ_f satisfying

$$\|f(t,x)\|\le\rho_f(1+\|x\|).\tag{25}$$

It follows that

$$\begin{aligned}&\left\|x_{uv}-x^{\text{ref}}\right\|\\&\quad=h\left\|f\left(t,\theta x^{\text{ref}}+(1-\theta)x_{uv}\right)+B_1\left(t_{\text{ref}},x^{\text{ref}}\right)u\right.\\&\qquad\left.+B_2\left(t_{\text{ref}},x^{\text{ref}}\right)v\right\|\\&\quad\le h\rho_f\left(1+\left\|\theta x^{\text{ref}}+(1-\theta)x_{uv}\right\|\right)+h\sigma_{B_1}\|u\|+h\sigma_{B_2}\|v\|\\&\quad\le h\rho_f\left(1+(1-\theta)\left\|x_{uv}-x^{\text{ref}}\right\|+\left\|x^{\text{ref}}\right\|\right)\\&\qquad+h\sigma_{B_1}\|u\|+h\sigma_{B_2}\|v\|\end{aligned}\tag{26}$$

and so

$$\left\|x_{uv}-x^{\text{ref}}\right\|\le\frac{h\rho_f\left(1+\left\|x^{\text{ref}}\right\|\right)+h\sigma_{B_1}\|u\|+h\sigma_{B_2}\|v\|}{1-h\rho_f(1-\theta)}.\tag{27}$$

This completes the proof. □

Lemma 14. *Let (f,B_1,B_2,G_1,G_2) satisfy conditions (A) and (B). Suppose that $SOL(K,q_1+F_1)$ and $SOL(K,q_2+F_2)$ satisfy the linear growth properties*

$$\begin{aligned}&\sup\{\|u\|:u\in SOL(K,q_1+F_1)\}\le\rho_1(1+\|q_1\|),\\&\qquad\qquad\forall q_1\in G_1(\Omega),\\&\sup\{\|u\|:u\in SOL(K,q_2+F_2)\}\le\rho_2(1+\|q_2\|),\\&\qquad\qquad\forall q_2\in G_2(\Omega).\end{aligned}\tag{28}$$

Then there exist positive scalars $C_{0x},C_{1x},C_{0u},C_{1u},C_{0v},C_{1v}$, and h_1 such that, for any $h\in(0,h_1]$ and $i=0,1,\dots,N_h$,

$$\begin{aligned}\left\|x^{h,i+1}\right\|&\le C_{0x}+C_{1x}\left\|x^0\right\|,\\\left\|u^{h,i+1}\right\|&\le C_{0u}+C_{1u}\left\|x^0\right\|,\\\left\|v^{h,i+1}\right\|&\le C_{0v}+C_{1v}\left\|x^0\right\|.\end{aligned}\tag{29}$$

Proof. Throughout the proof below, the scalar $h>0$ is taken to be sufficiently small. Let

$$\rho_x=\frac{\rho_f+\sigma_{B_1}+\sigma_{B_2}}{1-h(1-\theta)\rho_f}.\tag{30}$$

It follows from Lemma 13 that

$$\left\|x^{h,i+1}-x^{h,i}\right\|\le h\rho_x\left(1+\left\|x^{h,i}\right\|+\left\|u^{h,i+1}\right\|+\left\|v^{h,i+1}\right\|\right).\tag{31}$$

By the linear growth of solutions to VI, we have

$$\begin{aligned}&\left\|u^{h,i+1}\right\|\\&\quad\le\rho_1\left(1+\left\|G_1\left(t_{h,i+1},x^{h,i+1}\right)\right\|\right)\\&\quad\le\rho_1\left(1+\rho_{G_1}\left(1+\left\|x^{h,i+1}\right\|\right)\right)\\&\quad\le\rho_1\left[1+\rho_{G_1}\left\{1+\left\|x^{h,i}\right\|+h\rho_x\right.\right.\\&\qquad\qquad\left.\left.\times\left(1+\left\|x^{h,i}\right\|+\left\|u^{h,i+1}\right\|+\left\|v^{h,i+1}\right\|\right)\right\}\right]\\&\quad\le\left(\rho_1+\rho_1\rho_{G_1}+h\rho_1\rho_{G_1}\rho_x\right)\\&\qquad+\left(\rho_1\rho_{G_1}+h\rho_1\rho_{G_1}\rho_x\right)\left\|x^{h,i}\right\|+h\rho_1\rho_{G_1}\rho_x\left\|u^{h,i+1}\right\|\\&\qquad+h\rho_1\rho_{G_1}\rho_x\left\|v^{h,i+1}\right\|,\\&\left\|v^{h,i+1}\right\|\\&\quad\le\rho_2\left(1+\left\|G_2\left(t_{h,i+1},x^{h,i+1}\right)\right\|\right)\\&\quad\le\rho_2\left(1+\rho_{G_2}\left(1+\left\|x^{h,i+1}\right\|\right)\right)\\&\quad\le\rho_2\left[1+\rho_{G_2}\left\{1+\left\|x^{h,i}\right\|+h\rho_x\right.\right.\\&\qquad\qquad\left.\left.\times\left(1+\left\|x^{h,i}\right\|+\left\|u^{h,i+1}\right\|+\left\|v^{h,i+1}\right\|\right)\right\}\right]\\&\quad\le\left(\rho_2+\rho_2\rho_{G_2}+h\rho_2\rho_{G_2}\rho_x\right)\\&\qquad+\left(\rho_2\rho_{G_2}+h\rho_2\rho_{G_2}\rho_x\right)\left\|x^{h,i}\right\|+h\rho_2\rho_{G_2}\rho_x\left\|u^{h,i+1}\right\|\\&\qquad+h\rho_2\rho_{G_2}\rho_x\left\|v^{h,i+1}\right\|.\end{aligned}\tag{32}$$

Let

$$\begin{aligned}M_1&=\rho_1+\rho_1\rho_{G_1}+h\rho_1\rho_{G_1}\rho_x, & N_1&=\rho_1\rho_{G_1}\rho_x,\\M_2&=\rho_2+\rho_2\rho_{G_2}+h\rho_2\rho_{G_2}\rho_x, & N_2&=\rho_2\rho_{G_2}\rho_x.\end{aligned}\tag{33}$$

Then, we have

$$\begin{aligned}(1-hN_1)\left\|u^{h,i+1}\right\|&\le M\left(1+\left\|x^{h,i}\right\|\right)+hN_1\left\|v^{h,i+1}\right\|,\\(1-hN_2)\left\|v^{h,i+1}\right\|&\le M\left(1+\left\|x^{h,i}\right\|\right)+hN_2\left\|u^{h,i+1}\right\|.\end{aligned}\tag{34}$$

Letting $0 < h < \min\{1/N_1, 1/N_2\}$, one has

$$\left\|u^{h,i+1}\right\| \leq \frac{1}{1-hN_1} \times \left[M\left(1+\left\|x^{h,i}\right\|\right)+hN_1 \times \left(\frac{1}{1-hN_2}\left(M\left(1+\left\|x^{h,i}\right\|\right)+hN_2\left\|u^{h,i+1}\right\|\right)\right)\right]. \tag{35}$$

When h is sufficiently small, there exists $\rho_{M_1} > 0$ such that

$$\left\|u^{h,i+1}\right\| \leq \rho_{M_1}\left(1+\left\|x^{h,i}\right\|\right). \tag{36}$$

In a similar way, we can prove that there exists $\rho_{M_2} > 0$ such that

$$\left\|v^{h,i+1}\right\| \leq \rho_{M_2}\left(1+\left\|x^{h,i}\right\|\right). \tag{37}$$

It follows from (31) that

$$\begin{aligned}\left\|x^{h,i+1}-x^{h,i}\right\| &\leq h\rho_x\left(1+\left\|x^{h,i}\right\|+\rho_{M_1}\left(1+\left\|x^{h,i}\right\|\right)+\rho_{M_2}\left(1+\left\|x^{h,i}\right\|\right)\right)\\ &= \left(h\rho_x+h\rho_x\rho_{M_1}+h\rho_x\rho_{M_2}\right)\left(1+\left\|x^{h,i}\right\|\right).\end{aligned} \tag{38}$$

Let

$$\psi_x = \rho_x + \rho_x\rho_{M_1} + \rho_x\rho_{M_2}. \tag{39}$$

Then

$$\left\|x^{h,i+1}-x^{h,i}\right\| \leq h\psi_x\left(1+\left\|x^{h,i}\right\|\right). \tag{40}$$

It follows from Lemma 7.2 in [1] that there exist positive scalars $C_{0x}, C_{1x}, C_{0u}, C_{1u}, C_{0v}, C_{1v}$, and h_1 such that, for any $h \in (0, h_1]$ and $i = 0, 1, \ldots, N_h$,

$$\begin{aligned}\left\|x^{h,i+1}\right\| &\leq C_{0x}+C_{1x}\left\|x^0\right\|,\\ \left\|u^{h,i+1}\right\| &\leq C_{0u}+C_{1u}\left\|x^0\right\|,\\ \left\|v^{h,i+1}\right\| &\leq C_{0v}+C_{1v}\left\|x^0\right\|.\end{aligned} \tag{41}$$

This completes the proof. □

Lemma 15. *Let $K \subset R^n$ be a nonempty, closed, and convex set and let (f, B_1, B_2, G_1, G_2) satisfy conditions (A) and (B). Suppose that the set-valued maps F_1, F_2 are upper semicontinuous with nonempty compact values such that $q_i + F_i$ $(i = 1, 2)$ are pseudo monotone on R^n for each $q_i \in G_i(\Omega)$ $(i = 1, 2)$. For some constant $\rho > 0$, $SOL(K, q_1 + F_1)$ and $SOL(K, q_2 + F_2)$ satisfy the linear growth properties*

$$\sup\left\{\|u\| : u \in SOL\left(K, q_1+F_1\right)\right\} \leq \rho\left(1+\left\|q_1\right\|\right), \quad \forall q_1 \in G_1(\Omega), \tag{42}$$

$$\sup\left\{\|u\| : u \in SOL\left(K, q_2+F_2\right)\right\} \leq \rho\left(1+\left\|q_2\right\|\right), \quad \forall q_2 \in G_2(\Omega). \tag{43}$$

Then there exists a scalar $h_R > 0$ such that, for any $h \in (0, h_R]$ with $\theta \in [0, 1]$ and $x^0 \in R$, there exists $(x^{h,i+1}, u^{h,i+1}, v^{h,i+1})$ satisfying (15) for every $i = 0, 1, \ldots, N_h$.

Proof. Assume that ψ_x is defined by (39). For any scalar $h > 0$ sufficiently small, we define the scalars $\rho_1, \rho_2, \ldots, \rho_{N_h+1}$ by

$$\rho_{i+1} \equiv \left(1+h\psi_x\right)\rho_i + h\psi_x, \quad i = 0, 1, \ldots, N_h, \tag{44}$$

where ρ_0 is arbitrary. By the proof of Lemma 7.2 in [1], we can show that

$$\rho_i \leq e^{T\psi_x}\rho_0 + e^{T\psi_x} - 1, \quad \forall i = 0, 1, \ldots, N_i + 1. \tag{45}$$

Let α denote the quantity on the right-hand side, which depends on ρ_0 but is independent of h. Let $0 < h_R < \min\{h_0, h_1\}$ satisfy

$$h_R \frac{\rho_f(1+\alpha)+\left(\sigma_{B_1}+\sigma_{B_2}\right)\rho\rho_{G_1}(1+2\alpha)}{1-h_R(1-\theta)\rho_f} < \alpha, \tag{46}$$

where h_0 and h_1 are as described in Lemmas 13 and 14, respectively.

Next we show that, for any fixed $h \in (0, h_R]$, there exists a triple $(x^{h,i+1}, u^{h,i+1}, v^{h,i+1})$ satisfying (15) with $\|x^{h,i+1}\| \leq \rho_{i+1}$ for all $i = 0, 1, \ldots, N_h$. Let B_α denote the Euclidean ball in R^n with center at the origin and radius 2α. For any $x \in B_\alpha$, let $S_j(t, x)$ denote the nonempty set $\mathrm{SOL}(K, G_j(t, x) + F_j)$. Since G_j is Lipschitz continuous on Ω, we know that G_j have linear growth on Ω in x; that is, for some positive constants ρ_{G_j} and for all $(t, x) \in \Omega$,

$$\left\|G_j(t, x)\right\| \leq \rho_{G_j}(1+\|x\|). \tag{47}$$

By the linear growth assumption, for any $x \in B_\alpha$, we have

$$\begin{aligned}\sup\left\{\|u\| : u \in S_j(t, x)\right\} &\leq \rho\left(1+\left\|G_j(t, x)\right\|\right)\\ &\leq \rho\left(1+\rho_{G_j}(1+\|x\|)\right)\\ &\leq \rho\left(1+\rho_{G_j}\right)(1+2\alpha), \quad j = 1, 2.\end{aligned} \tag{48}$$

Define mappings S^i from B_α to subset of B_α as follows: for any $x \in B_\alpha$,

$$\begin{aligned}S^i(x) \equiv{} & \left(I - hf\left(t_{h,i+1}, \theta x^{h,i} + (1-\theta)x\right)\right)^{-1}\\ & \times\left[x^{h,i} + hB_1\left(t_{h,i}, x^{h,i}\right)S_1\left(t_{h,i+1}, x\right)\right.\\ & \left.+ hB_2\left(t_{h,i}, x^{h,i}\right)S_2\left(t_{h,i+1}, x\right)\right].\end{aligned} \tag{49}$$

Since F_1 and F_2 are upper semicontinuous with nonempty compact values such that $q_i + F_i$ $(i = 1, 2)$ are pseudo monotone on R^n for each $q_i \in G_i(\Omega)$ $(i = 1, 2)$, it follows from Lemmas 3.3 and 3.4 in [19] that $\mathrm{SOL}(K, G_1(t, x) + F_1)$ and $\mathrm{SOL}(K, G_2(t, x) + F_2)$ are nonempty, closed, and convex sets. By (48), we obtain that $\mathrm{SOL}(K, G_1(t, x) + F_1)$ and

$\mathrm{SOL}(K, G_2(t,x)+F_2)$ are compact and convex. Consider the map

$$(x, y) \longmapsto x^{h,i} + hB_1\left(t_{h,i}, x^{h,i}\right)x + hB_2\left(t_{h,i}, x^{h,i}\right)y. \quad (50)$$

It is easy to see that this map is continuous. Therefore, by the Tychonoff theorem, we know that $S_1(t,x)\times S_2(t,x)$ is compact and so

$$x^{h,i} + hB_1\left(t_{h,i}, x^{h,i}\right)S_1\left(t_{h,i+1}, x\right) + hB_2\left(t_{h,i}, x^{h,i}\right)S_2\left(t_{h,i+1}, x\right) \quad (51)$$

is compact. Since the mapping $(I - hf(t_{h,i+1}, \theta x^{h,i} + (1-\theta)\cdot))^{-1}$ is a homeomorphism for all $h > 0$ sufficiently small, it follows that $S^i(x)$ is a compact acyclic set. We need to show that $S^i(x)$ is a subset of B_α. Let $\tilde{x}$ be an arbitrary element in $S^i(x)$ and let $u \in S_1(t_{h,i+1}, x)$, $v \in S_2(t_{h,i+1}, x)$ be such that

$$\tilde{x} = x^{h,i} + h\left[f\left(t_{h,i+1}, \theta x^i + (1-\theta)\tilde{x}\right) + B_1\left(t_{h,i}, x^{h,i}\right)u + B_2\left(t_{h,i}, x^{h,i}\right)v\right]. \quad (52)$$

From Lemma 13, we have

$$\left\|\tilde{x} - x^{h,i}\right\| \le h\frac{\rho_f\left(1 + \left\|x^{h,i}\right\|\right) + \sigma_{B_1}\|u\| + \sigma_{B_2}\|v\|}{1 - h(1-\theta)\rho_f}. \quad (53)$$

By induction hypothesis and $\|x^{h,i}\| \le \rho_i \le \alpha$, one has

$$\|\tilde{x}\| \le \rho_i + h\frac{\rho_f(1+\rho_i) + \left(\sigma_{B_1} + \sigma_{B_2}\right)\rho\rho_{G_1}(1+2\alpha)}{1 - h(1-\theta)\rho_f} < 2\alpha. \quad (54)$$

Now we need to prove that the solution mapping $S_1(t_{h,i+1}, x)$ is upper semicontinuous. To prove the upper semicontinuity of $S_1(t_{h,i+1}, x)$, it suffices to show that $S_1(t_{h,i+1}, x)$ is closed. Suppose that $\{x_n\} \subset R^n$ is a sequence converging to $x_0 \in R^n$ and $u_n \in S_1(t_{h,i+1}, x_n)$. Then the linear growth condition implies that $\{u_n\}$ is bounded and so it has a convergent subsequence with a limit u_0. Since $u_n \in S_1(t_{h,i+1}, x_n)$, there exists $u'_n \in F_1(u_n)$ such that

$$\left\langle G_1\left(t_{h,i+1}, x_n\right) + u'_n, u' - u_n\right\rangle, \quad \forall u' \in K. \quad (55)$$

Since F_1 is upper semicontinuous on R^n with compact values, it follows that there exists a subsequence of $\{u'_n\}$, denoted again by $\{u'_n\}$, such that $u'_n \to u'_0 \in F_1(u_0)$. Letting $n \to \infty$, we have

$$\left\langle G_1\left(t_{h,i+1}, x_0\right) + u'_0, u' - u_0\right\rangle \ge 0, \quad \forall u' \in K \quad (56)$$

and so $u_0 \in S_1(t_{h,i+1}, x_0)$. It follows that $S_1(t_{h,i+1}, x)$ is closed and so upper semicontinuous. In a similar way, we can prove that $S_2(t_{h,i+1}, x)$ is upper semicontinuous. Thus, we know that $S^i : B_\alpha \to B_\alpha$ is a closed set-valued mapping with compact acyclic values. By Lemma 8, S^i has a fixed point and so there exists a triple $(x^{h,i+1}, u^{h,i+1}, v^{h,i+1})$ satisfying (15). Now we show that $\|x^{h,i+1}\| \le \rho_{i+1}$. In fact, by (40) and Lemma 7.2 in [1], one has

$$\left\|x^{h,i+1}\right\| \le e^{T\psi_x}\left\|x^0\right\| + e^{T\psi_x} - 1. \quad (57)$$

The definition of ρ_{i+1} implies that $\|x^{h,i+1}\| \le \rho_{i+1}$. This completes the proof. □

Let $\hat{x}^h(\cdot)$ be the continuous piecewise linear interpolant of the family $\{x^{h,i+1}\}$, $\hat{u}^h(\cdot)$ the constant piecewise interpolant of the family $\{u^{h,i+1}\}$, and $\hat{v}^h(\cdot)$ the constant piecewise interpolant of the family $\{v^{h,i+1}\}$; that is,

$$\begin{aligned}
\hat{x}^h(t) &= x^{h,i} + \frac{t - t_i}{h}\left(x^{h,i+1} - x^{h,i}\right), \quad \forall t \in \left[t_{h,i}, t_{h,i+1}\right], \\
\hat{u}^h(t) &= u^{h,i+1}, \quad \forall t \in \left(t_i, t_{i+1}\right], \\
\hat{v}^h(t) &= v^{h,i+1}, \quad \forall t \in \left(t_i, t_{i+1}\right],
\end{aligned} \quad (58)$$

for $i = 0, 1, \ldots, N_h$.

Theorem 16. *Let (f, B_1, B_2, G_1, G_2) satisfy conditions (A) and (B) and let $K \subset R^n$ be a nonempty, closed, and convex set. Suppose that $\mathrm{SOL}(K, q_1 + F_1)$ and $\mathrm{SOL}(K, q_2 + F_2)$ satisfy the linear growth properties. Then there exists a sequence $\{h_n\} \downarrow 0$ such that $\hat{x}^{h_n} \to \hat{x}$ uniformly on $[0,T]$ and $\hat{u}^{h_n} \to \hat{u}$ weakly in $L^2[0,T]$ with $\hat{v}^{h_n} \to \hat{v}$ weakly in $L^2[0,T]$. Furthermore, assume that $F_1(u) = \psi_1(E_1 u)$, $F_2(v) = \psi_2(E_2 v)$, $E_j \in R^{m\times m}$, $j = 1, 2$ and*

$$\psi_j : L^2\left([0,T], R^m\right) \rightrightarrows L^2\left([0,T], R^m\right), \quad j = 1, 2 \quad (59)$$

are upper semicontinuous set-valued mappings with nonempty compact values and there exist constants C_1, C_2 such that, for any h sufficiently small,

$$\left\|E_1 u^{h,i+1} - E_1 u^{h,i}\right\| \le hC_1, \qquad \left\|E_2 v^{h,i+1} - E_2 v^{h,i}\right\| \le hC_2. \quad (60)$$

Then the limit $(\hat{x}, \hat{u}, \hat{v})$ is a weak solution of the system (1).

Proof. By (31) and Lemma 14, we deduce that, for $h > 0$ sufficiently small, there exists an $L_{x_0} > 0$, which is independent of h, such that

$$\left\|x^{h,i+1} - x^{h,i}\right\| \le L_{x_0}h, \quad i = 0, 1, \ldots, N_h. \quad (61)$$

It follows from (58) that $\hat{x}^h$ is also Lipschitz continuous on $[0,T]$ and the Lipschitz constant is independent of h. Thus, there exists an $h_0 > 0$ such that the family of functions $\{\hat{x}^h\}$ $(h \in (0, h_0])$ is an equicontinuous family of functions. Let

$$\left\|\hat{x}^h\right\|_{L^\infty} = \sup_{t\in[0,T]}\left\|\hat{x}^h(t)\right\|. \quad (62)$$

From (58) and Lemma 14, we deduce that $\{\hat{x}^h\}$ is uniformly bounded. By using the Arzelá-Ascoli theorem, there exists a sequence $\{h_n\}$ with $h_n \downarrow 0$ such that $\{\hat{x}^{h_n}\}$ converges in the supremum norm to a Lipschitz function $\hat{x}$ on $[0,T]$. Since $\mathrm{SOL}(K, q_1 + F_1)$ and $\mathrm{SOL}(K, q_2 + F_2)$ satisfy the linear growth properties, it follows from Lemma 14 that $\{u^{h,i+1}\}$ is uniformly bounded in the L^∞ norm on $[0,T]$. From (58), we know that $\{\hat{u}^h\}$ is uniformly bounded in the L^∞ norm on $[0,T]$, which means that there exists a scalar $\gamma > 0$ such that

$$\left\|\hat{u}^h\right\|_{L^\infty} \leq \gamma. \tag{63}$$

Since $L^2[0,T]$ is a reflective Banach space, every bounded sequence has a weakly convergent subsequence and so there is a sequence $\{h_n\} \downarrow 0$ such that $\hat{u}^{h_n} \to \hat{u}$ weakly in $L^2[0,T]$. In a similar way, we obtain that $\hat{v}^{h_n} \to \hat{v}$ weakly in $L^2[0,T]$.

Next, we show that $(\hat{x}, \hat{u}, \hat{v})$ is a weak solution of the system (1). By Lemma 11, it is sufficient to prove the following three assertions:

(i) for any $0 \leq s \leq t \leq T$

$$\begin{aligned} \hat{x}(t) - \hat{x}(s) &= \int_s^t \left[f(\tau, \hat{x}(\tau)) + B_1(\tau, \hat{x}(\tau))\,\hat{u}(\tau) \right. \\ &\quad \left. + B_2(\tau, \hat{x}(\tau))\,\hat{v}(\tau) \right] d\tau; \end{aligned} \tag{64}$$

(ii) there exist $u_0^* \in F_1(\hat{u})$ and $v_0^* \in F_2(\hat{v})$ such that, for all continuous functions: $u : [0,T] \to K$,

$$\begin{aligned} &\int_0^T \left\langle G_1(t, \hat{x}(t)) + u_0^*(t), u(t) - \hat{u}(t) \right\rangle dt \geq 0, \\ &\int_0^T \left\langle G_2(t, \hat{x}(t)) + u_0^*(t), u(t) - \hat{v}(t) \right\rangle dt \geq 0; \end{aligned} \tag{65}$$

(iii) the initial condition $\hat{x}(0) = x_0$.

Since

$$\begin{aligned} x^{h,i+1} - x^{h,i} &= h\left[f\left(t_{h,i+1}, \theta x^{h,i} + (1-\theta)\, x^{h,i+1}\right) \right. \\ &\qquad \left. + B_1\left(t_{h,i}, x^{h,i}\right) u^{h,i+1} + B_2\left(t_{h,i}, x^{h,i}\right) v^{h,i+1} \right] \\ &= \int_{t_{h,i}}^{t_{h,i+1}} \left[f\left(\tau, \hat{x}^h(\tau)\right) + B_1\left(\tau, \hat{x}^h(\tau)\right) u^{h,i+1} \right. \\ &\qquad \left. + B_2\left(\tau, \hat{x}^h(\tau)\right) v^{h,i+1} \right] d\tau + h^2 \xi, \end{aligned} \tag{66}$$

where

$$\|\xi\| \leq L_f + L_f L_x + L_{B_1} \psi_u + L_{B_2} \psi_u, \tag{67}$$

L_x and ψ_u are the same as described in Theorem 7.1 in [1]; it follows that, for any $0 \leq s \leq t \leq T$,

$$\begin{aligned} x^h(t) - x^h(s) &= \int_s^t \left[f\left(\tau, \hat{x}^h(\tau)\right) + B_1\left(\tau, \hat{x}^h(\tau)\right) \hat{u}^h(\tau) \right. \\ &\qquad \left. + B_2\left(\tau, \hat{x}^h(\tau)\right) \hat{v}^h(\tau) \right] d\tau + O(h). \end{aligned} \tag{68}$$

By a similar proof to that in Theorem 7.1 of [1], we can obtain that

$$\begin{aligned} &\lim_{h\to 0} \int_s^t f\left(\tau, \hat{x}^h(\tau)\right) d\tau = \int_s^t f(\tau, \hat{x}(\tau))\, d\tau; \\ &\lim_{h\to 0} \int_s^t B_1\left(\tau, \hat{x}^h(\tau)\right) \hat{u}^h(\tau)\, d\tau = \int_s^t B_1(\tau, \hat{x}(\tau))\, \hat{u}(\tau)\, d\tau; \\ &\lim_{h\to 0} \int_s^t B_2\left(\tau, \hat{x}^h(\tau)\right) \hat{v}^h(\tau)\, d\tau = \int_s^t B_2(\tau, \hat{x}(\tau))\, \hat{v}(\tau)\, d\tau. \end{aligned} \tag{69}$$

Noting the proof of Theorem 7.1 in [1], we have $\hat{x}^{h_n} \to \hat{x}$ and $E_1\hat{u}^{h_n} \to E_1\hat{u}$ as $n \to \infty$. Let $u_n^* \in \psi_1(E_1\hat{u}^{h_n})$. Since ψ_1 is upper semicontinuous with nonempty compact values, there exists a subsequence of $\{u_n^*\}$, denoted again by $\{u_n^*\}$, such that $u_n^* \to u_0^*$ with $u_0^* \in \psi_1(E_1\hat{u})$. This implies that, for any continuous functions: $\tilde{u} : [0,T] \to K$,

$$\begin{aligned} &\lim_{n\to\infty} \int_0^T \left\langle G_1\left(t, \hat{x}^{h_n}(t)\right) + u_n^*(t), \tilde{u}(t) - \hat{u}^{h_n}(t) \right\rangle dt \\ &\quad = \int_0^T \left\langle G_1(t, \hat{x}) + u_0^*(t), \tilde{u} - \hat{u} \right\rangle dt. \end{aligned} \tag{70}$$

Then, in a similar way of Theorem 7.1 in [1], we can prove that

$$\int_0^T \left\langle G_1(t, \hat{x}) + u_0^*(t), \tilde{u} - \hat{u} \right\rangle dt \geq 0. \tag{71}$$

The proof in the case $j = 2$ is similar and so we omit it here. This completes the proof. □

Remark 17. Theorem 16 generalizes Theorem 7.1 in [1] from the differential variational inequality to the system of differential set-valued variational inequalities.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

The authors are grateful to the editor and the referees for their valuable comments and suggestions. This work was supported by the National Natural Science Foundation of China (11171237) and the Key Program of NSFC (Grant no. 70831005).

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