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Research Article

The Improvement on the Boundedness and Norm of a Class of Integral Operators on L^p Space

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We prove the condition "c is neither 0 nor a negative integer" can be dropped on the boundedness of a class of integral operators $S_{a,b,c}$ on L^p space, which improves the result by Krues and Zhu. Besides, the exact norm of $S_{a,b,c}$ on L^p space is also obtained under the assumption $c = n + 1 + a + b$.

1. Introduction

Let $\mathbb{B}_n$ be the open unit ball in the complex space $\mathbb{C}^n$. The measure,

$$dv_t = \left(1 - |z|^2\right)^t dv(z), \tag{1}$$

denotes the weighted Lebesgue measure on $\mathbb{B}_n$, where t is real parameter and v is the normalized Lebesgue measure on $\mathbb{B}_n$ such that $v(\mathbb{B}_n) = 1$. It is easy to know dv_t is finite if and only if $t > -1$. Suppose $1 \le p < \infty$; to simplify the notation, we write $L_t^p := L^p(\mathbb{B}_n, v_t)$ for the weighted L^p-space under the measure v_t on $\mathbb{B}_n$ and $L^p := L_0^p$ for the usual L^p-space under the measure v.

Suppose a, b, c are real numbers, and a class of integral operators is defined by

$$S_{a,b,c} f(z) = \left(1 - |z|^2\right)^a \int_{\mathbb{B}_n} \frac{\left(1 - |w|^2\right)^b}{|1 - \langle z, w\rangle|^c} f(w)\, dv(w). \tag{2}$$

The class of integral operators is introduced by Kures and Zhu [1]. And it is closely related to "maximal Bergman projection" and Berezin transform. In fact, the boundedness of Bergman projection on L_α^p comes from the boundedness of the operator

$$P_\alpha^\sharp f(z) = \frac{\Gamma(n + \alpha + 1)}{n!\Gamma(\alpha + 1)} \int_{\mathbb{B}_n} \frac{f(w)}{|1 - \langle z, w\rangle|^{n+1+\alpha}} dv_\alpha(w), \quad \alpha > -1, \tag{3}$$

on L_α^p; see [2]. Therefore, we can call $P_\alpha^\sharp$ by "maximal Bergman projection," which is the particular case of $S_{a,b,c}$. Berezin transforms, whatever the case of the unit disk [3, page 141] or the case of unit ball ([4, page 76], [5, page 383]), are all concluded in the form of $S_{a,b,c}$ with special a, b, c.

In [1], Krues and Zhu gave the sufficient and necessary conditions of the boundedness of operator $S_{a,b,c}$.

Theorem A (see [1]). *Suppose c is neither 0 nor a negative integer.*

(1) *The operator $S_{a,b,c}$ is bounded on L_t^p $(1 < p < \infty)$ if and only if $-pa < t + 1 < p(b + 1)$, $c \le n + 1 + a + b$.*

(2) *The operator $S_{a,b,c}$ is bounded on L_t^1 if and only if $-a < t + 1 < b + 1, c = n + 1 + a + b$ or $-a < t + 1 \le b + 1, c < n + 1 + a + b$.*

The main purposes of this note contain two parts. One part is to prove the condition "c is neither 0 nor a negative integer" in Theorem A can be removed; see Section 3.

The other part is to give the accurate norm of the operator $S_{a,b,c}$ on L_t^p under the assumption $c = n+1+a+b$, which can be seen from the following two theorems.

Theorem 1. *Suppose $c = n + 1 + a + b$. If $1 \le p < \infty$ and $-pa < t + 1 < p(b + 1)$, then*

$$\left\|S_{a,b,c}\right\|_{L_t^p \to L_t^p} = \frac{n!\Gamma\left(a + (t+1)/p\right)\Gamma\left(b + 1 - (t+1)/p\right)}{\Gamma^2\left((n+1+a+b)/2\right)}. \tag{4}$$

Else, we also give the sufficient and necessary conditions of the operator $S_{a,b,c}$ on L^∞ and the accurate norm under $c = n + 1 + a + b$ of this case, where L^∞ denotes the set of all essentially bounded and measurable functions under the measure ν_t on $\mathbb{B}_n$.

Theorem 2. *The operator $S_{a,b,c}$ is bounded on L^∞ if and only if $a > 0$, $b > -1$, and $c = n + 1 + a + b$ or $a \ge 0$, $b > -1$, and $c < n + 1 + a + b$. Moreover, when $c = n + 1 + a + b$, we have*

$$\left\|S_{a,b,c}\right\|_{L^\infty \to L^\infty} = \frac{n!\Gamma(a)\Gamma(1+b)}{\Gamma^2\left((n+1+a+b)/2\right)}. \tag{5}$$

Notice $S_{a,b,c}$ is the generalization of "maximal Bergman projection" and Berezin transform which was first introduced by Berezin [6]. The boundedness of Berezin transform of $f \in L^1(\mathbb{D})$ is a well-known fact; see [7, Proposition 2.2]. But the norm of it was not calculated out until 2008 by Dostanić; see [8, Corollary 2]. Recently, the result by Dostanić has been extended to several complex variables in [9, Theorem 1.1]. Thus, Theorems 1 and 2 promote the main results in [8, 9]. And they also imply the following corollary.

Corollary 3. *Suppose $1 \le p < \infty$, $\alpha > -1$, and the norm of $P_\alpha^\sharp$ on L_α^p can be*

$$\left\|P_\alpha^\sharp\right\|_{L_\alpha^p \to L_\alpha^p} = \frac{\Gamma\left((\alpha+1)/p\right)\Gamma\left((\alpha+1) - (\alpha+1)/p\right)\Gamma(n+\alpha+1)}{\Gamma^2\left((n+1+\alpha)/2\right)\Gamma(\alpha+1)}, \tag{6}$$

which implies $\|P_\alpha^\sharp\|_{L_\alpha^p \to L_\alpha^p}$ grows at most like $(\alpha+1)^{-1}$ as $\alpha \to -1$.

Next, we will see that the boundedness of an operator called Berezin-type transform on L_t^p can also be obtained from our main results. The Berezin-type transform is defined by

$$\mathscr{B}_{k,\alpha,\beta} f(z) = C_{k,\alpha,\beta} \times \int_{\mathbb{B}_n} \frac{\left(1-|z|^2\right)^{n+\alpha+\beta+k+1}\left(1-|w|^2\right)^k}{(1-\langle z,w\rangle)^{n+\alpha+k+1}(1-\langle w,z\rangle)^{n+\beta+k+1}} f(w)\, d\nu(w), \tag{7}$$

where

$$C_{k,\alpha,\beta} = \frac{\Gamma(n+\alpha+k+1)\Gamma\left(n+\beta+k+1\right)}{\Gamma(n+1)\Gamma(k+1)\Gamma\left(n+\alpha+\beta+k+1\right)}, \tag{8}$$

and $n + \alpha + \beta > 0$, $n + \alpha > 0$, $n + \beta > 0$, and $k > -1$. The transform was introduced by Li and Liu [10] when they discuss whether the mean-value property implies (α, β)-harmonicity for integrable functions on the unit ball in $\mathbb{C}^n$. Notice that

$$\left|\mathscr{B}_{k,\alpha,\beta} f(z)\right| \le C_{k,\alpha,\beta} S_{a,b,c}|f|(z) \tag{9}$$

with $a = n + \alpha + \beta + k + 1$, $b = k$, and $c = n + 1 + a + b$. And $\mathscr{B}_{k,\alpha,\alpha} f(z) = C_{k,\alpha,\alpha} S_{a,b,c} f(z)$ as $\alpha = \beta$. Therefore, the boundedness of Berezin-type transform $\mathscr{B}_{k,\alpha,\beta}$ on L_t^p comes from the boundedness of the operator $S_{a,b,c}$ on L_t^p. Thus, we have the following result, which extends Propositions 3.3 and 3.4 in [10] combining the fact of Lemma 2.4 in [10] therein.

Corollary 4. *If $1 \le p < \infty$ such that $-p(n+\alpha+\beta+k+1) < t + 1 < p(k+1)$, then the Berezin-type $\mathscr{B}_{k,\alpha,\beta}$ is bounded on L_t^p and*

$$\left\|\mathscr{B}_{k,\alpha,\beta}\right\|_{L_t^p \to L_t^p} \le \lambda_{k,\alpha,\beta,p} \frac{\Gamma(n+\alpha+k+1)\Gamma\left(n+\beta+k+1\right)}{\Gamma^2\left(n+1+\left(\alpha+\beta\right)/2+k\right)}, \tag{10}$$

where

$$\lambda_{k,\alpha,\beta,p} = \frac{\Gamma\left(n+\alpha+\beta+k+1+(t+1)/p\right)\Gamma\left(k+1-(t+1)/p\right)}{\Gamma\left(n+\alpha+\beta+k+1\right)\Gamma(k+1)}. \tag{11}$$

Moreover, the Berezin-type transform is bounded on L^∞, and

$$\left\|\mathscr{B}_{k,\alpha,\beta}\right\|_{L^\infty \to L^\infty} \le \frac{\Gamma(n+\alpha+k+1)\Gamma\left(n+\beta+k+1\right)}{\Gamma^2\left(n+1+\left(\alpha+\beta\right)/2+k\right)}. \tag{12}$$

2. Preliminaries

A number of hypergeometric functions will appear throughout. We use the classical notation ${}_2F_1(\alpha, \beta; \gamma; z)$ to denote

$${}_2F_1(\alpha,\beta;\gamma;z) = \sum_{k=0}^{\infty} \frac{(\alpha)_k(\beta)_k}{(\gamma)_k}\frac{z^k}{k!}, \tag{13}$$

with $\gamma \ne 0, -1, -2, \ldots$, where

$$(\alpha)_0 = 1, \qquad (\alpha)_k = \alpha(\alpha+1)\cdots(\alpha+k-1) \quad \text{for } k \ge 1. \tag{14}$$

And the hypergeometric series in (13) converges absolutely for all the value of $|z| < 1$. Moreover, as $|z| \to 1^-$, it is easy to know that

$${}_2F_1(\alpha,\beta;\gamma;z) \approx \begin{cases} 1, & \text{if } \gamma-\alpha-\beta > 0; \\ \log\dfrac{1}{1-|z|}, & \text{if } \gamma-\alpha-\beta = 0; \\ (1-|z|)^{\gamma-\alpha-\beta}, & \text{if } \gamma-\alpha-\beta < 0, \end{cases} \tag{15}$$

where $a(z) \approx b(z)$ represents the ratio and $a(z)/b(z)$ has a positive finite limit as $|z| \to 1^-$. Now we list a few formulas for easy reference (see [11, Chapter II]):

$$ {}_2F_1(\alpha,\beta;\gamma;1) = \frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}, \quad \operatorname{Re}(\gamma-\alpha-\beta) > 0, \tag{16} $$

$$ {}_2F_1(\alpha,\beta;\gamma;z) = (1-z)^{\gamma-\alpha-\beta}\, {}_2F_1(\gamma-\alpha,\gamma-\beta;\gamma;z), \tag{17} $$

$$ \begin{aligned} {}_2F_1(\alpha,\beta;\gamma;z) &= \frac{\Gamma(\gamma)}{\Gamma(\lambda)\Gamma(\gamma-\lambda)} \\ &\quad \times \int_0^1 t^{\lambda-1}(1-t)^{\gamma-\lambda-1}\, {}_2F_1(\alpha,\beta;\lambda;tz)\,dt, \end{aligned} \tag{18} $$

$$ \operatorname{Re}\gamma > \operatorname{Re}\lambda > 0; \quad |\arg(1-z)| < \pi; \quad z \neq 1. $$

Lemma 5. *Suppose* $\operatorname{Re}\delta > 0$ *and* $\operatorname{Re}(\lambda+\delta-\alpha-\beta) > 0$. *Then*

$$ \int_0^1 t^{\lambda-1}(1-t)^{\delta-1}\, {}_2F_1(\alpha,\beta;\lambda;t)\,dt = \frac{\Gamma(\lambda)\Gamma(\delta)\Gamma(\lambda+\delta-\alpha-\beta)}{\Gamma(\lambda+\delta-\alpha)\Gamma(\lambda+\delta-\beta)}. \tag{19} $$

Proof. Note that, under the assumption of the lemma, both sides of (18) are continuous at $z = 1$. The lemma then follows by letting $z \to 1$ in (18) and applying (16). □

The following integral formulae concerning the hypergeometric function are significant for our main results. And all these formulae are contained in [12]. Now we list them.

Lemma 6 (see [12, Corollary 2.4]). *For* $\alpha \in \mathbb{R}$ *and* $\gamma > -1$, *we have*

$$ \int_{\mathbb{B}_n} \frac{(1-|w|^2)^\gamma}{|1-\langle z,w\rangle|^{2\alpha}}\,d\nu(w) = \frac{n!\Gamma(1+\gamma)}{\Gamma(n+1+\gamma)}\, {}_2F_1(\alpha,\alpha;n+1+\gamma;|z|^2). \tag{20} $$

Lemma 6 is also contained implicitly in the proof of Theorem 1.4.10 in [13] (see the formula in page 19, line 5 of [13]).

Lemma 7 (see [12, Corollary 2.5]). *Suppose that* $\alpha, \beta > 0$, $\gamma \in \mathbb{R}$, *and* $n+\alpha+\beta-2\gamma > 0$. *Then*

$$ \int_{\mathbb{B}_n} |z|^{2\beta}(1-|z|^2)^{\alpha-1} \left\{ \int_{\mathbb{B}_n} \frac{(1-|w|^2)^{\beta-1}}{|1-\langle z,w\rangle|^{2\gamma}}\,d\nu(w) \right\} d\nu(z) = \frac{n(n!)\Gamma(\alpha)\Gamma(\beta)\Gamma(n+\alpha+\beta-2\gamma)}{\Gamma^2(n+\alpha+\beta-\gamma)}. \tag{21} $$

Proof. Using Lemma 6 in the inner integral, we have

$$ \begin{aligned} &\frac{n!\Gamma(\beta)}{\Gamma(n+\beta)} \int_{\mathbb{B}_n} |z|^{2\beta}(1-|z|^2)^{\alpha-1} \times {}_2F_1(\gamma,\gamma;n+\beta;|z|^2)\,d\nu(z) \\ &= \frac{n(n!)\Gamma(\beta)}{\Gamma(n+\beta)} \int_0^1 r^{n+\beta-1}(1-r)^{\alpha-1} \times {}_2F_1(\gamma,\gamma;n+\beta;|z|^2)\,dr. \end{aligned} \tag{22} $$

Then (19) gives the result. □

The following result, usually called Schur's test, is a very effective tool in proving the L^p-boundedness of integral operators. See, for example, [3].

Lemma 8. *Suppose that* (X,μ) *is a* σ*-finite measure space,* $K(x,y)$ *is a nonnegative measurable function on* $X \times X$*, and* T *is the associated integral operator:*

$$ Tf(x) = \int_X K(x,y) f(y)\,d\mu(y). \tag{23} $$

Let $1 < p < \infty$ *and* $1/p + 1/q = 1$. *If there exist a positive constant* C *and a positive measurable function* u *on* X *such that*

$$ \int_X K(x,y) u(y)^q d\mu(y) \le Cu(x)^q, \tag{24} $$

for almost every x *in* X*, and*

$$ \int_X K(x,y) u(x)^p d\mu(x) \le Cu(y)^p, \tag{25} $$

for almost every y *in* X*, then* T *is bounded on* $L^p(X,\mu)$ *with* $\|T\| \le C$.

3. The Improvement

The section mainly proposes the condition "c is neither 0 nor a negative integer" can be omitted in Theorem A. Notice the condition is only used to give $c \le n+1+a+b$ while proving the necessity for the boundedness of the operator $S_{a,b,c}$ on L_t^p $(1 \le p < \infty)$; see [1, lemma 12]. Now we will give a new proof of the necessity for the boundedness of $S_{a,b,c}$ on L_t^p in Propositions 9 and 11 to introduce the condition can be put off.

Proposition 9. *Suppose the operator* $S_{a,b,c}$ *is bounded on* L_t^p $(1 < p < \infty)$*, and then* $-pa < t+1 < p(b+1)$, $c \le n+1+a+b$.

Proof. Let q be the number such that $1/p + 1/q = 1$. For any fixed $\epsilon > 0$, define

$$ \begin{aligned} g_\epsilon(w) &= C_1(\epsilon)(1-|w|^2)^{(\epsilon-(t+1))/p}, \\ h_\epsilon(z) &= C_2(\epsilon)(1-|z|^2)^{(\epsilon-(t+1))/q}|z|^{2(b+1+(\epsilon-t-1)/p)}, \end{aligned} \tag{26} $$

where

$$C_1(\epsilon) = \left\{ \frac{\Gamma(\epsilon)\Gamma(n+1)}{\Gamma(n+\epsilon)} \right\}^{-1/p}, \quad (27)$$

$$C_2(\epsilon) = \left\{ \frac{n\Gamma(\epsilon)\Gamma(n+q(b+1)+q(\epsilon-(t+1))/p)}{\Gamma(n+q(b+1)+q(\epsilon-(t+1))/p+\epsilon)} \right\}^{-1/q}. \quad (28)$$

Easy calculation shows $\|g_\epsilon\|_{p,t} = \|h_\epsilon\|_{q,t} = 1$. Notice the fact

$$\begin{aligned} &\|S_{a,b,c}\|_{L_t^p \to L_t^p} \\ &= \sup_{\substack{\|f\|_{p,t}=1 \\ \|g\|_{q,t}=1}} \\ &\times \left\{ \left| \int_{\mathbb{B}_n} \left(\int_{\mathbb{B}_n} (1-|z|^2)^a \right. \right. \right. \\ &\left. \left. \left. \times \frac{(1-|w|^2)^{b-t}}{|1-\langle z,w\rangle|^c} f(w)\, dv_t(w) \right) \overline{g(z)} dv_t(z) \right| \right\}. \end{aligned} \quad (29)$$

Then the boundedness of the operator $S_{a,b,c}$ on L_t^p leads to the integral

$$\begin{aligned} &\left| \int_{\mathbb{B}_n} \left\{ \int_{\mathbb{B}_n} \frac{(1-|z|^2)^a (1-|w|^2)^{b-t}}{|1-\langle z,w\rangle|^c} g_\epsilon(w)\, dv_t(w) \right\} \right. \\ &\left. \times \overline{h_\epsilon(z)} dv_t(z) \right| \\ &\leq \|S_{a,b,c}\|_{L_t^p \to L_t^p} < +\infty. \end{aligned} \quad (30)$$

Hence, using Lemma 7 with $\alpha = a + \epsilon/q + (t+1)/p$, $\beta = b + 1 + (\epsilon - (t+1))/p$, and $\gamma = c/2$, we can conclude that

$$\begin{gathered} a + \frac{\epsilon}{q} + \frac{t+1}{p} > 0, \qquad b + 1 + \frac{\epsilon - (t+1)}{p} > 0, \\ n + 1 + a + b + \epsilon - c > 0. \end{gathered} \quad (31)$$

Then the arbitrariness of ϵ gives

$$-pa \leq t + 1 \leq p(b+1), \qquad c \leq n + 1 + a + b. \quad (32)$$

Now, we will give the proof by dividing into the following two cases.

When $c = n + 1 + a + b$, by Lemma 7, the integral in (30) equals

$$\begin{aligned} &\frac{n!\Gamma(a + (\epsilon/q) + ((t+1)/p))\Gamma(b + 1 + ((\epsilon - (t+1))/p))}{\Gamma^2((n+1+a+b)/2+\epsilon)} \\ &\times \left\{ \frac{\Gamma(n+\epsilon)}{\Gamma(n)} \right\}^{1/p} \\ &\times \left\{ \frac{\Gamma(n+q(b+1)+q(\epsilon-(t+1))/p+\epsilon)}{\Gamma(n+q(b+1)+q(\epsilon-(t+1))/p)} \right\}^{1/q}. \end{aligned} \quad (33)$$

Then letting $\epsilon \to 0^+$, by (30), we can know the limits

$$\begin{aligned} 0 &\leq \lim_{\epsilon \to 0^+} \frac{n!\Gamma(a + (\epsilon/q) + ((t+1)/p))\Gamma(b + 1 + ((\epsilon - (t+1))/p))}{\Gamma^2((n+1+a+b)/2+\epsilon)} \\ &\leq \|S_{a,b,c}\|_{L_t^p \to L_t^p}. \end{aligned} \quad (34)$$

Then the boundedness of the operator $S_{a,b,c}$ gives $-pa < t + 1 < p(b+1)$.

When $c < n + 1 + a + b$, take the function

$$f_\lambda(z) = (1-|z|^2)^\lambda, \quad (35)$$

with $\lambda > a$. The condition (32) implies the function $f_\lambda \in L_t^p$. And using Lemma 6, we have

$$\begin{aligned} S_{a,b,c} f_\lambda(z) &= (1-|z|^2)^a \int_{\mathbb{B}_n} \frac{(1-|w|^2)^b}{|1-\langle z,w\rangle|^c} f_\lambda(w)\, dv(w) \\ &= (1-|z|^2)^a \frac{n!\Gamma(1+b+\lambda)}{\Gamma(n+1+b+\lambda)} \\ &\quad \times {}_2F_1\left(\frac{c}{2}, \frac{c}{2}; n+1+b+\lambda; |z|^2\right). \end{aligned} \quad (36)$$

According to (15), we can obtain that $S_{a,b,c} f_\lambda(z) \approx (1-|z|^2)^a$. Thus the boundedness of the operator $S_{a,b,c}$ on $L_t^p(\mathbb{B}_n)$ gives that $pa + t > -1$; that is, $-pa < t + 1$. Now we consider the adjoint operator $S^*_{a,b,c}$ of the operator $S_{a,b,c}$; that is,

$$S^*_{a,b,c} f(z) = (1-|z|^2)^{b-t} \int_{\mathbb{B}_n} \frac{(1-|w|^2)^{a+t}}{|1-\langle z,w\rangle|^c} f(w)\, dv(w). \quad (37)$$

The boundedness of $S_{a,b,c}$ on L_t^p implies the boundedness of $S^*_{a,b,c}$ on L_t^q. With the similar discussion above, we can obtain that $q(b-t) + t > -1$; that is, $t + 1 < p(b+1)$. □

When $c = n + 1 + a + b$, (34) implies the following result.

Corollary 10. *Suppose $c = n + 1 + a + b$ and $1 < p < \infty$, $-pa < t + 1 < p(b+1)$, and then*

$$\|S_{a,b,c}\|_{L_t^p \to L_t^p} \geq \frac{n!\Gamma(a + (t+1)/p)\Gamma(b + 1 - (t+1)/p)}{\Gamma^2((n+1+a+b)/2)}. \quad (38)$$

Proposition 11. *The operator $S_{a,b,c}$ is bounded on L_t^1 if and only if $-a < t + 1 < b + 1, c = n + 1 + a + b$ or $-a < t + 1 \leq b + 1, c < n + 1 + a + b$. And when $c = n + 1 + a + b$, we have*

$$\|S_{a,b,c}\|_{L_t^1 \to L_t^1} = \frac{n!\Gamma(1+a+t)\Gamma(b-t)}{\Gamma^2((n+1+a+b)/2)}. \quad (39)$$

When $c < n + 1 + a + b$, $-a < t + 1 = b + 1$, we have

$$\|S_{a,b,c}\|_{L_t^1 \to L_t^1} = \frac{n!\Gamma(1+a+b)\Gamma(\sigma)}{\Gamma^2((n+1+a+b+\sigma)/2)}, \quad (40)$$

where $\sigma = (n+1+a+b) - c$.

Proof. By Lemma 6, we have

$$\begin{aligned}\|S_{a,b,c}\|_{L^1_t\to L^1_t} &= \|S^*_{a,b,c}\|_{L^\infty\to L^\infty}\\ &= \sup_{z\in\mathbb{B}_n}\left(1-|z|^2\right)^{b-t}\int_{\mathbb{B}_n}\frac{\left(1-|w|^2\right)^{a+t}}{|1-\langle z,w\rangle|^c}d\nu(w)\\ &= \frac{n!\Gamma(1+a+t)}{\Gamma(n+1+a+t)}\sup_{z\in\mathbb{B}_n}\left(1-|z|^2\right)^{b-t}\\ &\quad\times {}_2F_1\left(\frac{c}{2},\frac{c}{2};n+1+a+t;|z|^2\right),\end{aligned} \tag{41}$$

where $S^*_{a,b,c}$ denotes the adjoint operator of $S_{a,b,c}$. Then, using (15), we can obtain that the operator $S_{a,b,c}$ is bounded on L^1_t if and only if

$$\begin{gathered}1+a+t>0,\\ b-t>0,\\ n+1+a+t-c\geq t-b,\end{gathered} \tag{42}$$

or

$$\begin{gathered}1+a+t>0,\\ b-t=0,\\ n+1+a+t-c>0,\end{gathered} \tag{43}$$

which gives the first part of the proposition.

Now we will give the second part. When $c<n+1+a+b$ and $-a<t+1=b+1$, the hypergeometric function in (41) is increasing since its Taylor coefficients are all positive. Applying (16), we have (40). When $c=n+1+a+b$, (17) gives

$$\begin{aligned}&{}_2F_1\left(\frac{n+1+a+b}{2},\frac{n+1+a+b}{2};n+1+a+t;|z|^2\right)\\ &\quad=\left(1-|z|^2\right)^{t-b}\\ &\qquad\times {}_2F_1\left(\frac{n+1+a-b}{2}+t,\frac{n+1+a-b}{2}+t;\right.\\ &\qquad\left. n+1+a+t;|z|^2\right).\end{aligned} \tag{44}$$

Thus (41), the increase of the last hypergeometric function, and (16) lead to

$$\begin{aligned}&\|S_{a,b,c}\|_{L^1_t\to L^1_t}\\ &\quad=\|S^*_{a,b,c}\|_{L^\infty\to L^\infty}\\ &\quad=\frac{n!\Gamma(1+a+t)}{\Gamma(n+1+a+t)}\\ &\qquad\times {}_2F_1\left(\frac{n+1+a-b}{2}+t,\frac{n+1+a-b}{2}+t;\right.\\ &\qquad\left. n+1+a+t;1\right)\\ &\quad=\frac{n!\Gamma(1+a+t)\Gamma(b-t)}{\Gamma^2((n+1+a+b)/2)}.\end{aligned} \tag{45}$$

□

4. The Proof of Theorems 1 and 2

Proof of Theorems 1 and 2. Since

$$\|S_{a,b,c}\|_{L^\infty\to L^\infty}=\sup_{z\in\mathbb{B}_n}\left(1-|z|^2\right)^a\int_{\mathbb{B}_n}\frac{\left(1-|w|^2\right)^b}{|1-\langle z,w\rangle|^c}d\nu(w), \tag{46}$$

therefore Theorem 2 comes out as the same discussion as Proposition 11.

Next, we will concentrate on the proof of Theorem 1. Remember the hypothesis $c=n+1+a+b$ throughout the following proof. Since (39) gives the case of $p=1$, for the case $1<p<\infty$, Corollary 10 gives the lower bound of $\|S_{a,b,c}\|_{L^p_t\to L^p_t}$. Thus we only show the fact

$$\|S_{a,b,c}\|_{L^p_t\to L^p_t}\leq\frac{n!\Gamma(a+(t+1)/p)\Gamma(b+1-(t+1)/p)}{\Gamma^2((n+1+a+b)/2)}. \tag{47}$$

To this end, we will use Schur's test (Lemma 8) with

$$K(z,w)=\frac{\left(1-|z|^2\right)^a\left(1-|w|^2\right)^{b-t}}{|1-\langle z,w\rangle|^{n+1+a+b}}. \tag{48}$$

Set

$$u_t(z)=\left(1-|z|^2\right)^{-(t+1)/(pq)}, \tag{49}$$

where q is the conjugate exponent of p such that $1/p+1/q=1$. It then suffices to show

$$\begin{aligned}&\left(1-|z|^2\right)^a\int_{\mathbb{B}_n}\frac{\left(1-|w|^2\right)^{b-t}}{|1-\langle z,w\rangle|^{n+1+a+b}}u_t(w)^q d\nu_t(w)\\ &\quad\leq\frac{n!\Gamma(a+(t+1)/p)\Gamma(b+1-(t+1)/p)}{\Gamma^2((n+1+a+b)/2)}u_t(z)^q,\end{aligned} \tag{50}$$

for all $z\in\mathbb{B}_n$, and

$$\begin{aligned}&\left(1-|w|^2\right)^{b-t}\int_{\mathbb{B}_n}\frac{\left(1-|z|^2\right)^a}{|1-\langle z,w\rangle|^{n+1+a+b}}u_t(z)^p d\nu_t(z)\\ &\quad\leq\frac{n!\Gamma(a+(t+1)/p)\Gamma(b+1-(t+1)/p)}{\Gamma^2((n+1+a+b)/2)}u_t(w)^p\end{aligned} \tag{51}$$

for all $w \in \mathbb{B}_n$. We only prove (50), since (51) comes from the same way as (50). Applying Lemma 6 and (17), we have

$$\begin{aligned}
&\left(1-|z|^2\right)^a \int_{\mathbb{B}_n} \frac{\left(1-|w|^2\right)^{b-t}}{|1-\langle z,w\rangle|^{n+1+a+b}} u_t(w)^q d\nu_t(w) \\
&= \left(1-|z|^2\right)^a \frac{n!\Gamma\left(b+1-(t+1)/p\right)}{\Gamma\left(n+b+1-(t+1)/p\right)} \\
&\quad \times {}_2F_1\left(\frac{n+1+a+b}{2}, \frac{n+1+a+b}{2}; \right. \\
&\quad \left. n+b+1-\frac{t+1}{p}; |z|^2\right) \\
&= \frac{n!\Gamma\left(b+1-(t+1)/p\right)}{\Gamma\left(n+b+1-(t+1)/p\right)} \left(1-|z|^2\right)^{-(t+1)/p} \\
&\quad \times {}_2F_1\left(\frac{n+1+b-a}{2}-\frac{t+1}{p}, \frac{n+1+b-a}{2} \right. \\
&\quad \left. -\frac{t+1}{p}; n+1+b-\frac{t+1}{p}; |z|^2\right).
\end{aligned} \tag{52}$$

By (16), the last hypergeometric function is bounded from the above by

$$\begin{aligned}
&{}_2F_1\left(\frac{n+1+b-a}{2}-\frac{t+1}{p}, \frac{n+1+b-a}{2} \right. \\
&\quad \left. -\frac{t+1}{p}; n+1+b-\frac{t+1}{p}; 1\right) \\
&= \frac{\Gamma\left(n+1+b-(t+1)/p\right)\Gamma\left(a+(t+1)/p\right)}{\Gamma^2\left((n+1+a+b)/2\right)},
\end{aligned} \tag{53}$$

since it is increasing on the interval $[0,1)$. This proves (50), which in turn implies (47). The proof is completed. □

5. Remark

The topic on the exact norm of an operator is an interesting but difficult problem. In this note, we only give the accurate norm of the generalized operator $S_{a,b,c}$ on L_t^p under $c = n+1+a+b$. But for other cases, except the particular case (40), we can give an upper bound of $\|S_{a,b,c}\|_{L_t^p \to L_t^p}$ by Theorem 1 according to the fact

$$\frac{\left(1-|z|^2\right)^a \left(1-|w|^2\right)^{b-t}}{|1-\langle z,w\rangle|^c} \leq \frac{2^\sigma \left(1-|z|^2\right)^a \left(1-|w|^2\right)^{b-t}}{|1-\langle z,w\rangle|^{n+1+a+b}} \tag{54}$$

and a lower bound for one fixed $\epsilon > 0$ by (30) and Lemma 7; thus the problem of the norm of other cases may be left as an open problem to consider.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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