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Research Article

Choi-Davis-Jensen Inequalities in Semifinite von Neumann Algebras

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We prove the Choi-Davis-Jensen type submajorization inequalities on semifinite von Neumann algebras for concave functions and convex functions.

1. Introduction

Let $\mathbb{A}$ and $\mathscr{B}$ be C^*-algebras, and let Φ be a linear map between $\mathscr{A}$ and $\mathscr{B}$. It is said to be positive, if for all positive operators $A \in \mathscr{A}$ we have $\Phi(A) \geq 0$. If for all strictly positive operators $A \in \mathscr{A}$ $(A > 0)$, it follows that $\Phi(A)$ is strictly positive, then Φ is said to be strictly positive. Φ is called unital if $\Phi(1) = 1$, where 1 denotes the unities of the algebras.

Davis [1] and Choi [2] showed that if Φ is a unital positive linear map on $B(\mathbb{H})$ and if f is an operator convex function on an interval I, then the so-called Choi-Davis-Jensen inequality

$$f(\Phi(A)) \leq \Phi(f(A)) \tag{1}$$

holds for every self-adjoint operator A on $\mathbb{H}$ whose spectrum is contained in I, where $B(\mathbb{H})$ is the C^*-algebra of all bounded linear operators on Hilbert space $\mathbb{H}$. Khosravi et al. [3] proved that (1) still holds for positive linear map $\Phi : \mathscr{A} \to B(\mathbb{H})$ with $0 < \Phi(1) \leq I$. Antezana et al. [4] obtained the following type of Choi-Davis-Jensen inequality. Let $\mathscr{A}, \mathscr{B}$ be unital C^*-algebras, $\Phi : A \to B$ a positive unital linear map, f a convex function defined on an open interval I, and $A \in \mathscr{A}$ such that A is self-adjoint and $\sigma(a) \subset I$. If $\mathscr{B}$ is a von Neumann algebra and f is monotone, then $f(\Phi(A)) \preceq \Phi(f(A))$ (spectral preorder). One can find some related results to these topics in [5–8].

In [3], the authors proved the following refinement of the Choi-Davis-Jensen inequality: let $\Phi_1, \ldots, \Phi_n$ be strictly positive linear maps from a unital C^*-algebra $\mathscr{A}$ into a unital C^*-algebra $\mathscr{B}$ and let $\Phi = \sum_{j=1}^{n} \Phi_j$ be unital. If f is an operator convex function on an interval I, then for every self-adjoint operator $A \in \mathscr{A}$ with spectrum contained in I,

$$\begin{aligned} f(\Phi(A)) &\leq \sum_{j=1}^{n} \Phi_j(1)^{1/2} \\ &\quad \cdot f\left(\Phi_j(1)^{-1/2} \Phi_j(A) \Phi_j(1)^{-1/2}\right) \Phi_j(1)^{1/2} \\ &\leq \Phi(f(A)). \end{aligned} \tag{2}$$

The purpose of this paper is to extend (2) for measurable operators and for convex functions. Let $\mathscr{N}$, $\mathscr{M}$ be semifinite von Neumann algebras, $\Phi_1, \ldots, \Phi_n$ positive linear continuous maps from $L_0(\mathscr{N})$ into $L_0(\mathscr{M})$ such that $\Phi_1|_{\mathscr{N}}, \ldots, \Phi_n|_{\mathscr{N}}$ are positive linear maps from $\mathscr{N}$ into $\mathscr{M}$, and $\Phi = \sum_{j=1}^{n} \Phi_j$ unital, and let $x_k \in L_0(\mathscr{N})^+$ $(k = 1, 2, \ldots, n)$. We will prove

$$\sum_{k=1}^{n} \Phi_k(f(x_k)) \preccurlyeq f\left(\sum_{k=1}^{n} \Phi_k(x_k)\right) \tag{3}$$

for any concave function $f : [0, \infty) \to [0, \infty)$ and

$$g\left(\sum_{k=1}^{n} \Phi_k(x_k)\right) \preccurlyeq \sum_{k=1}^{n} \Phi_k(g(x_k)) \tag{4}$$

for any convex function $g : [0,\infty) \to [0,\infty)$ with $g(0) = 0$, where "$\preccurlyeq$" mains submajorization.

This paper is organized as follows. Section 2 contains some preliminary definitions. In Section 3, we prove the main result and related results.

2. Preliminaries

We use standard notions from theory of noncommutative L_p-spaces. Our main references are [9, 10] (see also [9] for more historical references). Throughout the paper, let $\mathscr{M}$ be a semifinite von Neumann algebra acting on a Hilbert space $\mathbb{H}$ with a normal semifinite faithful trace τ. Let $L_0(\mathscr{M})$ denote the topological $*$-algebra of measurable operators with respect to $(\mathscr{M},\tau)$. The topology of $L_0(\mathscr{M})$ is determined by the convergence in measure. The trace τ can be extended to the positive cone $L_0^+(\mathscr{M})$ of $L_0(\mathscr{M})$:

$$\tau(x) = \int_0^\infty \lambda \, d\tau\left(e_\lambda(x)\right), \tag{5}$$

where $x = \int_0^\infty \lambda \, de_\lambda(x)$ is the spectral decomposition of x.

For $x \in L_0(\mathscr{M})$ we define

$$\begin{aligned} \lambda_s(x) &= \tau\left(e_s^\perp(|x|)\right) \quad (s > 0), \\ \mu_t(x) &= \inf\{s > 0 : \lambda_s(x) \le t\} \quad (t > 0), \end{aligned} \tag{6}$$

where $e_s^\perp(|x|) = e_{(s,\infty)}(|x|)$ is the spectral projection of $|x|$ associated with the interval (s,∞). The function $s \mapsto \lambda_s(x)$ is called the *distribution function* of x and $\mu_t(x)$ is the *generalized s-number* of x. We will denote simply by $\lambda(x)$ and $\mu(x)$ the functions $s \mapsto \lambda_s(x)$ and $t \mapsto \mu_t(x)$, respectively. It is easy to check that both are decreasing and continuous from the right on $(0,\infty)$. For further information we refer the reader to [11].

If $x, y \in L_0(\mathscr{M})$, then we say that y is submajorised by x (in the sense of Hardy, Littlewood, and Polya) and write $y \preccurlyeq x$ if and only if

$$\int_0^t \mu_s(y)\, ds \le \int_0^t \mu_s(x)\, ds, \quad \forall t > 0. \tag{7}$$

We remark that if $\mathscr{M} = \mathbb{M}_n$ and τ is the standard trace, then

$$\mu_t(x) = s_j(x), \quad t \in [j-1, j), \; j = 1, 2, \ldots \tag{8}$$

and if $x, y \in \mathscr{M}$, then $y \preccurlyeq x$ is equivalent to

$$\sum_{j=1}^k s_j(y) \le \sum_{j=1}^k s_j(x), \quad 1 \le k \le n. \tag{9}$$

For further information we refer the reader to [11–13].

Let x, y be self-adjoint elements of $\mathscr{M}$; we say that x spectrally dominates y, denoted by $y \preceq x$, if $e_{(\alpha,\infty)}(y)$ is equivalent, in the sense of Murray-von Neumann, to a subprojection of $e_{(\alpha,\infty)}(x)$ for every real number α (see [6]). It is clear that if x spectrally dominates y, then y is submajorised by x.

3. Main Results

Lemma 1. *Let $\mathscr{N}$ and $\mathscr{M}$ be semifinite von Neumann algebras. Let Φ be a positive linear continuous map from $L_0(\mathscr{N})$ into $L_0(\mathscr{M})$ such that the restriction of Φ on $\mathscr{N}$ is a unital positive linear map from $\mathscr{N}$ into $\mathscr{M}$.*

(i) *If $f : [0,+\infty) \to [0,+\infty)$ is a concave function, then*

$$\Phi\left(f(x)\right) \preccurlyeq f\left(\Phi(x)\right), \quad \forall x \in L_0(\mathscr{N})^+. \tag{10}$$

(ii) *If $g : [0,+\infty) \to [0,+\infty)$ is a convex function with $g(0) = 0$, then*

$$g\left(\Phi(x)\right) \preccurlyeq \Phi\left(g(x)\right), \quad \forall x \in L_0(\mathscr{N})^+. \tag{11}$$

Proof. (i) We may assume $f(0) = 0$. It implies f is nondecreasing. First assume that $x \in \mathscr{N}^+$. We use same method as in the proof of Theorem 2 [4] (see Remark 3.2 in [4]). Let $a > 0$. If $\xi \in e_{\{t:f(t)\le a\}}(\Phi(x))(\mathscr{H}) \cap e_{(a,\infty)}(\Phi(f(x)))(\mathscr{H})$ with $\|\xi\| = 1$, then $f(\langle\Phi(x)\xi,\xi\rangle) \le a$ and $\langle\Phi(f(x))\xi,\xi\rangle > a$. On the other hand, using Jensen's inequality for the state $\langle\Phi(\cdot)\xi,\xi\rangle$ and nondecreasing concave function f, we get $\langle\Phi(f(x))\xi,\xi\rangle \le f(\langle\Phi(x)\xi,\xi\rangle) \le a$. Therefore $e_{\{t:f(t)\le a\}}(\Phi(x))(\mathscr{H}) \cap e_{(a,\infty)}(\Phi(f(x)))(\mathscr{H}) = \{0\}$. Thus $e_{\{t:f(t)\le a\}}(\Phi(x)) \wedge e_{(a,\infty)}(\Phi(f(x))) = 0$ and

$$\begin{aligned} e_{(a,\infty)}\left(\Phi\left(f(x)\right)\right) &= e_{(a,\infty)}\left(\Phi\left(f(x)\right)\right) \\ &\quad - e_{\{t:f(t)\le a\}}\left(\Phi(x)\right) \\ &\quad \wedge e_{(a,\infty)}\left(\Phi\left(f(x)\right)\right) \\ &\sim e_{\{t:f(t)\le a\}}\left(\Phi(x)\right) \\ &\quad \vee e_{(a,\infty)}\left(\Phi\left(f(x)\right)\right) \\ &\quad - e_{\{t:f(t)\le a\}}\left(\Phi(x)\right) \\ &\le 1 - e_{\{t:f(t)\le a\}}\left(\Phi(x)\right) \\ &= e_{\{t:f(t)>a\}}\left(\Phi(x)\right) \\ &= e_{(a,+\infty)}\left(f\left(\Phi(x)\right)\right); \end{aligned} \tag{12}$$

that is, $\Phi(f(x)) \preceq f(\Phi(x))$. Hence (10) holds.

Now let $x \in L_0(\mathscr{N})^+$. For each $m = 1, 2, \ldots$, observe that $x \wedge m1 \in \mathscr{M}$, and so, using the first case, it follows that

$$\Phi\left(f(x \wedge m1)\right) \preccurlyeq f\left(\Phi(x \wedge m1)\right). \tag{13}$$

Using the functional calculus and Corollary 1.2 in [13] observe that

$$\begin{aligned} x \wedge m1 &\uparrow_m x, \\ f(x \wedge m1) &\uparrow_m f(x) \end{aligned} \tag{14}$$

and so, by continuousness of Φ, it follows that

$$\begin{aligned} \Phi(x \wedge m1) &\uparrow_m \Phi(x), \\ \Phi\left(f(x \wedge m1)\right) &\uparrow_m \Phi\left(f(x)\right). \end{aligned} \tag{15}$$

Using (vi) of Lemma 2.5 in [11] we obtain that

$$\begin{aligned}\int_0^t \mu_s\left(f\left(\Phi\left(x\right)\right)\right)ds &= \int_0^t f\left(\mu_s\left(\Phi\left(x\right)\right)\right)ds \\ &= \lim_{m\to\infty}\int_0^t f\left(\mu_s\left(\Phi\left(x\wedge m1\right)\right)\right)ds \\ &= \lim_{m\to\infty}\int_0^t \mu_s\left(f\left(\Phi\left(x\wedge m1\right)\right)\right)ds \\ &\geq \lim_{m\to\infty}\int_0^t \mu_s\left(\Phi\left(f\left(x\wedge m1\right)\right)\right)ds \\ &= \int_0^t \mu_s\left(\Phi\left(f\left(x\right)\right)\right)ds, \quad \forall t>0;\end{aligned} \tag{16}$$

that is, (10) holds.

(ii) The proof is similar to the proof of (i). □

Theorem 2. *Let $\mathcal{N}$, $\mathcal{M}$ be semifinite von Neumann algebras, $\Phi_1,\ldots,\Phi_n$ positive linear continuous maps from $L_0(\mathcal{N})$ into $L_0(\mathcal{M})$ such that the restriction of Φ_k on $\mathcal{N}$ is a positive linear map from $\mathcal{N}$ into $\mathcal{M}$ $(k=1,2,\ldots,n)$, and $\Phi=\sum_{j=1}^n\Phi_j$ unital.*

(i) *If $f:[0,+\infty)\to[0,+\infty)$ is a concave function, then, for $x_k\in L_0(\mathcal{N})^+$ $(k=1,2,\ldots,n)$,*

$$\sum_{k=1}^n\Phi_k\left(f\left(x_k\right)\right)\preceq f\left(\sum_{k=1}^n\Phi_k\left(x_k\right)\right). \tag{17}$$

(ii) *If $g:[0,+\infty)\to[0,+\infty)$ is a convex function with $g(0)=0$, then, for $x_k\in L_0(\mathcal{N})^+$ $(k=1,2,\ldots,n)$,*

$$g\left(\sum_{k=1}^n\Phi_k\left(x_k\right)\right)\preceq\sum_{k=1}^n\Phi_k\left(g\left(x_k\right)\right). \tag{18}$$

Proof. Let $\mathcal{N}_n$ be the von Neumann algebra:

$$\mathcal{N}_n=\left\{\begin{pmatrix} x_1 & 0 & \cdots & 0\\ 0 & x_2 & \cdots & 0\\ & & \ddots & \\ 0 & 0 & \cdots & x_n\end{pmatrix},\ x_k\in\mathcal{N},\ k=1,2,\ldots,n\right\} \tag{19}$$

on Hilbert space $\mathcal{H}\oplus\cdots\oplus\mathcal{H}$. Define $\Phi:\mathcal{N}_n\to\mathcal{M}$ by

$$\Phi\left(\begin{pmatrix} x_1 & 0 & \cdots & 0\\ 0 & x_2 & \cdots & 0\\ & & \ddots & \\ 0 & 0 & \cdots & x_n\end{pmatrix}\right)=\sum_{k=1}^n\Phi_k\left(x_k\right); \tag{20}$$

then Φ is a unital positive linear map from $\mathcal{N}_n$ into $\mathcal{M}$. By Lemma 1, we obtain desired result. □

Using Theorem 5.3 in [14] and Theorem 2 we obtain the following.

Proposition 3. *Let $\mathcal{N}, M$ be semifinite von Neumann algebras, $\Phi_1,\ldots,\Phi_n$ positive linear continuous maps from $L_0(\mathcal{N})$ into $L_0(\mathcal{M})$ such that the restriction of Φ_k on $\mathcal{N}$ is a positive linear map from $\mathcal{N}$ into $\mathcal{M}$ $(k=1,2,\ldots,n)$, and $\Phi=\sum_{j=1}^n\Phi_j$ unital.*

(i) *If $f:[0,+\infty)\to[0,+\infty)$ is a concave function, then, for $x_k\in\mathcal{N}^+$ $(k=1,2,\ldots,n)$,*

$$\sum_{k=1}^n\Phi_k\left(f\left(x_k\right)\right)\preccurlyeq f\left(\sum_{k=1}^n\Phi_k\left(x_k\right)\right)\preccurlyeq\sum_{k=1}^n f\left(\Phi_k\left(x_k\right)\right). \tag{21}$$

(ii) *If $g:[0,+\infty)\to[0,+\infty)$ is a convex function with $g(0)=0$, then, for $x_k\in\mathcal{N}^+$ $(k=1,2,\ldots,n)$,*

$$\sum_{k=1}^n g\left(\Phi_k\left(x_k\right)\right)\preccurlyeq g\left(\sum_{k=1}^n\Phi_k\left(x_k\right)\right)\preccurlyeq\sum_{k=1}^n\Phi_k\left(g\left(x_k\right)\right). \tag{22}$$

Proposition 4. *Let $\mathcal{N}$, $\mathcal{M}$ be semifinite von Neumann algebras and $\Phi_1,\ldots,\Phi_n$ positive linear continuous maps from $L_0(\mathcal{N})$ into $L_0(\mathcal{M})$ such that the restriction of Φ_k on $\mathcal{N}$ is a positive linear map from $\mathcal{N}$ into $\mathcal{M}$ $(k=1,2,\ldots,n)$. Suppose $\Phi=\sum_{j=1}^n\Phi_j$ is unital trace-preserving positive linear map from $\mathcal{M}$ into $\mathcal{M}$. If $g:[0,+\infty)\to[0,+\infty)$ is a convex function with $g(0)=0$, then*

$$\begin{aligned}\sum_{k=1}^n g\left(\Phi_k\left(x\right)\right)\preccurlyeq g\left(\sum_{k=1}^n\Phi_k\left(x\right)\right)\preccurlyeq\sum_{k=1}^n\Phi_k\left(g\left(x\right)\right)\\ \preccurlyeq g\left(x\right),\quad\forall x\in\mathcal{M}^+.\end{aligned} \tag{23}$$

Proof. By Corollary 2.9 in [15] we have that $\sum_{j=1}^n\Phi_j$ is a trace-preserving positive contraction. Using Theorem 5.3 in [14], Lemma 3.1 in [16] (it is also holds for the semifinite case), and Theorem 2 we obtain the desired result. □

Corollary 5. *Let $a_k\in\mathcal{M}$ $(k=1,2,\ldots,n)$ and $\sum_{k=1}^n a_k^*a_k=1$.*

(i) *If $f:[0,+\infty)\to[0,+\infty)$ is a concave function, then, for $x_k\in L_0(\mathcal{M})^+$ $(k=1,2,\ldots,n)$,*

$$\sum_{k=1}^n a_k^*f\left(x_k\right)a_k\preccurlyeq f\left(\sum_{k=1}^n a_k^*x_ka_k\right)\preccurlyeq\sum_{k=1}^n f\left(a_k^*x_ka_k\right). \tag{24}$$

(ii) *If $g:[0,+\infty)\to[0,+\infty)$ is a convex function with $g(0)=0$, then*

$$\begin{aligned}\sum_{k=1}^n g\left(a_k^*xa_k\right)\preccurlyeq g\left(\sum_{k=1}^n a_k^*xa_k\right)\preccurlyeq\sum_{k=1}^n a_k^*g\left(x\right)a_k\\ \preccurlyeq g\left(x\right),\quad\forall x\in L_0\left(\mathcal{M}\right)^+.\end{aligned} \tag{25}$$

Let $\mathbb{M}_n$ be von Neumann algebra of $n\times n$ complex matrices, and let $P_1,P_2,\ldots,P_r$ be a family of mutually orthogonal

projections in $\mathbb{C}^n$ such that $\oplus_{j=1}^{r} P_j = I$, where I is unit matrix in $\mathbb{M}_n$. Then the operation of taking A to $\mathscr{C}(A) = \sum_{j=1}^{r} P_j A P_j$ is called a pinching of A. The pinching $\mathscr{C} : \mathbb{M}_n \to \mathbb{M}_n$ is a trace-preserving positive map (see [17, 18]).

Proposition 6. *Let $\mathscr{C} : \mathbb{M}_n \to \mathbb{M}_n$ be a pinching.*

(i) *If $f : [0, +\infty) \to [0, +\infty)$ is a concave function, then*

$$\mathscr{C}\left(f(A)\right) \preccurlyeq f\left(\mathscr{C}(A)\right), \quad \forall A \in \mathbb{M}_n^+. \tag{26}$$

(ii) *If $g : [0, +\infty) \to [0, +\infty)$ is a convex function with $g(0) = 0$, then*

$$g\left(\mathscr{C}(A)\right) \preccurlyeq \mathscr{C}\left(g(A)\right) \preccurlyeq g(A), \quad \forall A \in \mathbb{M}_n^+. \tag{27}$$

Let $x = (x_k)_{k\geq 1}$ be a sequence in $L_0(\mathscr{M})$. Define

$$\operatorname{diag}(x_k) = \begin{pmatrix} x_1 & 0 & \cdots & 0 & \cdots \\ 0 & x_2 & \cdots & 0 & \cdots \\ & & \ddots & & \\ 0 & 0 & \cdots & x_k & \cdots \\ & & & & \ddots \end{pmatrix}. \tag{28}$$

Proposition 7. *Let $x = (x_k)_{k\geq 1}$ be a sequence in $L_0(\mathscr{M})$.*

(i) *If $f : [0, +\infty) \to [0, +\infty)$, where $f(\sqrt{t})$ is a concave function, then*

$$\left\| f\left(\left(\sum_{k\geq 1} |x_k|^2\right)^{1/2}\right)\right\|_p \leq \left(\sum_{k\geq 1} \left\| f\left(|x_k|\right)\right\|^p\right)^{1/p}, \quad \forall 0 < p \leq 1. \tag{29}$$

(ii) *If $g : [0, +\infty) \to [0, +\infty)$, where $g(\sqrt{t})$ is a convex function with $g(0) = 0$, then*

$$\left(\sum_{k\geq 1} \left\| g\left(|x_k|\right)\right\|^p\right)^{1/p} \leq \left\| g\left(\left(\sum_{k\geq 1} |x_k|^2\right)^{1/2}\right)\right\|_p, \quad \forall 1 \leq p < \infty. \tag{30}$$

Proof. (i) Since

$$\begin{aligned} &\begin{pmatrix} |x_1^*|^2 & 0 & \cdots \\ 0 & |x_2^*|^2 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \\ &= \mathscr{C}\left(\begin{pmatrix} |x_1^*|^2 & x_1 x_2^* & \cdots \\ x_2 x_1^* & |x_2^*|^2 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}\right) \\ &= \mathscr{C}\left(\begin{pmatrix} x_1 & 0 & \cdots \\ x_2 & 0 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}\begin{pmatrix} x_1^* & x_2^* & \cdots \\ 0 & 0 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}\right) \end{aligned} \tag{31}$$

and $f(\sqrt{t})^p$ is concave, by Lemma 1, we get

$$\begin{aligned} &\mathscr{C}\left(f^p\left(\left|\begin{pmatrix} x_1^* & x_2^* & \cdots \\ 0 & 0 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}\right|\right)\right) \\ &\preccurlyeq \begin{pmatrix} f\left(|x_1^*|\right)^p & 0 & \cdots \\ 0 & f\left(|x_2^*|\right)^p & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}. \end{aligned} \tag{32}$$

Hence

$$\begin{aligned} &\left\| f\left(\left(\sum_{k\geq 1} |x_k|^2\right)^{1/2}\right)\right\|_p^p \\ &= \tau\left(f^p\left(\left|\begin{pmatrix} x_1 & 0 & \cdots \\ x_2 & 0 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}\right|\right)\right) \\ &= \tau\left(f^p\left(\left|\begin{pmatrix} x_1^* & x_2^* & \cdots \\ 0 & 0 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}\right|\right)\right) \\ &= \tau\left(\mathscr{C}\left(f^p\left(\left|\begin{pmatrix} x_1^* & x_2^* & \cdots \\ 0 & 0 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}\right|\right)\right)\right) \\ &\leq \tau\left(\begin{pmatrix} f\left(|x_1^*|\right)^p & 0 & \cdots \\ 0 & f\left(|x_2^*|\right)^p & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}\right) \\ &= \sum_{k\geq 1} \left\| f\left(|x_k|\right)\right\|^p. \end{aligned} \tag{33}$$

Using the same arguments, we can prove (ii). □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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References

[1] C. Davis, "A Schwarz inequality for convex operator functions," *Proceedings of the American Mathematical Society*, vol. 8, pp. 42–44, 1957.

[2] M. D. Choi, "A Schwarz inequality for positive linear maps on C^*-algebras," *Illinois Journal of Mathematics*, vol. 18, pp. 565–574, 1974.

[3] M. Khosravi, J. S. Aujla, S. S. Dragomir, and M. S. Moslehian, "Refinements of Choi-Davis-Jensen's inequality," *Bulletin of Mathematical Analysis and Applications*, vol. 3, no. 2, pp. 127–133, 2011.

[4] J. Antezana, P. Massey, and D. Stojanoff, "Jensen's inequality for spectral order and submajorization," *Journal of Mathematical Analysis and Applications*, vol. 331, no. 1, pp. 297–307, 2007.

[5] T. Ando, *Topics on Operator Inequalities*, Hokkaido University, Sapporo, Japan, 1978.

[6] L. G. Brown and H. Kosaki, "Jensen's inequality in semi-finite von Neumann algebras," *Journal of Operator Theory*, vol. 23, no. 1, pp. 3–19, 1990.

[7] F. Hansen and G. K. Pedersen, "Jensen's operator inequality," *The Bulletin of the London Mathematical Society*, vol. 35, no. 4, pp. 553–564, 2003.

[8] F. Hansen and G. K. Pedersen, "Jensen's inequality for operators and Lowner's theorem," *Mathematische Annalen*, vol. 258, no. 3, pp. 229–241, 1982.

[9] G. Pisier and Q. Xu, "Noncommutative L_p-spaces," in *Handbook of the Geometry of Banach Spaces*, vol. 2, pp. 1459–1517, 2003.

[10] Q. Xu, *Noncommutative* L_p*-Spaces and Martingale Inequalities*, 2007.

[11] T. Fack and H. Kosaki, "Generalized s-numbers of τ-measure operators," *Pacific Journal of Mathematics*, vol. 123, no. 2, pp. 269–300, 1986.

[12] P. G. Dodds, T. K. Dodds, and B. de Pagter, "Noncommutative Banach function spaces," *Mathematische Zeitschrift*, vol. 201, no. 4, pp. 583–597, 1989.

[13] P. G. Dodds, T. K. Dodds, and B. de Pager, "Noncommutative Köthe duality," *Transactions of the American Mathematical Society*, vol. 339, pp. 717–750, 1993.

[14] P. G. Dodds and F. A. Sukochev, "Submajorisation inequalities for convex and concave functions of sums of measurable operators," *Positivity.*, vol. 13, no. 1, pp. 107–124, 2009.

[15] V. Paulsen, *Completely Bounded Maps and Operator Algebras*, vol. 78 of *Cambridge Studies in Advanced Mathematics*, Cambridge University Press, 2002.

[16] T. N. Bekjan and M. Raikhan, "An Hadamard-type inequality," *Linear Algebra and its Applications*, vol. 443, pp. 228–234, 2014.

[17] R. Bhatia, M. D. Choi, and C. Davis, "Comparing a matrix to its off-diagonal part," *Operator Theory: Advances and Applications*, vol. 40, pp. 151–163, 1989.

[18] J.-C. Bourin and M. Uchiyama, "A matrix subadditivity inequality for $f(A+B)$ and $f(A)+f(B)$," *Linear Algebra and its Applications*, vol. 423, no. 2-3, pp. 512–518, 2007.