

## Research Article

# Weak Precompactness in the Space of Vector-Valued Measures of Bounded Variation

Ioana Ghenciu

Mathematics Department, University of Wisconsin-River Falls, River Falls, WI 54022, USA

Correspondence should be addressed to Ioana Ghenciu; ioana.ghenciu@uwrf.edu

Received 16 October 2014; Revised 15 December 2014; Accepted 18 December 2014

Academic Editor: José Rodríguez

Copyright © 2015 Ioana Ghenciu. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

For a Banach space  $X$  and a measure space  $(\Omega, \Sigma)$ , let  $M(\Omega, X)$  be the space of all  $X$ -valued countably additive measures on  $(\Omega, \Sigma)$  of bounded variation, with the total variation norm. In this paper we give a characterization of weakly precompact subsets of  $M(\Omega, X)$ .

## 1. Introduction

Let  $X$  be a Banach space,  $(\Omega, \Sigma)$  a measure space, and  $M(\Omega, X)$  the space of all  $X$ -valued countably additive measures on  $(\Omega, \Sigma)$  of bounded variation (with the total variation norm). For  $m \in M(\Omega, X)$  we denote by  $|m|$  its variation. For  $\lambda$ , a probability measure on  $(\Omega, \Sigma)$ , we denote  $W(\lambda, X) = \{m \in M(\Omega, X) : |m| \leq \lambda\}$ .

Ulger [1] and Diestel et al. [2] gave a characterization of weakly compact subsets of  $L^1(\mu, X)$ , the Banach space of all  $X$ -valued Bochner integrable functions on a probability space  $(\Omega, \Sigma, \mu)$ . In [3] we gave a characterization of weakly precompact subsets of  $L^1(\mu, X)$ . Randrianantoanina and Saab [4] gave a characterization of relatively weakly compact subsets of  $M(\Omega, X)$ .

In this paper we use results of Talagrand [5], Ulger [6], and techniques of Randrianantoanina and Saab [4] to characterize weakly precompact subsets of  $M(\Omega, X)$ . The characterization is obtained in two steps. In the first step we characterize the weakly precompact subsets of  $W(\lambda, X)$ . We show that a subset  $A$  of  $W(\lambda, X)$  is weakly precompact if and only if for any sequence  $(m_n)$  in  $A$  and for any lifting  $\rho$  of  $L^\infty(\lambda)$  there exists a sequence  $(m'_n)$  with  $m'_n \in \text{co}\{m_i : i \geq n\}$  for each  $n$  such that, for a.e.  $\omega$ , the sequence  $(\rho(m'_n)(\omega))$  is weakly Cauchy. In the second step we show that a subset  $A$  of  $M(\Omega, X)$  is weakly precompact if and only if there is a probability measure  $\lambda$  on  $(\Omega, \Sigma)$  such that, for any sequence

$(m_n)$  in  $A$ , there is a sequence  $(m'_n)$  with  $m'_n \in \text{co}\{m_i : i \geq n\}$  for each  $n$  such that, for any  $\epsilon > 0$ , there is a positive integer  $N$  and a weakly precompact subset  $H_\epsilon$  of  $NW(\lambda, X)$  so that  $\{m'_n : n \geq 1\} \subseteq H_\epsilon + \epsilon B(0)$ , where  $B(0)$  denotes the unit ball of  $M(\Omega, X)$ .

This paper also contains several corollaries of these results. We show that if  $\ell_1 \not\hookrightarrow X^*$ , then a subset  $A$  of  $M(\Omega, X^*)$  is weakly precompact if and only if  $A$  is bounded and  $V(A) = \{|m| : m \in A\}$  is uniformly countably additive.

## 2. Definitions and Notation

Throughout this paper,  $X$  and  $Y$  will denote Banach spaces. The unit ball of  $X$  will be denoted by  $B_X$ . The unit basis of  $\ell_1$  will be denoted by  $(e_n^*)$ , and a continuous linear transformation  $T : X \rightarrow Y$  will be referred to as an operator. The set of all compact operators from  $X$  to  $Y$  will be denoted by  $K(X, Y)$ . The set of all  $w^*$ - $w$  continuous compact operators from  $X^*$  to  $Y$  will be denoted by  $K_{w^*}(X^*, Y)$ .

A bounded subset  $S$  of  $X$  is said to be weakly precompact provided that every sequence from  $S$  has a weakly Cauchy subsequence [5]. For a subset  $A$  of  $X$ , let  $\text{co}(A)$  denote the convex hull of  $A$ . A series  $\sum x_n$  in  $X$  is said to be weakly unconditionally convergent (wuc) if for every  $x^* \in X^*$  the series  $\sum |x^*(x_n)|$  is convergent. An operator  $T : X \rightarrow Y$  is weakly precompact if  $T(B_X)$  is weakly precompact and unconditionally converging if it maps weakly unconditionally convergent series to unconditionally convergent ones.

A bounded subset  $A$  of  $X$  (resp.,  $A$  of  $X^*$ ) is called a  $V^*$ -subset of  $X$  (resp., a  $V$ -subset of  $X^*$ ) provided that

$$\begin{aligned} \lim_n (\sup \{|x_n^*(x)| : x \in A\}) &= 0 \\ (\text{resp. } \lim_n (\sup \{|x^*(x_n)| : x^* \in A\}) &= 0) \end{aligned} \quad (1)$$

for each wuc series  $\sum x_n^*$  in  $X^*$  (resp., wuc series  $\sum x_n$  in  $X$ ).

In his fundamental paper [7], Pełczyński introduced property  $(V)$  and property  $(V^*)$ . The Banach space  $X$  has property  $(V)$  (resp.,  $(V^*)$ ) if every  $V$ -subset of  $X^*$  (resp.,  $V^*$ -subset of  $X$ ) is relatively weakly compact. The following results were also established in [7]:  $C(K)$  spaces have property  $(V)$ ;  $L^1$ -spaces have property  $(V^*)$ ; the Banach space  $X$  has property  $(V)$  if and only if every unconditionally converging operator  $T$  from  $X$  to any Banach space  $Y$  is weakly compact; if  $X$  has property  $(V)$ , then  $X^*$  has property  $(V^*)$ ; every weakly Cauchy sequence in  $X^*$  (resp., in  $X$ ) is a  $V$ -set (resp., a  $V^*$ -set); consequently, every bounded weakly precompact set in  $X^*$  (resp., in  $X$ ) is a  $V$ -set (resp., a  $V^*$ -set). A Banach space  $X$  has property weak  $(V)$  ( $wV$ ) if any  $V$ -subset of  $X^*$  is weakly precompact [8]. A Banach space  $X$  has property weak  $(V^*)$  ( $wV^*$ ) if every  $V^*$ -subset of  $X$  is weakly precompact [9].

The Banach-Mazur distance  $d(E, F)$  between two isomorphic Banach spaces  $E$  and  $F$  is defined by  $\inf(\|T\| \|T^{-1}\|)$ , where the infimum is taken over all isomorphisms  $T$  from  $E$  onto  $F$ . A Banach space  $E$  is called an  $\mathcal{L}_\infty$ -space (resp.,  $\mathcal{L}_1$ -space) [10] if there is a  $\lambda \geq 1$  so that every finite dimensional subspace of  $E$  is contained in another subspace  $N$  with  $d(N, \ell_\infty^n) \leq \lambda$  (resp.,  $d(N, \ell_1^n) \leq \lambda$ ) for some integer  $n$ . Complemented subspaces of  $C(K)$  spaces (resp.,  $L_1(\mu)$  spaces) are  $\mathcal{L}_\infty$ -spaces (resp.,  $\mathcal{L}_1$ -spaces) (see [10, Proposition 1.26]). The dual of an  $\mathcal{L}_1$ -space (resp.,  $\mathcal{L}_\infty$ -space) is an  $\mathcal{L}_\infty$ -space (resp.,  $\mathcal{L}_1$ -space) (see [10, Proposition 1.27]).

Suppose  $\Omega$  is a compact Hausdorff space,  $X$  and  $Y$  are Banach spaces,  $C(\Omega, X)$  is the Banach space of all continuous  $X$ -valued functions (with the supremum norm), and  $\Sigma$  is the  $\sigma$ -algebra of Borel subsets of  $\Omega$ . It is known from [11] that  $C(\Omega, X)^* \simeq M(\Omega, X^*)$ .

If  $m : \Sigma \rightarrow L(X, Y^{**})$  is a finitely additive vector measure and  $y^* \in Y^*$ , then  $m_{y^*}(A) = y^* m(A)(x)$ ,  $A \in \Sigma$ ,  $x \in A$ . For each  $y^* \in Y^*$ ,  $m_{y^*} : \Sigma \rightarrow X^*$  is a finitely additive vector measure.

Every continuous linear function  $T : C(\Omega, X) \rightarrow Y$  may be represented by a vector measure  $m : \Sigma \rightarrow L(X, Y^{**})$  of finite semivariation (see [11, 12], and [13, page 182]) such that

$$T(f) = \int_\Omega f dm, \quad f \in C(\Omega, X), \quad \|T\| = \bar{m}(\Omega), \quad (2)$$

and  $T^*(y^*) = m_{y^*}$ ,  $y^* \in Y^*$ , where  $\bar{m}$  denotes the semivariation of  $m$ . We denote this correspondence  $m \leftrightarrow T$ . We note that, for  $f \in C(\Omega, X)$ ,  $\int_\Omega f dm \in Y$  even if  $m$  is not  $L(X, Y)$ -valued. A representing measure  $m$  is called strongly bounded if  $(\bar{m}(A_i)) \rightarrow 0$  for every decreasing sequence  $(A_i) \rightarrow \emptyset$  in  $\Sigma$ , and an operator  $m \leftrightarrow T : C(\Omega, X) \rightarrow Y$  is called strongly bounded if  $m$  is strongly bounded [11].

By Theorem 4.4 of [11], a strongly bounded representing measure takes its values in  $L(X, Y)$ . It is well known that if

$T$  is unconditionally converging, then  $m$  is strongly bounded [14].

Let  $\lambda$  be a probability measure on  $\Sigma$ ,  $m \in M(\Omega, X)$  with  $|m| \leq \lambda$ , and let  $\rho$  be a lifting of  $L^\infty(\lambda)$  (see [12, 15]). For each  $x^* \in X^*$ , the scalar measure  $x^* \circ m$  has a density  $(d/d\lambda)(x^* \circ m) \in L^\infty(\lambda)$  (see [4, 5]). We define  $\rho(m)(\omega)$  to be the element of  $X^{**}$  defined by

$$\begin{aligned} \langle \rho(m)(\omega), x^* \rangle &= \rho\left(\frac{d}{d\lambda}(x^* \circ m)\right)(\omega), \\ \omega \in \Omega, \quad x^* \in X^*. \end{aligned} \quad (3)$$

It is well known (see [12, sect. 13, Theorem 5, page 269]) that,

- (i) for every  $x^* \in X^*$ , the map  $\omega \rightarrow \langle \rho(m)(\omega), x^* \rangle$  is  $\lambda$ -integrable;
- (ii) for every  $A \in \Sigma$  and all  $x^* \in X^*$ ,

$$\langle m(A), x^* \rangle = \int_A \langle \rho(m)(\omega), x^* \rangle d\lambda; \quad (4)$$

- (iii) the map  $\omega \rightarrow \|\rho(m)(\omega)\|$  is  $\lambda$ -integrable and for every  $A \in \Sigma$ ,

$$|m|(A) = \int_A \|\rho(m)(\omega)\| d\lambda. \quad (5)$$

If  $X = E^*$  is a dual space, then we define  $\rho(m)(\omega)$  to be the element of  $X = E^*$  defined by

$$\langle \rho(m)(\omega), x \rangle = \rho\left(\frac{d}{d\lambda}(x \circ m)\right)(\omega), \quad \omega \in \Omega, \quad x \in E. \quad (6)$$

### 3. Weak Precompactness in $M(\Omega, X)$

Let  $X$  be a Banach space,  $(\Omega, \Sigma)$  a measure space, and  $M(\Omega, X)$  the space of all  $X$ -valued countably additive measures on  $(\Omega, \Sigma)$  of bounded variation (with the total variation norm). Following [4], for  $\lambda$ , a probability measure on  $(\Omega, \Sigma)$ , we denote

$$W(\lambda, X) = \{m \in M(\Omega, X) : |m| \leq \lambda\}. \quad (7)$$

Let  $\rho$  be a lifting of  $L^\infty(\lambda)$  [12, 15]. For a subset  $A$  of  $W(\lambda, X)$  and  $\omega \in \Omega$ , let  $A(\omega) = \{\rho(m)(\omega) : m \in A\}$ .

The following results will be useful in our study.

**Lemma 1** (see [1, 6]). *Let  $A$  be a bounded subset of  $X$ . Then  $A$  is weakly precompact (resp., relatively weakly compact) if and only if for each sequence  $(x_n)$  in  $A$  there is a sequence  $(y_n)$  so that  $y_n \in \text{co}\{x_i : i \geq n\}$  for each  $n$  and  $(y_n)$  is weakly Cauchy (resp., weakly convergent).*

**Lemma 2** (see [5, Theorem 14]). *Let  $(m_n)$  be a sequence in  $M(\Omega, X)$ . Suppose there is a probability measure  $\lambda$  with  $|m_n| \leq \lambda$  for each  $n$ . Let  $\rho$  be a lifting of  $L^\infty(\lambda)$ . Then there is a sequence*

$(m'_n)$  with  $m'_n \in \text{co}\{m_i : i \geq n\}$  for each  $n$  and two measurable sets  $C$  and  $L$  such that  $\lambda(C \cup L) = 1$  and,

- (a) for  $\omega \in C$ ,  $(\rho(m'_n)(\omega))$  is weakly Cauchy;
- (b) for  $\omega \in L$ , there exists  $k \in \mathbb{N}$  such that  $(\rho(m'_n)(\omega))_{n \geq k}$  is equivalent to  $(e_n^*)$ .

The most general result of this paper is the following theorem.

**Theorem 3.** *Let  $A$  be a subset of  $W(\lambda, X)$ . Then  $A$  is weakly precompact if and only if for any sequence  $(m_n)$  in  $A$  and for any/some lifting  $\rho$  of  $L^\infty(\lambda)$  there exists a sequence  $(m'_n)$  with  $m'_n \in \text{co}\{m_i : i \geq n\}$  for each  $n$  such that, for a.e.  $\omega$ , the sequence  $(\rho(m'_n)(\omega))$  is weakly Cauchy in  $X^{**}$ .*

*Proof.* Suppose that  $A$  is a weakly precompact subset of  $W(\lambda, X)$  and let  $(m_n)$  be a sequence in  $A$ . By passing to a subsequence, we can assume that  $(m_n)$  is weakly Cauchy. Let  $\rho$  be a lifting of  $L^\infty(\lambda)$ . By Lemma 2, there is a sequence  $(m'_n)$  with  $m'_n \in \text{co}\{m_i : i \geq n\}$  for each  $n$  and two measurable sets  $C$  and  $L$  such that  $\lambda(C \cup L) = 1$  and,

- (a) for  $\omega \in C$ ,  $(\rho(m'_n)(\omega))$  is weakly Cauchy in  $X^{**}$ ;
- (b) for  $\omega \in L$ , there exists  $k \in \mathbb{N}$  such that  $(\rho(m'_n)(\omega))_{n \geq k}$  is equivalent to  $(e_n^*)$ .

If  $\lambda(L) > 0$ , then, by [5, Theorem 15], there exists  $k \in \mathbb{N}$  such that  $(m'_n)_{n \geq k}$  is equivalent to  $(e_n^*)$ . Since  $(m'_n)$  lies in the set  $\text{co}(A)$ , which is weakly precompact (see [16, page 377], [17, page 27]), one obtains a contradiction. Hence  $\lambda(L) = 0$ , and for a.e.  $\omega$ ,  $(\rho(m'_n)(\omega))$  is weakly Cauchy in  $X^{**}$ .

Conversely, let  $(m_n)$  be a sequence in  $A$ , and let  $\rho$  be a lifting of  $L^\infty(\lambda)$ . Let  $(m'_n)$  be a sequence with  $m'_n \in \text{co}\{m_i : i \geq n\}$  for each  $n$  such that, for a.e.  $\omega$ , the sequence  $(\rho(m'_n)(\omega))$  is weakly Cauchy in  $X^{**}$ . By [5, Theorem 15],  $(m'_n)$  is weakly Cauchy. By Lemma 1,  $A$  is weakly precompact. □

**Theorem 4.** *Let  $A$  be a subset of  $W(\lambda, X^*)$ . Then  $A$  is weakly precompact if and only if for any sequence  $(m_n)$  in  $A$  and for any/some lifting  $\rho$  of  $L^\infty(\lambda)$  there exists a sequence  $(m'_n)$  with  $m'_n \in \text{co}\{m_i : i \geq n\}$  for each  $n$  such that, for a.e.  $\omega$ , the sequence  $(\rho(m'_n)(\omega))$  is weakly Cauchy in  $X^*$ .*

*Proof.* The proof is similar to that of Theorem 3. □

**Corollary 5.** *Let  $A$  be a subset of  $W(\lambda, X)$ .*

- (i) *Suppose that, for a.e.  $\omega$ , the set  $A(\omega)$  is weakly precompact. Then  $A$  is weakly precompact.*
- (ii) *Suppose that, for a.e.  $\omega$ , the set  $A(\omega)$  is relatively weakly compact. Then  $A$  is relatively weakly compact.*

*Proof.* Let  $(m_n)$  be a sequence in  $A$ . By Lemma 2, there exist a sequence  $(m'_n)$  with  $m'_n \in \text{co}\{m_i : i \geq n\}$  for each  $n$  and two sets  $C$  and  $L$  in  $\Sigma$  with  $\lambda(C \cup L) = 1$ , such that conditions (a) and (b) of Lemma 2 are satisfied.

- (i) Since, for a.e.  $\omega$ , the set  $\text{co}(A(\omega))$  is weakly precompact (see [16, page 377], [17, page 27]) and

the sequence  $(\rho(m'_n)(\omega))$  lies in this set, we have  $\lambda(L) = 0$ . Then for a.e.  $\omega$ , the sequence  $(\rho(m'_n)(\omega))$  is weakly Cauchy in  $X^{**}$ . By Theorem 3,  $A$  is weakly precompact.

- (ii) By (i), for a.e.  $\omega$ , the sequence  $(\rho(m'_n)(\omega))$  is weakly Cauchy in  $X^{**}$ . Since, for a.e.  $\omega$ , the set  $\text{co}(A(\omega))$  is relatively weakly compact (Krein-Smulian's theorem) and the sequence  $(\rho(m'_n)(\omega))$  lies in this set, we have that  $(\rho(m'_n)(\omega))$  is relatively weakly compact. Then for a.e.  $\omega$ ,  $(\rho(m'_n)(\omega))$  is weakly convergent. By [4, Theorem 1],  $A$  is relatively weakly compact. □

**Corollary 6.** *Suppose that  $A$  is a subset of  $W(\lambda, X)$ .*

- (i) *If  $\ell_1 \nrightarrow X^{**}$ , then  $A$  is weakly precompact.*
- (ii) *If  $X$  is reflexive, then  $A$  is relatively weakly compact.*

*Proof.* Apply Corollary 5. □

**Corollary 7.** *Let  $A$  be a subset of  $W(\lambda, X^*)$ . If the set  $A(\omega)$  is weakly precompact for a.e.  $\omega$ , then  $A$  is weakly precompact.*

*Proof.* The proof is similar to that of Corollary 5. □

**Corollary 8.** *Suppose that  $A$  is a subset of  $W(\lambda, X^*)$ . If  $\ell_1 \nrightarrow X^*$ , then  $A$  is weakly precompact.*

*Proof.* Apply Corollary 7. □

The statements of Corollaries 6 and 8 can be considered just for  $W(\lambda, X)$  (resp.,  $W(\lambda, X^*)$ ).

For the proofs of the following two theorems we will need the following lemmas. The first lemma contains a well-known result due to Grothendieck about relatively weakly compact sets (see [18, page 227]).

**Lemma 9** (see [3, Lemma 2.15]; [18, page 227]; [9, Corollary 1.7]). *Let  $A$  be a bounded subset of  $X$ . If for any  $\epsilon > 0$  there exists a weakly precompact (relatively weakly compact, resp.,  $V^*$ ) subset  $A_\epsilon$  of  $X$  such that  $A \subseteq A_\epsilon + \epsilon B_X$ , then  $A$  is weakly precompact (relatively weakly compact, resp., a  $V^*$ -set).*

**Lemma 10** (see [3, Lemma 2.12]). *Let  $A$  be a bounded subset of  $X$ . Then  $A$  is a  $V^*$ -set if and only if for any sequence  $(x_n)$  in  $A$  there is a sequence  $(z_n)$  so that  $z_n \in \text{co}\{x_i : i \geq n\}$  for each  $n$  and  $\{z_n : n \geq 1\}$  is a  $V^*$ -set.*

**Lemma 11.** (i) (see [19, Lemma 3.7]) *Let  $A$  be a bounded subset of  $X^*$ . If for any  $\epsilon > 0$  there exists a  $V$ -subset  $A_\epsilon$  of  $X^*$  such that  $A \subseteq A_\epsilon + \epsilon B_{X^*}$ , then  $A$  is a  $V$ -set.*

(ii) (see [19, Proposition 3.6]) *Let  $A$  be a bounded subset of  $X^*$ . Then  $A$  is a  $V$ -set if and only if for any sequence  $(x_n^*)$  in  $A$  there is a sequence  $(z_n^*)$  so that  $z_n^* \in \text{co}\{x_i^* : i \geq n\}$  for each  $n$  and  $\{z_n^* : n \geq 1\}$  is a  $V$ -set.*

**Lemma 12.** (i) *If  $K$  is a  $V^*$ -set in  $M(\Omega, X)$ , then  $V(K) = \{|m| : m \in K\}$  is uniformly countably additive.*

(ii) *If  $K$  is a weakly precompact subset of  $M(\Omega, X)$ , then  $V(K) = \{|m| : m \in K\}$  is uniformly countably additive.*

*Proof.* (i) Suppose  $K$  is a  $V^*$ -set in  $M(\Omega, X)$ . Since each member of  $K$  is a countably additive measure on the  $\sigma$ -algebra  $\Sigma$ , the set  $V(K) = \{|m| : m \in K\}$  is uniformly countably additive if and only if  $\lim_n |m|(A_n) = 0$  uniformly for  $m \in K$  whenever  $(A_n)$  is a pairwise disjoint sequence in  $\Sigma$ .

Let  $(m_n)$  be a sequence in  $K$ . Without loss of generality suppose that  $|m_n| \leq 1$  for all  $n$ . Let  $(A_n)$  be a pairwise disjoint sequence in  $\Sigma$  and  $\epsilon > 0$  such that  $|m_n|(A_n) > \epsilon$  for all  $n$ . For each  $n \in \mathbb{N}$ , let  $(A_{n_i})_{i=1}^{k_n}$  be a partition of  $A_n$  and let  $(x_{n_i}^*)_{i=1}^{k_n}$  be points in  $B_{X^*}$  such that

$$\sum_{i=1}^{k_n} \langle x_{n_i}^*, m_n(A_{n_i}) \rangle > \epsilon. \quad (8)$$

Define the  $X^*$ -valued simple functions  $s_n$  by  $s_n = \sum_{i=1}^{k_n} \chi_{A_{n_i}} x_{n_i}^*$ . Note that  $\int s_n dm_n > \epsilon$  for all  $n \in \mathbb{N}$ . Define  $T : M(\Omega, X) \rightarrow \ell_1$  by

$$T(m) = \sum_n \left( \int s_n dm \right) e_n^*. \quad (9)$$

Note that  $T$  is a well-defined operator,  $\sum T^*(e_n)$  is wuc, and  $\langle m_n, T^*(e_n) \rangle = \int s_n dm_n > \epsilon$  for each  $n$ . Then  $\{m_n : n \geq 1\}$  is not a  $V^*$ -set. This contradiction concludes the proof.

(ii) If  $K$  is a weakly precompact set in  $M(\Omega, X)$ , then  $K$  is a  $V^*$ -set in  $M(\Omega, X)$  [7]. Apply (i).  $\square$

Let  $B(0)$  denote the unit ball of  $M(\Omega, X)$ .

**Theorem 13.** (i) Suppose  $A$  is a bounded subset of  $M(\Omega, X)$  such that  $V(A) = \{|m| : m \in A\}$  is uniformly countably additive in  $M(\Omega)$ . Then there is a probability measure  $\lambda$  on  $(\Omega, \Sigma)$  such that, for any sequence  $(m_n)$  in  $A$ , there is a sequence  $(m'_n)$  with  $m'_n \in \text{co}\{m_i : i \geq n\}$  for each  $n$  such that, for any  $\epsilon > 0$ , there is a positive integer  $N$  and a subset  $H_\epsilon$  of  $NW(\lambda, X)$  so that  $\{m'_n : n \geq 1\} \subseteq H_\epsilon + \epsilon B(0)$ .

(ii) Suppose  $A$  is a bounded subset of  $M(\Omega, X)$ . Then  $A$  is weakly precompact (resp., a  $V^*$ -set) if and only if there is a probability measure  $\lambda$  on  $(\Omega, \Sigma)$  such that, for any sequence  $(m_n)$  in  $A$ , there is a sequence  $(m'_n)$  with  $m'_n \in \text{co}\{m_i : i \geq n\}$  for each  $n$  such that, for any  $\epsilon > 0$ , there is a positive integer  $N$  and a weakly precompact (resp., a  $V^*$ -set) subset  $H_\epsilon$  of  $NW(\lambda, X)$  so that  $\{m'_n : n \geq 1\} \subseteq H_\epsilon + \epsilon B(0)$ .

*Proof.* (i) Let  $A$  be a bounded subset of  $M(\Omega, X)$  such that  $V(A) = \{|m| : m \in A\}$  is uniformly countably additive in  $M(\Omega)$ . Then  $V(A)$  is relatively weakly compact in  $M(\Omega)$  [18]. Hence there is a probability measure  $\lambda$  on  $(\Omega, \Sigma)$  so that  $V(A)$  is uniformly  $\lambda$ -continuous [18]; that is, for any  $\epsilon > 0$ , there is  $\delta > 0$  such that if  $B \in \Sigma$ ,  $\lambda(B) < \delta$ , then

$$|m|(B) < \epsilon, \quad (10)$$

for all  $m \in A$ .

Let  $(m_n)$  be a sequence in  $A$ . For each  $n$ , let  $f_n$  be the  $\lambda$ -density of  $|m_n|$ . Since  $\{|m_n| : n \geq 1\}$  is relatively weakly compact in  $M(\Omega)$ ,  $\{f_n : n \geq 1\}$  is relatively weakly compact in  $L^1(\lambda)$ . Choose a subsequence  $(f_{j_n})$  of  $(f_j)$  so that  $(f_{j_n})$

converges weakly to some function  $f \in L^1(\lambda)$ . By Mazur's theorem, there is a sequence  $(g_n)$  with  $g_n \in \text{co}\{f_{j_i} : i \geq n\}$  for each  $n$  so that  $\|g_n - f\| \rightarrow 0$ . By taking a subsequence, if necessary,  $\|g_n(\omega) - f(\omega)\| \rightarrow 0$ ,  $\lambda$ -a.e. Therefore  $\sup_j g_j(\omega) < \infty$   $\lambda$ -a.e., and thus  $\Omega = \cup_N \{\omega : \sup_j g_j(\omega) < N\} \cup Z$ , where  $Z$  is a set of measure zero.

Let  $\epsilon > 0$ . Choose  $\delta > 0$  from the definition of uniform  $\lambda$ -continuity. Choose a positive integer  $N$  so that

$$\lambda \left( \left\{ \omega : \sup_j g_j(\omega) > N \right\} \right) < \delta, \quad (11)$$

and let  $E = \{\omega : \sup_j g_j(\omega) \leq N\}$ .

For each  $n$ ,  $g_n \in \text{co}\{f_{j_i} : i \geq n\}$ . Suppose  $g_n = \sum_{k \geq n} a_k^n f_{j_k}$ , with  $a_k^n \geq 0$  and  $\sum_{k \geq n} a_k^n = 1$ , where the sums are finite. Let  $m'_n = \sum_{k \geq n} a_k^n m_{j_k}$ . Note that  $m'_n \in \text{co}\{m_i : i \geq n\}$  for each  $n$ . Define

$$m'_n \chi_E : \Sigma \rightarrow X \quad \text{by } m'_n \chi_E(B) = m'_n(E \cap B), \quad B \in \Sigma, \quad (12)$$

and let  $H_\epsilon = \{m'_n \chi_E : n \geq 1\}$ .

For each  $n$ ,

$$m'_n = m'_n \chi_E + m'_n \chi_{E^c}. \quad (13)$$

For  $B \in \Sigma$ ,

$$\begin{aligned} \|m'_n \chi_E(B)\| &= \|m'_n(E \cap B)\| \\ &\leq \int_{E \cap B} g_n(\omega) d\lambda \leq N\lambda(B). \end{aligned} \quad (14)$$

Then  $H_\epsilon \subseteq NW(\lambda, X)$ . For each  $n$ ,

$$|m'_n \chi_{E^c}| = |m'_n|(E^c) \leq \sum_{k \geq n} a_k^n |m_{j_k}|(E^c) \leq \epsilon, \quad (15)$$

and thus  $m'_n \chi_{E^c} \in \epsilon B(0)$  for each  $n$ . Therefore  $\{m'_n : n \geq 1\} \subseteq H_\epsilon + \epsilon B(0)$ .

(ii) Suppose  $A$  is a weakly precompact subset (resp., a  $V^*$ -subset) of  $M(\Omega, X)$ . By Lemma 12,  $V(A) = \{|m| : m \in A\}$  is uniformly countably additive in  $M(\Omega)$ . By (i), there is a probability measure  $\lambda$  on  $(\Omega, \Sigma)$  such that, for any sequence  $(m_n)$  in  $A$ , there is a sequence  $(m'_n)$  with  $m'_n \in \text{co}\{m_i : i \geq n\}$  for each  $n$  such that, for any  $\epsilon > 0$ , there is a positive integer  $N$  and a subset  $H_\epsilon$  of  $NW(\lambda, X)$  so that  $\{m'_n : n \geq 1\} \subseteq H_\epsilon + \epsilon B(0)$ . Since  $\{m'_n : n \geq 1\} \subseteq \text{co}(A)$ , which is weakly precompact (resp., a  $V^*$ -set) (see [16, page 377], [17, page 27], resp., [9]),  $H_\epsilon$  is weakly precompact (resp., a  $V^*$ -set).

Conversely, let  $A$  be a bounded subset of  $M(\Omega, X)$ . Choose  $\lambda$  a probability measure as in the statement. Let  $(m_n)$  be a sequence in  $A$ . Let  $(m'_n)$  be a sequence with  $m'_n \in \text{co}\{m_i : i \geq n\}$  for each  $n$  such that, for any  $\epsilon > 0$ , there is a positive integer  $N$  and a weakly precompact subset (resp., a  $V^*$ -subset)  $H_\epsilon$  of  $NW(\lambda, X)$  so that  $\{m'_n : n \geq 1\} \subseteq H_\epsilon + \epsilon B(0)$ . By Lemma 9,  $\{m'_n : n \geq 1\}$  is weakly precompact (resp., a  $V^*$ -set). By Lemma 1 (resp., 10),  $A$  is weakly precompact (resp., a  $V^*$ -subset).  $\square$

Let  $B$  denote the unit ball of  $M(\Omega, X^*)$ .

**Theorem 14.** (i) Suppose  $A$  is a bounded subset of  $M(\Omega, X^*)$  such that  $V(A) = \{|m| : m \in A\}$  is uniformly countably additive in  $M(\Omega)$ . Then there is a probability measure  $\lambda$  on  $(\Omega, \Sigma)$  such that, for any sequence  $(m_n)$  in  $A$ , there is a sequence  $(m'_n)$  with  $m'_n \in \text{co}\{m_i : i \geq n\}$  for each  $n$  such that, for any  $\epsilon > 0$ , there is a positive integer  $N$  and a subset  $H_\epsilon$  of  $NW(\lambda, X^*)$  so that  $\{m'_n : n \geq 1\} \subseteq H_\epsilon + \epsilon B$ .

(ii) Suppose  $A$  is a bounded subset of  $M(\Omega, X^*)$ . Then  $A$  is weakly precompact (resp., a  $V$ -subset) if and only if there is a probability measure  $\lambda$  on  $(\Omega, \Sigma)$  such that, for any sequence  $(m_n)$  in  $A$ , there is a sequence  $(m'_n)$  with  $m'_n \in \text{co}\{m_i : i \geq n\}$  for each  $n$  such that, for any  $\epsilon > 0$ , there is a positive integer  $N$  and a weakly precompact subset (resp., a  $V$ -subset)  $H_\epsilon$  of  $NW(\lambda, X^*)$  so that  $\{m'_n : n \geq 1\} \subseteq H_\epsilon + \epsilon B$ .

*Proof.* (i) The proof is similar to the proof of Theorem 13(i).

(ii) If  $A$  is a weakly precompact subset of  $M(\Omega, X^*)$ , then  $A$  is a  $V$ -subset of  $M(\Omega, X^*)$  [7]. Hence  $V(A)$  is uniformly countably additive in  $M(\Omega)$  (see [9, Proposition 2.1], [20, Proposition 2]). The remainder of the proof is similar to that of Theorem 13(ii), using Lemma 9 (resp., 11) and Lemma 1 (resp., 11).  $\square$

**Corollary 15.** (i) Assume  $\ell_1 \not\rightarrow X^{**}$ . Then a subset  $A$  of  $M(\Omega, X)$  is weakly precompact if and only if  $A$  is bounded and  $V(A)$  is uniformly countably additive.

(ii) Assume  $X$  is reflexive. Then a subset  $A$  of  $M(\Omega, X)$  is relatively weakly compact if and only if  $A$  is bounded and  $V(A)$  is uniformly countably additive.

*Proof.* If  $A$  is a weakly precompact subset of  $M(\Omega, X)$ , then  $V(A)$  is uniformly countably additive, by Lemma 12.

Now suppose  $A$  is a bounded subset of  $M(\Omega, X)$  and  $V(A)$  is uniformly countably additive. By Theorem 13(i), there is a probability measure  $\lambda$  on  $(\Omega, \Sigma)$  such that, for any sequence  $(m_n)$  in  $A$ , there is a sequence  $(m'_n)$  with  $m'_n \in \text{co}\{m_i : i \geq n\}$  for each  $n$  such that, for any  $\epsilon > 0$ , there is a positive integer  $N$  and a subset  $H_\epsilon$  of  $NW(\lambda, X)$  so that  $\{m'_n : n \geq 1\} \subseteq H_\epsilon + \epsilon B(0)$ .

(i) By Corollary 6,  $H_\epsilon$  is weakly precompact, since  $\ell_1 \not\rightarrow X^{**}$ . By Theorem 13(ii),  $A$  is weakly precompact.

(ii) By Corollary 6,  $H_\epsilon$  is relatively weakly compact, since  $X$  is reflexive. By Lemma 9,  $\{m'_n : n \geq 1\}$  is relatively weakly compact. By Lemma 1,  $A$  is relatively weakly compact.  $\square$

**Corollary 16.** (i) If  $\ell_1 \not\rightarrow X^{**}$ , then  $M(\Omega, X)$  has property  $(wV^*)$ .

(ii) If  $X$  is reflexive, then  $M(\Omega, X)$  has property  $(V^*)$ .

*Proof.* Let  $A$  be a  $V^*$ -subset of  $M(\Omega, X)$ . By Lemma 12,  $V(A)$  is uniformly countably additive.

(i) By Corollary 15(i),  $A$  is weakly precompact.

(ii) By Corollary 15(ii),  $A$  is relatively weakly compact.  $\square$

**Corollary 17.** Suppose  $\ell_1 \not\rightarrow X^*$ . Then a subset  $A$  of  $M(\Omega, X^*)$  is weakly precompact if and only if  $A$  is bounded and  $V(A)$  is uniformly countably additive.

*Proof.* The proof is similar to that of Corollary 15, using Theorem 14 and Corollary 8.  $\square$

**Corollary 18** (see [21], Theorem 3.12, [8]). Suppose  $\Omega$  is a compact Hausdorff space and  $Y$  is a Banach space. If  $\ell_1 \not\rightarrow X^*$  and  $m \leftrightarrow T : C(\Omega, X) \rightarrow Y$  is a strongly bounded operator, then  $T^* : Y^* \rightarrow C(\Omega, X)^*$  is weakly precompact.

*Proof.* Suppose that  $\ell_1 \not\rightarrow X^*$  and  $m \leftrightarrow T : C(\Omega, X) \rightarrow Y$  is a strongly bounded operator. We claim that  $T^*$  is weakly precompact. Recall that  $T^*$  takes values in  $M(\Omega, X^*)$  and that  $T^*(y^*) = m_{y^*} : \Sigma \rightarrow X^*$ . Let  $(y_n^*)$  be a sequence in  $B_{Y^*}$  and let  $m_n = m_{y_n^*} = T^*(y_n^*)$  for each  $n \in \mathbb{N}$ . Without loss of generality suppose that  $|m_n| \leq 1$  for each  $n$ . Since  $m$  is strongly bounded,  $\{|m_n| : n \geq 1\}$  is uniformly countably additive (see [11, Lemma 3.1]). By Corollary 17,  $\{m_n : n \geq 1\}$  is weakly precompact. Hence  $T^*$  is weakly precompact.  $\square$

**Corollary 19.** If  $\Omega$  is a compact Hausdorff space and  $\ell_1 \not\rightarrow X^*$ , then  $C(\Omega, X)$  has property  $(wV)$ .

*Proof.* Let  $m \leftrightarrow T : C(\Omega, X) \rightarrow Y$  be an unconditionally converging operator. Then  $T$  is strongly bounded [14]. By Corollary 18,  $T^*$  is weakly precompact. Then  $C(\Omega, X)$  has property  $(wV)$  [8].  $\square$

**Corollary 20.** Suppose  $\ell_1 \not\rightarrow X$ . Then the following are equivalent:

(i)  $c_0$  is not a quotient of  $X$ ;

(ii) for any compact Hausdorff space  $\Omega$  and any Banach space  $Y$ , an operator  $m \leftrightarrow T : C(\Omega, X) \rightarrow Y$  has weakly precompact adjoint whenever  $m$  is strongly bounded and  $m(A)^* : Y^* \rightarrow X^*$  is weakly precompact for every  $A \in \Sigma$ .

*Proof.* (i)  $\Rightarrow$  (ii) If  $c_0$  is not a quotient of  $X$  and  $\ell_1 \not\rightarrow X$ , then  $\ell_1 \not\rightarrow X^*$ , by [21, Proposition 3.8]. Apply Corollary 18.

(ii)  $\Rightarrow$  (i) Suppose  $L : X \rightarrow c_0$  is a surjection. By [21, Theorem 2.4], there is a compact space  $\Delta$  and a continuous linear surjection  $m \leftrightarrow T : C(\Delta, X) \rightarrow c_0$  so that  $m$  is strongly bounded and  $m(A) : X \rightarrow c_0$  is compact for all  $A \in \Sigma$ . Since  $T$  is a surjection onto  $c_0$ ,  $T^*$  is an isomorphism on  $\ell_1$ , and thus  $T^*$  is not weakly precompact.  $\square$

**Corollary 21.** (i) [6, 11] If  $X$  is reflexive, then every strongly bounded operator  $T : C(\Omega, X) \rightarrow Y$  is weakly compact.

(ii) [7] If  $X$  is reflexive, then  $C(\Omega, X)$  has property  $(V)$ .

*Proof.* (i) Let  $m \leftrightarrow T : C(\Omega, X) \rightarrow Y$  be a strongly bounded operator. Let  $(y_n^*)$  be a sequence in  $B_{Y^*}$  and  $m_n = T^*(y_n^*)$ ,  $n \in \mathbb{N}$ . By Corollary 18,  $\{m_n : n \geq 1\}$  is weakly precompact in  $M(\Omega, X^*)$ . Since  $X^*$  is weakly sequentially complete,  $M(\Omega, X^*)$  is weakly sequentially complete [5]. Hence  $\{m_n : n \geq 1\}$  is relatively weakly compact. Then  $T^*$  is weakly compact. Hence  $T$  is weakly compact.

(ii) Every unconditionally converging operator on  $C(\Omega, X)$  is strongly bounded [14] and thus weakly compact (by (i)). Then  $C(\Omega, X)$  has property  $(V)$  [7].  $\square$

A Banach space is injective if it is complemented in any superspace. We recall that property  $(wV)$  (resp., property  $(V)$ ) is stable under quotients.

**Corollary 22.** (i) Suppose that  $X$  is injective and  $\ell_1 \not\rightarrow Y^*$ . Then  $K_{w^*}(X^*, Y)$  has property  $(wV)$ .

(ii) Suppose that  $X$  is injective and  $Y$  is reflexive. Then  $K_{w^*}(X^*, Y)$  has property  $(V)$ .

*Proof.* The space  $K_{w^*}(X^*, Y)$  is isomorphic to  $K_{w^*}(Y^*, X)$ . Since  $X$  is injective,  $K_{w^*}(Y^*, X)$  is complemented in  $K_{w^*}(Y^*, C(B_{X^*}))$ . Now,  $K_{w^*}(Y^*, C(B_{X^*}))$  is isomorphic to  $C(B_{X^*}, Y)$  [22].

(i) By Corollary 19,  $C(B_{X^*}, Y)$  has property  $(wV)$ . Hence  $K_{w^*}(Y^*, X)$  has property  $(wV)$ .

(ii) By Corollary 21,  $C(B_{X^*}, Y)$  has property  $(V)$ . Hence  $K_{w^*}(Y^*, X)$  has property  $(V)$ .  $\square$

**Corollary 23.** (i) Suppose that  $Z$  is an  $\mathcal{L}_\infty$ -space and  $\ell_1 \not\rightarrow Y^*$ . Then  $K(Z^*, Y)$  has property  $(wV)$ .

(ii) Suppose that  $Z$  is an  $\mathcal{L}_\infty$ -space and  $Y$  is reflexive. Then  $K(Z^*, Y)$  has property  $(V)$ .

*Proof.* The space  $K(Z^*, Y)$  is isomorphic to  $K_{w^*}(Z^{***}, Y)$  [22]. Since  $Z$  is an  $\mathcal{L}_\infty$ -space,  $Z^*$  is an  $\mathcal{L}_1$ -space [10], and thus  $Z^{**}$  is injective [23]. Apply Corollary 22.  $\square$

## Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

## References

- [1] A. Ulger, "Weak compactness in  $L^1(\mu, X)$ ," *Proceedings of the American Mathematical Society*, vol. 103, pp. 143–149, 1991.
- [2] J. Diestel, W. M. Ruess, and W. Schachermayer, "Weak compactness in  $L^1(\mu, X)$ ," *Proceedings of the American Mathematical Society*, vol. 118, no. 2, pp. 447–453, 1993.
- [3] I. Ghenciu, "Weakly precompact subsets of  $L^1(\mu, X)$ ," *Colloquium Mathematicum*, vol. 129, pp. 133–143, 2012.
- [4] N. Randrianantoanina and E. Saab, "Weak compactness in the space of vector-valued measures of bounded variation," *The Rocky Mountain Journal of Mathematics*, vol. 24, no. 2, pp. 681–688, 1994.
- [5] M. Talagrand, "Weak Cauchy sequences in  $L^1(E)$ ," *American Journal of Mathematics*, vol. 106, no. 3, pp. 703–724, 1984.
- [6] A. Ulger, "Continuous linear operators on  $C(K, X)$  and pointwise weakly precompact subsets of  $C(K, X)$ ," *Mathematical Proceedings of the Cambridge Philosophical Society*, vol. 111, pp. 143–150, 1992.
- [7] A. Pełczyński, "Banach spaces on which every unconditionally converging operator is weakly compact," *Bulletin de l'Académie Polonaise des Sciences. Série des Sciences Mathématiques, Astronomiques, et Physiques*, vol. 10, pp. 641–648, 1962.
- [8] E. Saab and P. Saab, "On unconditionally converging and weakly precompact operators," *Illinois Journal of Mathematics*, vol. 35, no. 3, pp. 522–531, 1991.
- [9] F. Bombal, "On  $(V^*)$  sets and Pelczynski's property  $(V^*)$ ," *Glasgow Mathematical Journal*, vol. 32, no. 1, pp. 109–120, 1990.
- [10] J. Bourgain, *New Classes of  $L_p$ -Spaces*, vol. 889 of *Lecture Notes in Mathematics*, Springer, 1981.
- [11] J. K. Brooks and P. W. Lewis, "Linear operators and vector measures," *Transactions of the American Mathematical Society*, vol. 192, pp. 139–162, 1974.
- [12] N. Dinculeanu, *Vector Measures*, Pergamon Press, 1967.
- [13] J. Diestel and J. J. Uhl Jr., *Vector Measures*, vol. 15, American Mathematical Society, Providence, RI, USA, 1977.
- [14] I. Dobrakov, "On representation of linear operators on  $C_0(T, X)$ ," *Czechoslovak Mathematical Journal*, vol. 21, pp. 13–30, 1971.
- [15] A. Ionescu Tulcea and C. Ionescu Tulcea, *Topics in the Theory of Lifting*, vol. 48 of *Ergebnisse der Mathematik und ihrer Grenzgebiete*, Springer, New York, NY, USA, 1963.
- [16] H. P. Rosenthal, "Point-wise compact subsets of the first Baire class," *American Journal of Mathematics*, vol. 99, no. 2, pp. 362–378, 1977.
- [17] T. Schlumprecht, *Limited sets in Banach spaces [Ph.D. dissertation]*, University München, Fakultät für Mathematik, Munich, Germany, 1987.
- [18] J. Diestel, *Sequences and Series in Banach Spaces*, vol. 92 of *Graduate Texts in Mathematics*, Springer, Berlin, Germany, 1984.
- [19] E. Bator and P. Lewis, "Properties  $(V)$  and  $(wV)$  on  $C(\Omega, X)$ ," *Mathematical Proceedings of the Cambridge Philosophical Society*, vol. 117, pp. 469–477, 1995.
- [20] N. Randrianantoanina, "Pełczyński's property  $(V)$  on spaces of vector-valued functions," *Colloquium Mathematicum*, vol. 71, no. 1, pp. 63–78, 1996.
- [21] C. Abbott, E. Bator, and P. Lewis, "Strictly singular and strictly cosingular operators on spaces of continuous functions," *Mathematical Proceedings of the Cambridge Philosophical Society*, vol. 110, no. 3, pp. 505–521, 1991.
- [22] W. Ruess, "Duality and geometry of spaces of compact operators," in *Functional Analysis: Surveys and Recent Results III*, Mathematical Studies no. 90, pp. 59–78, North Holland, 1984, (Proceedings of the 3rd Paderborn Conference 1983).
- [23] J. Lindenstrauss and A. Pełczyński, "Absolutely summing operators in  $L_p$ -spaces, and their applications," *Studia Mathematica*, vol. 29, pp. 275–326, 1968.



# Hindawi

Submit your manuscripts at  
<http://www.hindawi.com>

