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Research Article

Invertible Weighted Composition Operators on the Fock Space of $\mathbb{C}^N$

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We give a complete characterization of bounded invertible weighted composition operators on the Fock space of $\mathbb{C}^N$.

1. Introduction

It seems that there are simple forms for the weighted composition operators on the Fock space as implied in [1], where bounded and compact weighted composition operators on the Fock space of complex plane $\mathbb{C}$ are characterized. Following the ideas in [1], in this paper a complete characterization of bounded invertible weighted composition operators on Fock space of $\mathbb{C}^N$ is given.

Recall that the Fock space $\mathcal{F}^2$ is the space of analytic functions f on $\mathbb{C}^N$ for which

$$\|f\|^2 = \frac{1}{(2\pi)^N} \int_{\mathbb{C}^N} |f(z)|^2 e^{-|z|^2/2} dm_{2N}(z), \tag{1}$$

where dm_{2N} is the usual Lebesgue measure on $\mathbb{C}^N$ and $|z|$ denotes the norm for $z \in \mathbb{C}^N$. $\mathcal{F}^2$ is a reproducing kernel Hilbert space with inner product

$$\langle f, g \rangle = \frac{1}{(2\pi)^N} \int_{\mathbb{C}^N} f(z) \overline{g(z)} e^{-|z|^2/2} dm_{2N}(z), \quad f, g \in \mathcal{F}^2 \tag{2}$$

and reproducing kernel function

$$K_w(z) = \exp\left(\frac{\langle z, w \rangle}{2}\right), \quad w, z \in \mathbb{C}^N, \tag{3}$$

where $\langle z, w \rangle$ denotes the inner product for $z, w \in \mathbb{C}^N$ and $|z|^2 = \langle z, z \rangle$. Note that it is unnecessary to distinguish the symbols of inner product in $\mathcal{F}^2$ and inner product in $\mathbb{C}^N$.

Let k_w be the normalization of K_w; then

$$k_w(z) = \exp\left(\frac{\langle z, w \rangle}{2} - \frac{|w|^2}{4}\right). \tag{4}$$

For analytic function ψ on $\mathbb{C}^N$ and analytic self-mapping φ on $\mathbb{C}^N$, the weighted composition operator $C_{\psi,\varphi}$ on $\mathcal{F}^2$ is defined as

$$C_{\psi,\varphi} f = \psi (f \circ \varphi), \quad f \in \mathcal{F}^2. \tag{5}$$

For an operator A on $\mathbb{C}^N$, denote by $|A|$ the norm of A. We have the following main result.

Theorem 1. *Let ψ be an analytic function on $\mathbb{C}^N$ and let φ be an analytic self-mapping on $\mathbb{C}^N$. Then $C_{\psi,\varphi}$ is a bounded invertible operator on $\mathcal{F}^2$ if and only if*

$$\varphi(z) = Az + b \tag{6}$$

for some invertible operator A on $\mathbb{C}^N$ with $|A| = 1$, $b \in \mathbb{C}^N$, and there exist positive constants M, L such that

$$L \le |\psi(z)|^2 \exp\left(\frac{|\varphi(z)|^2 - |z|^2}{2}\right) \le M, \quad z \in \mathbb{C}^N. \tag{7}$$

Weighted composition operators on various function spaces have been studied intensively and extensively, which reflects the perfect combination of operator theory and function theory. For related topic of (weighted) composition operators on the Fock space, see [2–9] and so forth.

2. Proof of the Main Results

In this section, we present the proof of the main results. First, we list some known results.

Lemma 2. *Let $\psi_1, \ldots, \psi_n$ be analytic functions on $\mathbb{C}^N$ and let $\varphi_1, \ldots, \varphi_n$ be analytic self-mapping on $\mathbb{C}^N$. If $C_{\psi_1,\varphi_1}, \ldots, C_{\psi_n,\varphi_n}$ are bounded operators on $\mathcal{F}^2$, then*

$$C_{\psi_1,\varphi_1} C_{\psi_2,\varphi_2} \cdots C_{\psi_n,\varphi_n} = C_{\psi_1(\psi_2\circ\varphi_1)\cdots(\psi_n\circ\varphi_{n-1}\circ\cdots\circ\varphi_1),\varphi_n\circ\varphi_{n-1}\circ\cdots\circ\varphi_1}. \tag{8}$$

Lemma 3. *Let ψ be an analytic function on $\mathbb{C}^N$ and let φ be an analytic self-mapping on $\mathbb{C}^N$. If $C_{\psi,\varphi}$ is a bounded operator on $\mathcal{F}^2$, then, for $z \in \mathbb{C}^N$,*

$$C^*_{\psi,\varphi} K_z = \overline{\psi(z)} K_{\varphi(z)}. \tag{9}$$

For $p \in \mathbb{C}^N$, denote $\varphi_p(z) = z - p, U_p = C_{k_p,\varphi_p}$.

Lemma 4 (see [8, Proposition 2.3]). *U_p is a unitary operator on $\mathcal{F}^2$ and $U_p^{-1} = U_{-p}$.*

The following lemma is a modification of Proposition 2.1 in [1] since the reproducing kernel function in this paper is a little different from the reproducing kernel function in [1].

Lemma 5. *Let ψ, φ be entire functions on $\mathbb{C}$ with $\psi \neq 0$. If there exists a positive constant M such that*

$$|\psi(u)|^2 \exp\left(\frac{|\varphi(u)|^2 - |u|^2}{2}\right) \leq M, \quad u \in \mathbb{C}, \tag{10}$$

then

$$\varphi(u) = au + b \tag{11}$$

for some constants a, b with $|a| \leq 1$. If $|a| = 1$, then $\psi(u) = sk_c(u)$ for some nonzero constant s and $c = -\overline{a}b$.

Let f be an analytic function on $\mathbb{C}^N$; for any $\xi \in \mathbb{S} = \{z \in \mathbb{C}^N, |z| = 1\}$, denote

$$f_\xi(u) = f(u\xi), \quad u \in \mathbb{C}, \tag{12}$$

where f_ξ is called the slice function of f in ξ and f_ξ is an analytic function on $\mathbb{C}$.

Now, we extend Lemma 5 to the case of $\mathbb{C}^N$ in some sense.

Lemma 6. *Let ψ be an analytic function on $\mathbb{C}^N$ with $\psi(0) \neq 0$ and let φ be an analytic self-mapping on $\mathbb{C}^N$. If there exists a positive constant M such that*

$$|\psi(z)|^2 \exp\left(\frac{|\varphi(z)|^2 - |z|^2}{2}\right) \leq M, \quad z \in \mathbb{C}^N, \tag{13}$$

then

$$\varphi(z) = Az + b, \tag{14}$$

where A is an operator on $\mathbb{C}^N$ with $|A| \leq 1$ and $b \in \mathbb{C}^N$.

Moreover, there exists a constant $s \in \mathbb{C}$ such that

$$\psi_\xi(u) = s \exp\left(-\frac{u\langle A\xi, b\rangle}{2} - \frac{|b|^2}{4}\right), \quad u \in \mathbb{C} \tag{15}$$

whenever $|A\xi| = |\xi|$ for $\xi \in \mathbb{C}^N$.

In particular, when A is a unitary operator on $\mathbb{C}^N$, then there exists a constant $s \in \mathbb{C}$ such that

$$\psi(z) = sk_c(z), \tag{16}$$

with $c = -A^*b$.

Proof. Since $\varphi : \mathbb{C}^N \to \mathbb{C}^N$ is an analytic mapping, assume

$$\varphi(z) = (\varphi_1(z), \ldots, \varphi_N(z)), \quad z \in \mathbb{C}^N, \tag{17}$$

where $\varphi_j, 1 \leq j \leq N$, are analytic functions on $\mathbb{C}^N$.

By formula (13), for any $\xi \in \mathbb{S}$,

$$|\psi_\xi(u)|^2 \exp\left(\frac{|\varphi_\xi(u)|^2 - |\xi u|^2}{2}\right) \leq M, \quad u \in \mathbb{C}, \tag{18}$$

where $\varphi_\xi(u) = (\varphi_{1,\xi}(u), \ldots, \varphi_{N,\xi}(u))$.

For any $j, 1 \leq j \leq N$, we have

$$|\psi_\xi(u)|^2 \exp\left(\frac{|\varphi_{j,\xi}(u)|^2 - |u|^2}{2}\right) \leq M, \quad u \in \mathbb{C}, \tag{19}$$

since

$$|\varphi_{j,\xi}(u)|^2 \leq |\varphi_{1,\xi}(u)|^2 + \cdots + |\varphi_{N,\xi}(u)|^2 = |\varphi_\xi(u)|^2. \tag{20}$$

Since $\psi_\xi(0) = \psi(0) \neq 0$, it follows from Lemma 5 that there exist constants $A_j(\xi), b_j(\xi) \in \mathbb{C}$ such that

$$\varphi_{j,\xi}(u) = A_j(\xi)u + b_j(\xi), \quad u \in \mathbb{C}. \tag{21}$$

Let $u = 0$ in the formula above; then

$$b_j(\xi) = \varphi_{j,\xi}(0) = \varphi_j(0), \tag{22}$$

which implies that $b_j(\xi)$ is a constant. Assume

$$b_j = b_j(\xi) = \varphi_j(0); \tag{23}$$

then

$$\varphi_{j,\xi}(u) = A_j(\xi)u + b_j, \quad u \in \mathbb{C}. \tag{24}$$

Let $\varphi_j(z) = \sum_{n=0}^{\infty} \varphi_{j,n}(z)$, $z \in \mathbb{C}^N$, be the homogeneous expansion of φ_j, where $\varphi_{j,n}$ is homogeneous of degree n; then

$$\varphi_{j,\xi}(u) = \varphi_j(u\xi) = \sum_{n=0}^{\infty} \varphi_{j,n}(u\xi) = \sum_{n=0}^{\infty} u^n \varphi_{j,n}(\xi), \quad u \in \mathbb{C}. \tag{25}$$

Comparing formulas (24) and (25), it follows from the arbitrary of u that

$$\begin{aligned} \varphi_{j,0}(\xi) &= b_j, \\ A_j(\xi) &= \varphi_{j,1}(\xi), \\ \varphi_{j,n}(\xi) &= 0, \quad n \geq 2. \end{aligned} \tag{26}$$

By the arbitrary of ξ and homogeneity of $\varphi_{j,n}$, $0 \leq n < \infty$, we have

$$\begin{aligned} \varphi_{j,0} &= b_j, \\ \varphi_{j,n} &= 0, \quad n \geq 2. \end{aligned} \tag{27}$$

Since $\varphi_{j,1}$ is homogeneous of degree 1, assume

$$\begin{aligned} \varphi_{j,1}(z) &= a_{j,1}z_1 + \cdots + a_{j,N}z_N, \\ z &= (z_1, \ldots, z_N) \in \mathbb{C}^N,\ a_{j,m} \in \mathbb{C},\ 1 \leq m \leq N; \end{aligned} \tag{28}$$

then

$$\begin{aligned} \varphi_j(z) &= a_{j,1}z_1 + \cdots + a_{j,N}z_N + b_j, \\ z &= (z_1, \ldots, z_N) \in \mathbb{C}^N. \end{aligned} \tag{29}$$

Let $A = (a_{j,m})_{1 \leq j,m \leq N}$, $b = (b_1, \ldots, b_N)$; then

$$\varphi(z) = Az + b, \quad z \in \mathbb{C}^N. \tag{30}$$

The following reasoning is similar as Proposition 2.1 in [1].

Taking logarithms in both sides of formula (18), we have

$$4\log\left|\psi_\xi(u)\right| + \left|\varphi_\xi(u)\right|^2 - |u|^2 \leq 2\log M, \quad u \in \mathbb{C}. \tag{31}$$

Put $u = re^{i\theta}$ and, integrating with respect to θ on $[-\pi, \pi]$, we obtain

$$\begin{aligned} &\int_{-\pi}^{\pi} \left|\varphi_\xi\left(re^{i\theta}\right)\right|^2 \frac{d\theta}{2\pi} - r^2 + 4\int_{-\pi}^{\pi} \log\left|\psi_\xi\left(re^{i\theta}\right)\right| \frac{d\theta}{2\pi} \\ &\quad \leq 2\log M. \end{aligned} \tag{32}$$

Since $\log|\psi_\xi|$ is subharmonic,

$$\log\left|\psi(0)\right| = \log\left|\psi_\xi(0)\right| \leq \int_{-\pi}^{\pi} \log\left|\psi_\xi\left(re^{i\theta}\right)\right| \frac{d\theta}{2\pi}. \tag{33}$$

So

$$\int_{-\pi}^{\pi} \left|\varphi_\xi\left(re^{i\theta}\right)\right|^2 \frac{d\theta}{2\pi} - r^2 + 4\log\left|\psi(0)\right| \leq 2\log M. \tag{34}$$

Since

$$\begin{aligned} \left|\varphi_\xi\left(re^{i\theta}\right)\right|^2 &= \left|\varphi\left(re^{i\theta}\xi\right)\right|^2 = \left|A\left(re^{i\theta}\xi\right) + b\right|^2 \\ &= \left|re^{i\theta}(A\xi) + b\right|^2 \\ &= \left|re^{i\theta}\left(a_{1,1}\xi_1 + \cdots + a_{1,N}\xi_N\right) + b_1\right|^2 + \cdots \\ &\quad + \left|re^{i\theta}\left(a_{N,1}\xi_1 + \cdots + a_{N,N}\xi_N\right) + b_N\right|^2, \end{aligned} \tag{35}$$

where $\xi = (\xi_1, \ldots, \xi_N) \in \mathbb{S}$, we have

$$\begin{aligned} &\int_{-\pi}^{\pi} \left|\varphi_\xi\left(re^{i\theta}\right)\right|^2 \frac{d\theta}{2\pi} \\ &\quad = r^2\left|a_{1,1}\xi_1 + \cdots + a_{1,N}\xi_N\right|^2 \\ &\qquad + \left|b_1\right|^2 + \cdots + r^2\left|a_{N,1}\xi_1 + \cdots + a_{N,N}\xi_N\right|^2 \\ &\qquad + \left|b_N\right|^2 = r^2\left|A\xi\right|^2 + |b|^2. \end{aligned} \tag{36}$$

By formula (34) again, we get

$$\left(\left|A\xi\right|^2 - 1\right)r^2 + |b|^2 + 4\log\left|\psi(0)\right| \leq 2\log M. \tag{37}$$

Since $\psi(0) \neq 0$, it follows from the arbitrary of r that

$$\left|A\xi\right|^2 - 1 \leq 0 \tag{38}$$

for all $\xi \in \mathbb{S}$, which implies that

$$|A| \leq 1. \tag{39}$$

When $|A\xi| = |\xi|$ for some $\xi \in \mathbb{C}^N$, without loss of generality, assume that $\xi \in \mathbb{S}$; then $|A\xi| = |\xi| = 1$ and

$$\begin{aligned} &\exp\left(\frac{\left|A(u\xi) + b\right|^2 - |u|^2}{2}\right) \\ &\quad = \exp\left(\frac{\langle uA\xi, b\rangle + \langle b, uA\xi\rangle}{2} + \frac{|b|^2}{2}\right) \\ &\quad = \left|\exp\left(\frac{u\langle A\xi, b\rangle}{2} + \frac{|b|^2}{4}\right)\right|^2. \end{aligned} \tag{40}$$

It follows from (18) that

$$\begin{aligned} &\left|\psi_\xi(u)\exp\left(\frac{u\langle A\xi, b\rangle}{2} + \frac{|b|^2}{4}\right)\right|^2 \\ &\quad = \left|\psi_\xi(u)\right|^2 \exp\left(\frac{\left|A(u\xi) + b\right|^2 - |u|^2}{2}\right) \leq M, \end{aligned} \tag{41}$$

which implies that $\psi_\xi(u)\exp(u\langle A\xi, b\rangle/2 + |b|^2/4)$ is a bounded analytic function on $\mathbb{C}$. By Liouville theorem, there exists a constant $s(\xi) \in \mathbb{C}$ such that

$$\psi_\xi(u)\exp\left(\frac{u\langle A\xi, b\rangle}{2} + \frac{|b|^2}{4}\right) = s(\xi). \tag{42}$$

Let $u = 0$; then $s(\xi) = \psi_\xi(0)e^{|b|^2/4} = \psi(0)e^{|b|^2/4}$, which implies that $s(\xi)$ is a constant. Assume $s = s(\xi)$; then

$$\psi_\xi(u) = s\exp\left(-\frac{u\langle A\xi, b\rangle}{2} - \frac{|b|^2}{4}\right), \quad u \in \mathbb{C}. \tag{43}$$

If A is a unitary operator on $\mathbb{C}^N$, then, for any $z \in \mathbb{C}^N$, $|Az| = |z|$. Taking $\xi \in \mathbb{S}$, $u \in \mathbb{C}$ such that $z = u\xi$, then

$$\begin{aligned}\psi(z) &= \psi(u\xi) = \psi_\xi(u) \\ &= s\exp\left(-\frac{\langle A(u\xi), b\rangle}{2} - \frac{|b|^2}{4}\right) \\ &= s\exp\left(\frac{\langle z, -A^*b\rangle}{2} - \frac{|A^*b|^2}{4}\right) = sk_{-A^*b}(z).\end{aligned} \tag{44}$$

□

Proposition 7. *Let ψ be a nonzero analytic function on $\mathbb{C}^N$ and let φ be an analytic self-mapping on $\mathbb{C}^N$. If $C_{\psi,\varphi}$ is a bounded operator on $\mathscr{F}^2$, then there exists an operator A on $\mathbb{C}^N$, $|A| \le 1$, $b \in \mathbb{C}^N$, such that*

$$\begin{aligned}\varphi(z) &= Az + b, \\ \sup_{z\in\mathbb{C}^N} |\psi(z)|^2 \exp\left(\frac{|\varphi(z)|^2 - |z|^2}{2}\right) &< \infty.\end{aligned} \tag{45}$$

In particular, when A is invertible, condition (45) is sufficient also.

Proof. If $C_{\psi,\varphi}$ is a bounded operator on $\mathscr{F}^2$, then there exists a positive constant M such that, for any $z \in \mathbb{C}^N$,

$$\left\|C^*_{\psi,\varphi}K_z\right\|^2 \le M\|K_z\|^2. \tag{46}$$

Since $C^*_{\psi,\varphi}K_z = \overline{\psi(z)}K_{\varphi(z)}$ and $\|K_z\|^2 = e^{|z|^2/2}$, we have

$$|\psi(z)|^2 \exp\left(\frac{|\varphi(z)|^2 - |z|^2}{2}\right) \le M, \quad z \in \mathbb{C}^N. \tag{47}$$

Take $p \in \mathbb{C}^N$ such that $\psi(-p) \neq 0$. Since $C_{\psi,\varphi}$ is a bounded operator on $\mathscr{F}^2$, so is $U_pC_{\psi,\varphi}$. Denote $\psi_1 = k_p \cdot (\psi \circ \varphi_p)$, $\varphi_1 = \varphi \circ \varphi_p$; then

$$U_pC_{\psi,\varphi} = C_{\psi_1,\varphi_1}. \tag{48}$$

Since C_{ψ_1,φ_1} is a bounded operator on $\mathscr{F}^2$ also, we obtain

$$\sup_{z\in\mathbb{C}^N} |\psi_1(z)|^2 \exp\left(\frac{|\varphi_1(z)|^2 - |z|^2}{2}\right) < \infty. \tag{49}$$

Note that $\psi_1(0) = k_p(0)\psi(-p) \neq 0$. It follows from Lemma 6 that

$$\varphi_1(z) = Az + d, \quad z \in \mathbb{C}^N \tag{50}$$

for some operator A on $\mathbb{C}^N$ with $|A| \le 1$ and $d \in \mathbb{C}^N$.

Since $\varphi(z) = (\varphi_1 \circ \varphi_{-p})(z)$, we have

$$\varphi(z) = Az + b \tag{51}$$

with $b = Ap + d$.

When A is invertible, we have

$$\varphi^{-1}(w) = A^{-1}w - A^{-1}b, \quad w \in \mathbb{C}^N. \tag{52}$$

If ψ, φ satisfy condition (45), then, for any $f \in \mathscr{F}^2$,

$$\begin{aligned}\left\|C_{\psi,\varphi}f\right\|^2 &= \frac{1}{(2\pi)^N}\int_{\mathbb{C}^N} |\psi(z)|^2 |f(\varphi(z))|^2 \\ &\cdot e^{-|z|^2/2}dm_{2N}(z) = \frac{1}{(2\pi)^N}\int_{\mathbb{C}^N} |f(\varphi(z))|^2 |\psi(z)|^2 \\ &\cdot e^{(|\varphi(z)|^2 - |z|^2)/2}e^{-|\varphi(z)|^2/2}dm_{2N}(z) \le \frac{1}{(2\pi)^N} \\ &\cdot \sup_{z\in\mathbb{C}^N}\left(|\psi(z)|^2 e^{(|\varphi(z)|^2 - |z|^2)/2}\right)\int_{\mathbb{C}^N} |f(\varphi(z))|^2 \\ &\cdot e^{-|\varphi(z)|^2/2}dm_{2N}(z) = \frac{\left|\det A^{-1}\right|^2}{(2\pi)^N} \\ &\cdot \sup_{z\in\mathbb{C}^N}\left(|\psi(z)|^2 e^{(|\varphi(z)|^2 - |z|^2)/2}\right)\int_{\mathbb{C}^N} |f(w)|^2 \\ &\cdot e^{-|w|^2/2}dm_{2N}(w) = \left|\det A^{-1}\right|^2 \\ &\cdot \sup_{z\in\mathbb{C}^N}\left(|\psi(z)|^2 e^{(|\varphi(z)|^2 - |z|^2)/2}\right)\|f\|^2,\end{aligned} \tag{53}$$

which implies that $C_{\psi,\varphi}$ is a bounded operator on $\mathscr{F}^2$ and

$$\left\|C_{\psi,\varphi}\right\|^2 \le \left|\det A^{-1}\right|^2 \sup_{z\in\mathbb{C}^N}\left(|\psi(z)|^2 e^{(|\varphi(z)|^2 - |z|^2)/2}\right). \tag{54}$$

□

Now we restate the main result and present the proof.

Theorem 8. *Let ψ be an analytic function on $\mathbb{C}^N$ and let φ be an analytic self-mapping on $\mathbb{C}^N$. Then $C_{\psi,\varphi}$ is a bounded invertible operator on $\mathscr{F}^2$ if and only if there exist an invertible operator A on $\mathbb{C}^N$, with $|A| = 1$, $b \in \mathbb{C}^N$, and positive constants M, L such that*

$$\begin{aligned}\varphi(z) &= Az + b, \\ L \le |\psi(z)|^2 \exp\left(\frac{|\varphi(z)|^2 - |z|^2}{2}\right) &\le M, \\ z &\in \mathbb{C}^N.\end{aligned} \tag{55}$$

Proof. Assume that $C_{\psi,\varphi}$ is a bounded invertible operator on $\mathscr{F}^2$. By the boundedness of $C_{\psi,\varphi}$, it follows from Proposition 7 that

$$\varphi(z) = Az + b \tag{56}$$

for some operator A on $\mathbb{C}^N$ with $|A| \le 1$, and there exists a positive constant M such that

$$|\psi(z)|^2 \exp\left(\frac{|\varphi(z)|^2 - |z|^2}{2}\right) \le M, \quad z \in \mathbb{C}^N. \tag{57}$$

Since $C_{\psi,\varphi}$ is a bounded invertible operator on $\mathscr{F}^2$, so is $C^*_{\psi,\varphi}$. Hence there exists a positive constant L such that, for any $z \in \mathbb{C}^N$,

$$\left\| C^*_{\psi,\varphi} K_z \right\|^2 \geq L \left\| K_z \right\|^2. \tag{58}$$

It follows from $C^*_{\psi,\varphi} K_z = \overline{\psi(z)} K_{\varphi(z)}$ and $\|K_z\|^2 = e^{|z|^2/2}$ that

$$\left|\psi(z)\right|^2 \exp\left(\frac{\left|\varphi(z)\right|^2 - |z|^2}{2} \right) \geq L, \quad z \in \mathbb{C}^N, \tag{59}$$

which implies that ψ has no zeroes in $\mathbb{C}^N$.

Let $z, w \in \mathbb{C}^N$ with $\varphi(z) = \varphi(w)$; then

$$C^*_{\psi,\varphi}\left(\frac{K_z}{\overline{\psi(z)}} - \frac{K_w}{\overline{\psi(w)}} \right) = K_{\varphi(z)} - K_{\varphi(w)} = 0, \tag{60}$$

which implies that

$$\frac{K_z}{\overline{\psi(z)}} - \frac{K_w}{\overline{\psi(w)}} = 0. \tag{61}$$

Since $K_z(0) = K_w(0) = 1$, we have

$$\psi(z) = \psi(w), \tag{62}$$

and hence $K_z = K_w$. So $z = w$. It follows that φ is an injective mapping on $\mathbb{C}^N$.

Since $\varphi(z) = Az + b$, A is an injective mapping on $\mathbb{C}^N$ also and hence a bijection, which implies that A is invertible on $\mathbb{C}^N$. So

$$\varphi^{-1}(w) = A^{-1}w - A^{-1}b, \quad w \in \mathbb{C}^N. \tag{63}$$

Taking $\varphi^{-1}(w)$ to place z in formula (59), we have

$$\frac{1}{\left|\psi\left(\varphi^{-1}(w)\right)\right|^2} \exp\left(\frac{\left|\varphi^{-1}(w)\right|^2 - |w|^2}{2} \right) \leq \frac{1}{L}, \quad w \in \mathbb{C}^N. \tag{64}$$

It follows from Lemma 6 that $|A^{-1}| \leq 1$ and from Proposition 7 that $C_{1/(\psi\circ\varphi^{-1}),\varphi^{-1}}$ is a bounded operator on $\mathscr{F}^2$.

Since $|A| \leq 1$, $|A^{-1}| \leq 1$, and $AA^{-1} = I$, the identity operator on $\mathbb{C}^N$, we have

$$|A| = \left|A^{-1}\right| = 1. \tag{65}$$

On the other hand, if ψ, φ satisfy the condition stated in the theorem, by Proposition 7, $C_{\psi,\varphi}$ and $C_{1/(\psi\circ\varphi^{-1}),\varphi^{-1}}$ are bounded operators on $\mathscr{F}^2$. Direct computation shows that

$$C_{\psi,\varphi} C_{1/(\psi\circ\varphi^{-1}),\varphi^{-1}} = C_{1/(\psi\circ\varphi^{-1}),\varphi^{-1}} C_{\psi,\varphi} = I, \tag{66}$$

the identity operator on $\mathscr{F}^2$.

So $C_{\psi,\varphi}$ is a bounded invertible operator on $\mathscr{F}^2$. □

For $N = 1$, combined with Lemma 5, [8, Corollary 1.2], we have the following corollary.

Corollary 9. *Let ψ, φ be entire functions on $\mathbb{C}$. Then $C_{\psi,\varphi}$ is a bounded invertible operator on $\mathscr{F}^2$ of $\mathbb{C}$ if and only if $C_{\psi,\varphi}$ is a nonzero constant multiple of a unitary operator on $\mathscr{F}^2$.*

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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