Hindawi Publishing Corporation
Journal of Function Spaces
Volume 2015, Article ID 305961, 7 pages
http://dx.doi.org/10.1155/2015/305961

Research Article

Decompositions of g-Frames and Duals and Pseudoduals of g-Frames in Hilbert Spaces

Xunxiang Guo

Department of Mathematics, Southwestern University of Finance and Economics, Chengdu 611130, China

Correspondence should be addressed to Xunxiang Guo; guoxunxiang@yahoo.com

Received 7 July 2014; Accepted 25 August 2014

Academic Editor: Nelson José Merentes Díaz

Firstly, we study the representation of g-frames in terms of linear combinations of simpler ones such as g-orthonormal bases, g-Riesz bases, and normalized tight g-frames. Then, we study the dual and pseudodual of g-frames, which are critical components in reconstructions. In particular, we characterize the dual g-frames in a constructive way; that is, the formulae for dual g-frames are given. We also give some g-frame like representations for pseudodual g-frame pairs. The operator parameterizations of g-frames and decompositions of bounded operators are the key tools to prove our main results.

1. Introduction

A sequence $(f_i)_{i\in I}$ of elements of a Hilbert space H is called a *frame* for H if there are constants $A, B > 0$ so that

$$A\|f\|^2 \le \sum_{i\in I}\left|\langle f, f_i\rangle\right|^2 \le B\|f\|^2, \quad \forall f \in H. \tag{1}$$

The numbers A and B are called the *lower* (resp., *upper*) frame bounds. The frame is a *tight frame* if $A = B$ and a *normalized tight frame* if $A = B = 1$.

The concept of frame first appeared in the late 40s and early 50s (see [1–3]). The development and study of wavelet theory during the last decades also brought new ideas and attention to frames because of their close connections. There are many related references on this topic, see [4–8].

In [9], Sun raised the concept of g-frame as follows, which generalized the concept of frame extensively. A sequence $\{\Lambda_i \in B(H, H_i) : i \in \mathcal{N}\}$ is called a *g-frame* for H with respect to $\{H_i : i \in \mathcal{N}\}$, which is a sequence of closed subspaces of a Hilbert space K, if there exist two positive constants A and B such that, for any $f \in H$,

$$A\|f\|^2 \le \sum_{i\in\mathcal{N}}\|\Lambda_i f\|^2 \le B\|f\|^2, \tag{2}$$

where A is called the *lower g-frame bound* and B is called the *upper g-frame bound*. The largest lower frame bound and the smallest upper frame bound are called the *optimal lower g-frame bound* and the *optimal upper g-frame bound*, respectively. We simply call $\{\Lambda_i : i \in \mathcal{N}\}$ a g-frame for H whenever the space sequence $\{H_i : i \in \mathcal{N}\}$ is clear. The tight g-frame and normalized tight g-frame are defined similarly. We call $\{\Lambda_i : i \in \mathcal{N}\}$ a *g-frame sequence*, if it is a g-frame for $\overline{\operatorname{span}}\{\Lambda_i^*(H_i)\}_{i\in\mathcal{N}}$. We call $\{\Lambda_i : i \in \mathcal{N}\}$ a *g-Bessel sequence*, if only the right inequality is satisfied. A g-frame $\{\Gamma_j : j \in \mathcal{N}\}$ for H is called an *alternate dual g-frame* of $\{\Lambda_j : j \in \mathcal{N}\}$, if for every $f \in H$, we have

$$f = \sum_{j\in N}\Lambda_j * \Gamma_j f = \sum_{j\in\mathcal{N}}\Gamma_j^*\Lambda_j f. \tag{3}$$

If $\{\Lambda_j : j \in \mathcal{N}\}$ is a g-frame for H, then the operator $S \in B(H)$ such that

$$Sf = \sum_{j\in\mathcal{N}}\Lambda_j^*\Lambda_j f, \quad \forall f \in H \tag{4}$$

is called the *g-fame operator* associated with $\{\Lambda_j : j \in \mathcal{N}\}$. It is well-known that $\{\Lambda_j S^{-1} : j \in \mathcal{N}\}$ is a dual g-frame of $\{\Lambda_j : j \in \mathcal{N}\}$, which is called the *canonical dual g-frame* associated with $\{\Lambda_j : j \in \mathcal{N}\}$. In this paper, we use dual of g-frames to denote any of the duals. Recently, g-frames in Hilbert spaces have been studied intensively; for more details, see [10–16] and the references therein.

Frames and g-frames have advantages of allowing decomposing and reconstructing elements in Hilbert spaces, in which the dual and pseudodual of frames (g-frames) play important roles. Characterizing dual frames and general frame decompositions is an important problem in pure and applied fields. In [17–22], the authors study the dual frames and frame-like decompositions in Hilbert spaces. In particular, Li derived a general parametric and algebraic formula for all duals of a frame in [17] and introduced the pseudoframe decompositions in [18]. In this paper, motivated by these works on frames, we consider similar problems on g-frames in Hilbert spaces and generalize the corresponding results on frames to g-frames. Another interesting problem in frame theory is representing general g-frames in terms of special and more simpler g-frames such as g-orthonormal bases, g-Riesz bases, and normalized tight g-frames. In [23], the authors study similar questions on frames in Hilbert spaces by applying the techniques of decomposing linear bounded operators. In this paper, we will study the decompositions of g-frames in Hilbert spaces by using similar techniques combing with what we have obtained on the operator parameterizations for g-frames in [24].

Throughout this paper, we use $\mathcal{N}$ to denote the set of natural numbers and $\mathcal{C}$ to denote the complex plane. All Hilbert spaces in this paper are assumed to be separable complex Hilbert spaces. This paper is organized as follows. In Section 2, we give some definitions and lemmas which are needed to understand the following sections. In Section 3, we consider the decomposition of g-frames. In Section 4, the dual and pseudodual of g-frames are considered.

2. Preliminary Definitions and Lemmas

In this section, we introduce some basic definitions and lemmas which are necessary for the following sections.

Definition 1. Let $\Lambda_i \in B(H, H_i)$, $i \in \mathcal{N}$.

(i) If $\{f : \Lambda_i f = 0, i \in \mathcal{N}\} = \{0\}$, then we say that $\{\Lambda_i : i \in \mathcal{N}\}$ is *g-complete*.

(ii) If $\{\Lambda_i : i \in \mathcal{N}\}$ is g-complete and there are positive constants A and B such that, for any finite subset $J \subset \mathcal{N}$ and $g_j \in H_j$, $j \in J$,

$$A\sum_{j\in J}\left\|g_j\right\|^2 \le \left\|\sum_{j\in J}\Lambda_j^* g_j\right\|^2 \le B\sum_{j\in J}\left\|g_j\right\|^2, \quad (5)$$

then we say that $\{\Lambda_i : i \in \mathcal{N}\}$ is a *g-Riesz basis* for H with respect to $\{H_i : i \in \mathcal{N}\}$.

(iii) We say $\{\Lambda_i : i \in \mathcal{N}\}$ is a *g-orthonormal basis* for H with respect to $\{H_i : i \in \mathcal{N}\}$ if it satisfies the following:

$$\begin{gathered}\left\langle \Lambda_i^* g_i, \Lambda_j^* g_j \right\rangle = \delta_{ij}\left\langle g_i, g_j\right\rangle, \\ \forall i, j \in \mathcal{N}, \quad g_i \in H_i, \quad g_j \in H_j, \\ \sum_{j\in\mathcal{N}}\left\|\Lambda_j f\right\|^2 = \|f\|^2, \quad \forall f \in H.\end{gathered} \quad (6)$$

Remark 2. It is obvious that any g-frame $\{\Lambda_i : i \in \mathcal{N}\}$ is g-complete and any g-orthonormal basis is a normalized tight g-frame.

Definition 3. Suppose that $\Lambda_j \in B(H, H_j)$, $\Gamma_j \in B(H, H_j)$ for any $j \in \mathcal{N}$. If, for any $x, y \in H$, we have $\langle x, y\rangle = \sum_{j\in\mathcal{N}}\langle \Lambda_j x, \Gamma_j y\rangle$, then we call $\{\Lambda_j : j \in \mathcal{N}\}$ and $\{\Gamma_j : j \in \mathcal{N}\}$ *a pair of pseudodual g-frames* for H. In particular, if $\{\Lambda_j : j \in \mathcal{N}\}$ is a g-frame for H, we call $\{\Gamma_j : j \in \mathcal{N}\}$ a *pseudodual g-frame* of $\{\Lambda_j : j \in \mathcal{N}\}$.

Lemma 4 (see [25]). *Let H be a Hilbert space. Then,*

(1) *for every invertible operator $U \in B(H)$, there exists a unique decomposition $U = WP$, where W is a unitary operator and P is a positive operator.*

(2) *for every positive operator $P \in B(H)$ with $\|P\| \le 1$, $P = (1/2)(W + W^*)$, where $W = P + i\sqrt{I - P^2}$ is a unitary operator.*

Lemma 5 (see [26]). *Given Hilbert space H and a sequence of closed subspaces $\{H_j : j \in \mathcal{N}\}$ of a Hilbert space K, then there exists a g-orthonormal basis $\{\Lambda_j \in B(H, H_j) : j \in \mathcal{N}\}$ for H with respect to $\{H_j : j \in \mathcal{N}\}$ if and only if $\sum_{j\in\mathcal{N}}\dim H_j = \dim H$.*

Lemma 6 (see [24]). *Let $\{\theta_j \in B(H, H_j) : j \in \mathcal{N}\}$ be a g-orthonormal basis for H with respect to $\{H_j : j \in \mathcal{N}\}$. Then, the sequence $\{\Lambda_j : j \in \mathcal{N}\}$ is a g-Bessel sequence for H if and only if there is a unique bounded operator $V \in B(H)$ such that $\Lambda_j = \theta_j V^*$ for all $j \in \mathcal{N}$.*

Remark 7. Given the g-orthonormal basis $\{\theta_i \in B(H, H_i) : i \in N\}$, the operator V in Lemma 6 is called the *g-preframe operator* associated with $\{\Lambda_i \in B(H, H_i) : i \in N\}$.

Lemma 8 (see [24]). *Suppose that $\{\theta_i \in B(H, H_i) : i \in N\}$ is a g-orthonormal basis for H, $\{\Lambda_i \in B(H, H_i) : i \in N\}$ is a g-Bessel sequence for H, and V and S are the g-preframe operator and g-frame operator associated with $\{\Lambda_i \in B(H, H_i) : i \in N\}$, respectively. Then*

(1) *$\{\Lambda_i \in B(H, H_i) : i \in N\}$ is a g-frame if and only if V is onto;*

(2) *$\{\Lambda_i \in B(H, H_i) : i \in N\}$ is a normalized tight g-frame if and only if V is a coisometry;*

(3) *$\{\Lambda_i \in B(H, H_i) : i \in N\}$ is a g-Riesz basis if and only if V is invertible;*

(4) *$\{\Lambda_i \in B(H, H_i) : i \in N\}$ is a g-orthonormal basis if and only if V is unitary.*

Lemma 9 (see [23]). *Let $T \in B(H)$ be onto; then T can be written as a linear combination of two unitary operators if and only if T is invertible.*

3. Decompositions of g-Frames

In this section, we do some research on the decompositions of g-frames in Hilbert spaces by using similar techniques in [23] combing with what we have established on the operator parameterizations for g-frames in [24].

Theorem 10. *Suppose that $\{\Lambda_j \in B(H, H_j) : j \in \mathcal{N}\}$ is a g-Bessel sequence for H. Let T be the g-preframe operator associated with $\{\Lambda_j : j \in \mathcal{N}\}$. Then, for any $\varepsilon \in (0,1)$, there exist three g-orthonormal bases $\{\theta_j^l : j \in \mathcal{N}\}$ $(l = 1,2,3)$ such that $\Lambda_j = (1-\varepsilon)^{-1}\|T\|(\theta_j^1 + \theta_j^2 + \theta_j^3)$ for any $j \in \mathcal{N}$.*

Proof. Since we have assumed that all Hilbert spaces are separable, the g-orthonormal bases for H with respect to $\{H_j : j \in \mathcal{N}\}$ exist by Lemma 5. Let $\{\theta_j \in B(H, H_j) : j \in \mathcal{N}\}$ be a g-orthonormal basis for H with respect to $\{H_j : j \in \mathcal{N}\}$. Since $\{\Lambda_j \in B(H, H_j) : j \in \mathcal{N}\}$ is a g-Bessel sequence for H, there exists a bounded operator $T \in B(H)$ such that $\Lambda_j = \theta_j T^*$ for any $j \in \mathcal{N}$ by Lemma 6. Define an operator $U \in B(H)$ by $U = (1/2)I + ((1-\varepsilon)/2) \cdot (T^*/\|T\|)$, where I is the identity operator on H. Since

$$\begin{aligned} \|I - U\| &= \left\| \frac{1}{2}I - \frac{1-\varepsilon}{2} \cdot \frac{T^*}{\|T\|} \right\| \\ &\le \left\| \frac{1}{2}I \right\| + \left\| \frac{1-\varepsilon}{2} \cdot \frac{T^*}{\|T\|} \right\| \\ &= \frac{1}{2} + \frac{1-\varepsilon}{2} = 1 - \frac{\varepsilon}{2} < 1, \end{aligned} \tag{7}$$

U is invertible. By Lemma 4, there exist a unitary V and a positive operator P such that $U = VP$. Since

$$\begin{aligned} \|P\| &= \left\| V^{-1}U \right\| \le \|U\| \\ &\le \left\| \frac{1}{2}I \right\| + \left\| \frac{1-\varepsilon}{2} \cdot \frac{T^*}{\|T\|} \right\| \\ &= \frac{1}{2} + \frac{1-\varepsilon}{2} \le 1, \end{aligned} \tag{8}$$

$P = (1/2)(W^* + W)$, where $W = P + i\sqrt{I - P^2}$ is a unitary operator by Lemma 4. So

$$\begin{aligned} T^* &= \frac{2\|T\|}{1-\varepsilon}\left(U - \frac{1}{2}I\right) \\ &= \frac{2\|T\|}{1-\varepsilon}\left(\frac{V}{2}(W + W^*) - \frac{I}{2}\right) \\ &= \frac{\|T\|}{1-\varepsilon}(VW + VW^* - I). \end{aligned} \tag{9}$$

Hence,

$$\begin{aligned} \Lambda_j &= \theta_j T^* = \theta_j \cdot \frac{\|T\|}{1-\varepsilon}(VW + VW^* - I) \\ &= \frac{\|T\|}{1-\varepsilon}\left(\theta_j VW + \theta_j VW^* - \theta_j\right). \end{aligned} \tag{10}$$

Denote $\theta_j^1 = \theta_j VW$, $\theta_j^2 = \theta_j VW^*$, and $\theta_j^3 = -\theta_j$ for any $j \in \mathcal{N}$. Then, it is easy to see that $\{\theta_j^l : j \in \mathcal{N}\}$ $(l = 1,2,3)$ are g-orthonormal bases for H, since V and W are unitary operators. So $\Lambda_j = (1-\varepsilon)^{-1}\|T\|(\theta_j^1 + \theta_j^2 + \theta_j^3)$ for any $j \in \mathcal{N}$. □

Since a g-frame is of course a g-Bessel sequence, the following corollary is obvious.

Corollary 11. *Every g-frame can be represented as a multiple of sum of three g-orthonormal bases.*

Theorem 12. *A g-frame $\{\Lambda_j \in B(H, H_j) : j \in \mathcal{N}\}$ for H can be written as a linear combination of two g-orthonormal bases for H if and only if $\{\Lambda_j : j \in \mathcal{N}\}$ is a g-Riesz basis for H.*

Proof. $\Rightarrow$: Suppose that $\{\Gamma_j : j \in \mathcal{N}\}$ and $\{L_j : j \in \mathcal{N}\}$ are g-orthonormal bases for H such that $\Lambda_j = a \cdot \Gamma_j + b \cdot L_j$ for any $j \in \mathcal{N}$. By Lemma 8, there exist surjective operator T and unitary operator U such that $\Lambda_j = \Gamma_j T^*$ and $L_j = \Gamma_j U$ for any $j \in \mathcal{N}$. So, $\Gamma_j T^* = a \cdot \Gamma_j + b \cdot \Gamma_j U$, $\forall j \in \mathcal{N}$. Hence, $T\Gamma_j^* = \overline{a} \cdot \Gamma_j^* + \overline{b} \cdot U^* \Gamma_j^*$, $\forall j \in \mathcal{N}$. It implies that $T = \overline{a} \cdot I + \overline{b} \cdot U^*$, since $\overline{\operatorname{Span}}\{\Lambda_j^*(H_j) : j \in \mathcal{N}\} = H$. So T is invertible by Lemma 9. Hence, $\{\Lambda_j : j \in \mathcal{N}\}$ is a g-Riesz basis for H.

$\Leftarrow$: Since $\{\Lambda_j : j \in \mathcal{N}\}$ is a g-Riesz basis for H, there exist a g-orthonormal basis $\{\theta_j \in B(H, H_j) : j \in \mathcal{N}\}$ and an invertible operator $T \in B(H)$ such that $\Lambda_j = \theta_j T^*$ for any $j \in \mathcal{N}$ by Lemma 8. There exist two unitary operators U_1 and U_2 in $B(H)$ and constants a, b such that $T^* = a \cdot U_1 + b \cdot U_2$ by Lemma 9. So $\Lambda_j = \theta_j T^* = \theta_j(a \cdot U_1 + b \cdot U_2) = a \cdot \theta_j U_1 + b \cdot \theta_j U_2$ for any $j \in \mathcal{N}$. Since $\{\theta_j U_1 : j \in \mathcal{N}\}$ and $\{\theta_j U_2 : j \in \mathcal{N}\}$ are g-orthonormal bases for H by Lemmas 8 and 9, the result follows. □

Theorem 13. *Every g-frame for H is a multiple of two normalized tight g-frames for H.*

Proof. Suppose that $\{\Lambda_j \in B(H, H_j) : j \in \mathcal{N}\}$ is a g-frame for H and $\{\theta_j \in B(H, H_j) : j \in \mathcal{N}\}$ is a g-orthonormal basis for H. Then, there exists a surjective operator $T \in B(H)$ such that $\Lambda_j = \theta_j T^*$ for any $j \in \mathcal{N}$ by Lemma 8. Let $U = T/2\|T\|$. Then, $\|U\| = 1/2 < 1$ and U is also surjective. Suppose that $U = VP$ is the polar decomposition of U, where V is a coisometry and P is a positive operator in $B(H)$. Since $\|P\| = \|V^*U\| \le \|U\| < 1$, then $P = (1/2)(W + W^*)$ with $W = P + i\sqrt{I - P^2}$ being a unitary operator. So $U = VP = (1/2)V(W + W^*)$. It follows that $T = 2\|T\|U = \|T\|(VW + VW^*)$. So

$$\begin{aligned} \Lambda_j &= \theta_j T^* = \theta_j \|T\| (W^*V^* + WV^*) \\ &= \|T\| \left(\theta_j W^* V^* + \theta_j W V^*\right). \end{aligned} \tag{11}$$

Since VW and VW^* are coisometries, $\{\theta_j W^* V^* : j \in \mathcal{N}\}$ and $\{\theta_j W V^* : j \in \mathcal{N}\}$ are normalized tight g-frames for H by Lemma 8. This finishes the proof. □

Theorem 14. *Every g-frame for H is a multiple of the sum of a g-orthonormal basis for H and a g-Riesz basis for H.*

Proof. Suppose that $\{\Lambda_j \in B(H, H_j) : j \in \mathcal{N}\}$ is a g-frame for H and $\{\theta_j \in B(H, H_j) : j \in \mathcal{N}\}$ is a g-orthonormal basis for H. Let T be the g-preframe operator associated with $\{\Lambda_j : j \in \mathcal{N}\}$; then $\Lambda_j = \theta_j T^*$ for any $j \in \mathcal{N}$. Define operator S by $S = (3/4)I + (1/4)(1-\varepsilon)\cdot(T^*/\|T\|)$; then

$$\begin{aligned}\|I - S\| &= \left\|\frac{I}{4} - \frac{1}{4}\cdot(1-\varepsilon)\cdot\frac{T^*}{\|T\|}\right\| \\ &\le \left\|\frac{1}{4}\cdot I\right\| + \frac{1-\varepsilon}{4} < 1, \\ \|S\| &\le \frac{3}{4} + \frac{1-\varepsilon}{4} < 1.\end{aligned} \tag{12}$$

So S is invertible. Let $S = VP$ be the polar decomposition of S. Then, V is a unitary operator and P is a positive operator by Lemma 4. Since $P = V^*S$, $\|P\| = \|V^*S\| \le \|V^*\|\cdot\|S\| < 1$. So $P = (1/2)(W + W^*)$ by Lemma 4, where $W = P + i\sqrt{I - P^2}$ is a unitary operator. So $S = VP = (1/2)(VW + VW^*)$. It implies that

$$\begin{aligned}T^* &= \frac{4\|T\|}{1-\varepsilon}\left(S - \frac{3}{4}\cdot I\right) \\ &= \frac{4\|T\|}{1-\varepsilon}\left(\frac{1}{2}\cdot(VW + VW^*) - \frac{3}{4}\cdot I\right) \\ &= \frac{2\|T\|}{1-\varepsilon}\left(VW + VW^* - \frac{3}{2}\cdot I\right) \\ &= \frac{2\|T\|}{1-\varepsilon}(VW + R),\end{aligned} \tag{13}$$

where $R = VW^* - (3/2)\cdot I$. Since

$$\begin{aligned}\left\|I - \frac{-1}{2}\cdot R\right\| &= \left\|\frac{I}{4} + \frac{1}{2}VW^*\right\| \\ &\le \left\|\frac{I}{4}\right\| + \left\|\frac{1}{2}\cdot VW^*\right\| \\ &= \frac{1}{4} + \frac{1}{2} = \frac{3}{4} < 1,\end{aligned} \tag{14}$$

$-(1/2)\cdot R$ is invertible. Hence, R is invertible. So

$$\begin{aligned}\Lambda_j = \theta_j T^* &= \theta_j \cdot \frac{2\|T\|}{1-\varepsilon}\cdot(VW + R) \\ &= \frac{2\|T\|}{1-\varepsilon}\left(\theta_j VW + \theta_j R\right).\end{aligned} \tag{15}$$

Since $\{\theta_j VW : j \in \mathcal{N}\}$ is a g-orthonormal basis for H and $\{\theta_j R : j \in \mathcal{N}\}$ is a g-Riesz basis for H by Lemma 8, $\{\Lambda_j : j \in \mathcal{N}\}$ is a multiple of a sum of a g-orthonormal basis and a g-Riesz basis for H. □

4. Dual and Pseudodual g-Frames

In this section, we consider the characterizations of dual and pseudodual g-frames. The algebraic formula about the dual of g-frames for a given g-frame will be given and some properties on dual and pseudodual g-frames will be established.

Theorem 15. *Suppose that $\{\Lambda_j \in B(H, H_j) : j \in \mathcal{N}\}$ is a g-frame for H and $\{\theta_j \in B(H, H_j) : j \in \mathcal{N}\}$ is a g-orthonormal basis for H. Suppose that the g-preframe operator associated with $\{\Lambda_j : j \in \mathcal{N}\}$ is T; that is, $\Lambda_j = \theta_j T^*$ for any $j \in \mathcal{N}$. Then, $\{\Gamma_j \in B(H, H_j) : j \in \mathcal{N}\}$ is a dual g-frame of $\{\Lambda_j : j \in \mathcal{N}\}$ if and only if $\Gamma_j = \theta_j V^*$ for any $j \in \mathcal{N}$, where V is a bounded left inverse of T^*.*

Proof. ⇒: Suppose that $\{\Gamma_j : j \in \mathcal{N}\}$ is a dual g-frame of $\{\Lambda_j : j \in \mathcal{N}\}$. Let V be the g-preframe operator of $\{\Gamma_j : j \in \mathcal{N}\}$. Then, $\Gamma_j = \theta_j V^*$ for any $j \in \mathcal{N}$ and V is bounded. Since, for any $f \in H$, we have $f = \sum_{j\in\mathcal{N}} \Lambda_j^* \Gamma_j f$,

$$f = \sum_{j\in\mathcal{N}} T\theta_j^*\theta_j V^* f = T\sum_{j\in\mathcal{N}} \theta_j^*\theta_j V^* f = TV^* f. \tag{16}$$

It implies that $TV^* = I$. Hence, $VT^* = I$. It follows that V is a bounded left inverse of T^*.

⇐: Since $VT^* = I$, V is bounded surjective operator in $B(H)$. Hence, $\{\Gamma_j : j \in \mathcal{N}\}$ is a g-frame for H by Lemma 8. Since

$$\begin{aligned}\sum_{j\in\mathcal{N}} \Lambda_j^*\Gamma_j f &= \sum_{j\in\mathcal{N}} T\theta_j^*\theta_j V^* f = T\sum_{j\in\mathcal{N}} \theta_j^*\theta_j V^* f \\ &= TV^* f = f, \quad \forall f \in H,\end{aligned} \tag{17}$$

$\{\Gamma_j : j \in \mathcal{N}\}$ is a dual g-frame for H. □

Lemma 16. *Suppose that $\{\Lambda_j \in B(H, H_j) : j \in \mathcal{N}\}$ is a g-frame for H, whose g-preframe operator is T. Then, V is a linear bounded left inverse of T^* if and only if*

$$V = S^{-1}T + W\left(I - T^*S^{-1}T\right), \tag{18}$$

where S is the g-frame operator associated with $\{\Lambda_j : j \in \mathcal{N}\}$, $W \in B(H)$, and I is the identity operator in $B(H)$.

Proof. ⇒: Suppose V is a linear bounded left inverse of T^*. Let $W = V$. Then

$$\begin{aligned}S^{-1}T + V\left(I - T^*S^{-1}T\right) &= S^{-1}T + V - VT^*S^{-1}T \\ &= S^{-1}T + V - S^{-1}T = V.\end{aligned} \tag{19}$$

⇐: Suppose $V = S^{-1}T + W(I - T^*S^{-1}T)$. Then

$$\begin{aligned}VT^* &= S^{-1}TT^* + W\left(I - T^*S^{-1}T\right)T^* \\ &= S^{-1}S + W\left(T^* - T^*S^{-1}TT^*\right) = I.\end{aligned} \tag{20}$$

Hence, V is a linear bounded left inverse of T^*. □

Theorem 17. *Suppose $\{\Lambda_j \in B(H, H_j) : j \in \mathcal{N}\}$ is a g-frame for H, T is its g-preframe operator, and S is its g-frame operator. Let $\{\theta_j \in B(H, H_j) : j \in \mathcal{N}\}$ be a g-orthonormal basis for H. Then, $\{\Gamma_j \in B(H, H_j) : j \in \mathcal{N}\}$ is a dual g-frame of $\{\Lambda_j : j \in \mathcal{N}\}$ if and only if there exists a bounded operator $W \in B(H)$ such that*

$$\Gamma_j = \Lambda_j S^{-1} + \theta_j W^* - \Lambda_j S^{-1}TW^*, \quad \forall j \in \mathcal{N}. \tag{21}$$

Proof. $\Rightarrow$: Suppose that $\{\Gamma_j \in B(H,H_j) : j \in \mathcal{N}\}$ is a dual g-frame of $\{\Lambda_j \in B(H,H_j) : j \in \mathcal{N}\}$. Then, by Theorem 15, we know that $\Gamma_j = \theta_j V^*$ for any $j \in \mathcal{N}$, where V is a linear bounded left inverse of T^*. By Lemma 16, $V = S^{-1}T + W(I - T^*S^{-1}T)$ for some linear bounded operator $W \in B(H)$. Hence, for any $j \in \mathcal{N}$, we have

$$\begin{aligned}\Gamma_j &= \theta_j V^* = \theta_j\left(S^{-1}T + W\left(I - T^*S^{-1}T\right)\right)^* \\ &= \theta_j\left(T^*S^{-1} + W^* - T^*S^{-1}TW^*\right) \\ &= \theta_j T^*S^{-1} + \theta_j W^* - \theta_j T^*S^{-1}TW^* \\ &= \Lambda_j S^{-1} + \theta_j W^* - \Lambda_j S^{-1}TW^*.\end{aligned} \tag{22}$$

$\Leftarrow$: Suppose that there exists a linear bounded operator $W \in B(H)$ such that $\Gamma_j = \Lambda_j S^{-1} + \theta_j W^* - \Lambda_j S^{-1}TW^*$. Then,

$$\begin{aligned}\Gamma_j &= \theta_j T^*S^{-1} + \theta_j W^* - \theta_j T^*S^{-1}TW^* \\ &= \theta_j\left(T^*S^{-1} + W^* - T^*S^{-1}TW^*\right) \\ &= \theta_j\left(S^{-1}T + W - WT^*S^{-1}T\right)^*.\end{aligned} \tag{23}$$

So $\{\Gamma_j : j \in \mathcal{N}\}$ is a g-Bessel sequence for H and the g-preframe operator associated with $\{\Gamma_j : j \in \mathcal{N}\}$ is

$$\begin{aligned}V &= S^{-1}T + W - WT^*S^{-1}T \\ &= S^{-1}T + W\left(I - T^*S^{-1}T\right).\end{aligned} \tag{24}$$

Since V is a linear bounded left inverse of T^* by Lemma 16, $VT^* = I$. Therefore, $\{\Gamma_j : j \in \mathcal{N}\}$ is a dual g-frame of $\{\Lambda_j : j \in \mathcal{N}\}$ by Theorem 15. □

Theorem 18. *Suppose that $\{\Lambda_j \in B(H,H_j) : j \in \mathcal{N}\}$ is a g-frame for H. If $\{\Gamma_j \in B(H,H_j) : j \in \mathcal{N}\}$ is a pseudodual g-frame of $\{\Lambda_j : j \in \mathcal{N}\}$, then $\{\Gamma_j : j \in \mathcal{N}\}$ has lower g-frame bound.*

Proof. Since $\{\Gamma_j : j \in \mathcal{N}\}$ is a pseudodual g-frame of $\{\Lambda_j : j \in \mathcal{N}\}$, $\langle x, y\rangle = \sum_{j\in\mathcal{N}}\langle \Lambda_j x, \Gamma_j y\rangle$ for any $x, y \in H$. In particular, $\langle x, x\rangle = \sum_{j\in\mathcal{N}}\langle \Lambda_j x, \Gamma_j x\rangle$; that is, $\|x\|^2 = \sum_{j\in\mathcal{N}}\langle \Lambda_j x, \Gamma_j x\rangle$. Since

$$\begin{aligned}\sum_{j\in\mathcal{N}}\left\langle \Lambda_j x, \Gamma_j x\right\rangle &\le \sum_{j\in\mathcal{N}}\left\|\Lambda_j x\right\| \cdot \left\|\Gamma_j x\right\| \\ &\le \left(\sum_{j\in\mathcal{N}}\left\|\Lambda_j x\right\|^2\right)^{1/2} \cdot \left(\sum_{j\in\mathcal{N}}\left\|\Gamma_j x\right\|^2\right)^{1/2} \\ &\le \left(B\|x\|^2\right)^{1/2} \cdot \left(\sum_{j\in\mathcal{N}}\left\|\Gamma_j x\right\|^2\right)^{1/2},\end{aligned} \tag{25}$$

where B is the upper g-frame bound of $\{\Lambda_j : j \in \mathcal{N}\}$, hence,

$$\sum_{j\in\mathcal{N}}\left\|\Gamma_j x\right\|^2 \ge \frac{1}{B} \cdot \|x\|^2. \tag{26}$$

So $\{\Gamma_j : j \in \mathcal{N}\}$ has lower g-frame bound. □

Corollary 19. *Suppose that $\{\Lambda_j \in B(H,H_j) : j \in \mathcal{N}\}$ is a g-frame for H and $\{\Gamma_j : j \in \mathcal{N}\}$ is a pseudodual g-frame of $\{\Lambda_j : j \in \mathcal{N}\}$; then $\{\Gamma_j : j \in \mathcal{N}\}$ is g-complete.*

Proof. Since $\{\Gamma_j : j \in \mathcal{N}\}$ has lower g-frame bound by Theorem 18, there exists a constant $B > 0$ such that

$$\sum_{j\in\mathcal{N}}\left\|\Gamma_j x\right\|^2 \ge B\|x\|^2, \quad \forall x \in H. \tag{27}$$

If $\Gamma_j x = 0$, for all $j \in \mathcal{N}$, then $B\|x\|^2 = 0$; it follows that $x = 0$. So $\{\Gamma_j : j \in \mathcal{N}\}$ is g-complete. □

Theorem 20. *Suppose that $\{\Lambda_j \in B(H,H_j) : j \in \mathcal{N}\}$ and $\{\Gamma_j \in B(H,H_j) : j \in \mathcal{N}\}$ are a pair of pseudodual g-frames for H. Then, for any $x \in H$, $x = \sum_{j\in\mathcal{N}}\Lambda_j^*\Gamma_j x$ if and only if $x = \sum_{j\in\mathcal{N}}\Gamma_j^*\Lambda_j x$, where the series converge in norm of H.*

Proof. It is obvious that we only need to prove one direction; the other direction is identical. Now, suppose that $x = \sum_{j\in\mathcal{N}}\Lambda_j^*\Gamma_j x$. Since $\{\Lambda_j \in B(H,H_j) : j \in \mathcal{N}\}$ and $\{\Gamma_j \in B(H,H_j) : j \in \mathcal{N}\}$ are a pair of pseudodual g-frames for H, we have $\langle x, y\rangle = \sum_{j\in\mathcal{N}}\langle \Lambda_j x, \Gamma_j y\rangle \forall x, y \in H$. It is obvious that $f_N(y) = |\sum_{j=1}^{N}\langle \Lambda_j x, \Gamma_j y\rangle - \langle x, y\rangle|$ is a weakly continuous function on H and $\lim_{N\to\infty} f_N(y) = 0$ for each $x \in H$. Since the closed unit ball of H is weakly compact, for any $\varepsilon > 0$, there exists $N_0 > 0$ such that, for any $\|y\| \le 1$ and any $N > N_0$, we have $f_N(y) < \varepsilon$. So whenever $N > N_0$, we have

$$\begin{aligned}\left\|\sum_{j=1}^{N}\Gamma_j^*\Lambda_j x - x\right\| &= \sup_{\|y\|=1}\left|\left\langle \sum_{j=1}^{N}\Gamma_j^*\Lambda_j x - x, y\right\rangle\right| \\ &= \sup_{\|y\|=1}\left|\left\langle \sum_{j=1}^{N}\Gamma_j^*\Lambda_j x, y\right\rangle - \langle x, y\rangle\right| \\ &= \sup_{\|y\|=1} f_N(y) \le \varepsilon.\end{aligned} \tag{28}$$

Hence, $x = \sum_{j\in\mathcal{N}}\Gamma_j^*\Lambda_j x$. □

Corollary 21. *Suppose that $\{\Lambda_j \in B(H,H_j) : j \in \mathcal{N}\}$ and $\{\Gamma_j \in B(H,H_j) : j \in \mathcal{N}\}$are a pair of pseudo g-frames, $x_0 \in H$. If $\sum_{j\in\mathcal{N}}\Lambda_j^*\Gamma_j x_0$ is convergent, then*

$$x_0 = \sum_{j\in\mathcal{N}}\Lambda_j^*\Gamma_j x_0 = \sum_{j\in\mathcal{N}}\Gamma_j^*\Lambda_j x_0. \tag{29}$$

Proof. Since $\{\Lambda_j \in B(H,H_j) : j \in \mathcal{N}\}$ and $\{\Gamma_j \in B(H,H_j) : j \in \mathcal{N}\}$are a pair of pseudo g-frames, for any $y \in H$, we have

$$\left\langle y, x_0\right\rangle = \sum_{j\in\mathcal{N}}\left\langle \Lambda_j y, \Gamma_j x_0\right\rangle = \left\langle y, \sum_{j\in\mathcal{N}}\Lambda_j^*\Gamma_j x_0\right\rangle. \tag{30}$$

So $x_0 = \sum_{j\in\mathcal{N}}\Lambda_j^*\Gamma_j x_0$. It follows that $x_0 = \sum_{j\in\mathcal{N}}\Lambda_j^*\Gamma_j x_0 = \sum_{j\in\mathcal{N}}\Gamma_j^*\Lambda_j x_0$ by Theorem 20. □

Theorem 22. *Suppose that $\{\theta_j \in B(H, H_j) : j \in \mathcal{N}\}$ is a g-orthonormal basis for H and T_1 and T_2 are g-preframe operators associated with g-frames $\{\Lambda_j \in B(H, H_j) : j \in \mathcal{N}\}$ and $\{\Gamma_j \in B(H, H_j) : j \in \mathcal{N}\}$, respectively. Then, $\{\Lambda_j : j \in \mathcal{N}\}$ and $\{\Gamma_j : j \in \mathcal{N}\}$ are g-biorthogonal if and only if $T_1^* T_2 = I$, where I is the identity operator in $B(H)$.*

Proof. Since T_1 and T_2 are g-preframe operators associated with g-frames $\{\Lambda_j : j \in \mathcal{N}\}$ and $\{\Gamma_j : j \in \mathcal{N}\}$, respectively, $\Lambda_j = \theta_j T_1^*$ and $\Gamma_j = \theta_j T_2^*$ for any $j \in \mathcal{N}$. So, for any $i, j \in \mathcal{N}$ and any $g_i \in H_i$, $g_j \in H_j$, we have

$$\left\langle \Lambda_i^* g_i, \Gamma_j^* g_j \right\rangle = \left\langle T_1 \theta_i^* g_i, T_2 \theta_j^* g_j \right\rangle = \left\langle \theta_i^* g_i, T_1 T_2^* \theta_j^* g_j \right\rangle . \tag{31}$$

If $\{\Lambda_j : j \in \mathcal{N}\}$ and $\{\Gamma_j : j \in \mathcal{N}\}$ are g-biorthogonal, then

$$\begin{gathered} \left\langle \Lambda_i^* g_i, \Gamma_j^* g_j \right\rangle = \delta_{i,j} \left\langle g_i, g_j \right\rangle = \left\langle \theta_i^* g_i, \theta_j^* g_j \right\rangle , \\ \forall i, j \in \mathcal{N}, \quad \forall g_i \in H_i, \quad g_j \in H_j. \end{gathered} \tag{32}$$

So

$$\begin{gathered} \left\langle \theta_i^* g_i, T_1 T_2^* \theta_j^* g_j \right\rangle = \left\langle \theta_i^* g_i, \theta_j^* g_j \right\rangle , \\ \forall i, j \in \mathcal{N}, \quad \forall g_i \in H_i, \quad g_j \in H_j. \end{gathered} \tag{33}$$

It implies that $T_1 T_2^* = I$.

Conversely, if $T_1 T_2^* = I$, then

$$\begin{aligned} \langle \Lambda_i^* g_i, \Gamma_j^* g_j \rangle &= \langle \theta_i^* g_i, T_1 T_2^* \theta_j^* g_j \rangle \\ &= \langle \theta_i^* g_i, \theta_j^* g_j \rangle = \delta_{i,j} \langle g_i, g_j \rangle, \\ &\forall i, j \in \mathcal{N}, \quad \forall g_i \in H_i, \quad g_j \in H_j. \end{aligned} \tag{34}$$

So $\{\Lambda_j : j \in \mathcal{N}\}$ and $\{\Gamma_j : j \in \mathcal{N}\}$ are g-biorthogonal. □

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

Acknowledgment

This work was partially supported by the Fundamental Research Funds for the Central Universities (JBK140925).

References

[1] R. J. Duffin and A. C. Schaeffer, "A class of nonharmonic Fourier series," *Transactions of the American Mathematical Society*, vol. 72, pp. 341–366, 1952.

[2] B. de Sz. Nagy, "Expansion theorems of Paley-Wiener type," *Duke Mathematical Journal*, vol. 14, pp. 975–978, 1947.

[3] R. Young, *An Introduction t Nonharmonic Fourier Series*, Academic Press, New York, NY, USA, 1980.

[4] C. K. Chui, *An Introduction to Wavelets*, Academic Press, New York, NY, USA, 1992.

[5] I. Daubechies, *Ten Lectures on Wavelets*, vol. 61 of *CBMS-NSF Regional Conference Series in Applied Mathematics*, SIAM, 1992.

[6] E. Hernández and G. Weiss, *A First Course on Wavelets*, CRC Press, Boca Raton, Fla, USA, 1996.

[7] S. G. Mallat, "Multiresolution approximations and wavelet orthonormal basis of $L^2(R)$," *Transactions of the American Mathematical Society*, vol. 315, pp. 69–87, 1989.

[8] D. Han and D. Larson, "Bases, frames and group representations," *Memoirs of the American Mathematical Society*, vol. 147, no. 697, 2000.

[9] W. Sun, "G-frames and g-Riesz bases," *Journal of Mathematical Analysis and Applications*, vol. 322, no. 1, pp. 437–452, 2006.

[10] Y. C. Zhu, "Characterizations of g-frames and g-Riesz bases in Hilbert spaces," *Acta Mathematica Sinica*, vol. 24, no. 10, pp. 1727–1736, 2008.

[11] A. Najati, M. Faroughi, and A. Rahimi, "g-frames and stability of g-frames in Hilbert spaces," *Methods of Functional Analysis and Topology*, vol. 14, no. 3, pp. 271–286, 2008.

[12] Y. J. Wang and Y. C. Zhu, "G-frames and g-frame sequences in Hilbert spaces," *Acta Mathematica Sinica*, vol. 25, no. 12, pp. 2093–2106, 2009.

[13] A. Khosravi and K. Musazadeh, "Fusion frames and g-frames," *Journal of Mathematical Analysis and Applications*, vol. 342, no. 2, pp. 1068–1083, 2008.

[14] M. L. Ding and Y. C. Zhu, "g-Besselian frames in Hilbert spaces," *Acta Mathematica Sinica*, vol. 26, no. 11, pp. 2117–2130, 2010.

[15] J.-Z. Li and Y.-C. Zhu, "Exact g-frames in Hilbert spaces," *Journal of Mathematical Analysis and Applications*, vol. 374, no. 1, pp. 201–209, 2011.

[16] L. Zang, W. Sun, and D. Chen, "Excess of a class of g-frames," *Journal of Mathematical Analysis and Applications*, vol. 352, no. 2, pp. 711–717, 2009.

[17] S. Li, "On general frame decompositions," *Numerical Functional Analysis and Optimization*, vol. 16, no. 9-10, pp. 1181–1191, 1995.

[18] S. Li and H. Ogawa, "Pseudo-duals of frames with applications," *Applied and Computational Harmonic Analysis*, vol. 11, no. 2, pp. 289–304, 2001.

[19] S. Li and H. Ogawa, "Pseudoframes for subspaces with applications," *The Journal of Fourier Analysis and Applications*, vol. 10, no. 4, pp. 409–431, 2004.

[20] M. Bownik and J. Lemvig, "The canonical and alternate duals of a wavelet frame," *Applied and Computational Harmonic Analysis*, vol. 23, no. 2, pp. 263–272, 2007.

[21] O. Christensen, "Frames and pseudo-inverses," *Journal of Mathematical Analysis and Applications*, vol. 195, no. 2, pp. 401–414, 1995.

[22] O. Christensen, A. M. Powell, and X. . Xiao, "A note on finite dual frame pairs," *Proceedings of the American Mathematical Society*, vol. 140, no. 11, pp. 3921–3930, 2012.

[23] P. G. Casazza, "Every frame is a sum of three (but not two) orthonormal bases—and other frame representations," *The Journal of Fourier Analysis and Applications*, vol. 4, no. 6, pp. 727–732, 1998.

[24] X. Guo, "Operator characterizations and some properties of g-frames on Hilbert spaces," *Journal of Function Spaces and Applications*, vol. 2013, Article ID 931367, 9 pages, 2013.

[25] J. Conway, *A Course in Functional Analysis*, Graduate Texts in Mathematics, Springer, New York, NY, USA, 1990.

[26] X. Guo, "g-orthonormal basis in Hilbert spaces," *Acta Mathematica Sinica*, vol. 43, pp. 1047–1058, 2013 (Chinese).