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Research Article

Operators on Spaces of Bounded Vector-Valued Continuous Functions with Strict Topologies

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Let X be a completely regular Hausdorff space, and let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be Banach spaces. Let $C_b(X, E)$ be the space of all E-valued bounded, continuous functions defined on X, equipped with the strict topologies β_z, where $z = \sigma, \infty, p, \tau, t$. General integral representation theorems of $(\beta_z, \|\cdot\|_F)$-continuous linear operators $T : C_b(X, E) \to F$ with respect to the corresponding operator-valued measures are established. Strongly bounded and $(\beta_z, \|\cdot\|_F)$-continuous operators $T : C_b(X, E) \to F$ are studied. We extend to "the completely regular setting" some classical results concerning operators on the spaces $C(X, E)$ and $C_o(X, E)$, where X is a compact or a locally compact space.

1. Introduction and Terminology

Throughout the paper let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be real Banach spaces, and let E' and F' denote the Banach duals of E and F, respectively. By $B_{F'}$ and B_E we denote the closed unit ball in F' and E, respectively. By $\mathscr{L}(E, F)$ we denote the space of all bounded linear operators $U : E \to F$. Given a locally convex space (L, ξ) by $(L, \xi)'$ or L'_ξ we will denote its topological dual. We denote by $\sigma(L, K)$ the weak topology on L with respect to a dual pair $\langle L, K\rangle$.

Assume that X is a completely regular Hausdorff space. Let $C_b(X, E)$ stand for the Banach space of all bounded continuous, E-valued functions on X provided with the uniform norm $\|\cdot\|$. We write $C_b(X)$ instead of $C_b(X, \mathbb{R})$. By $C_b(X, E)'$ we denote the Banach dual of $C_b(X, E)$. For $f \in C_b(X, E)$ let $\tilde{f}(t) = \|f(t)\|_E$ for $t \in X$.

Let $\mathscr{B}$ (resp., $\mathscr{B}a$) be the algebra (resp., σ-algebra) of Baire sets in X, which is the algebra (resp., σ-algebra) generated by the class $\mathscr{Z}$ of all zero sets of functions of $C_b(X)$. By $\mathscr{P}$ we denote the family of all cozero sets in X. Let $B(\mathscr{B}, E)$ stand for the Banach space of all totally $\mathscr{B}$-measurable functions $f : X \to E$ (the uniform limits of sequences of E-valued $\mathscr{B}$-simple functions) provided with the uniform norm $\|\cdot\|$ (see [1, 2]). We will write $B(\mathscr{B})$ instead of $B(\mathscr{B}, \mathbb{R})$.

Strict topologies β_z on $C_b(X)$ and $C_b(X, E)$ (for $z = \sigma, \infty, p, \tau, t$) play an important role in the topological measure theory (see [3–12] for definitions and more details). Recall that a subset H of $C_b(X, E)$ is said to be *solid* if $f_1 \in C_b(X, E)$ and $f_2 \in H$ with $\tilde{f}_1(t) \le \tilde{f}_2(t)$ for $t \in X$ imply that $f_1 \in H$. Then β_z are locally convex-solid topologies on $C_b(X, E)$; that is, they have a local base at 0 consisting of convex and solid sets (see [6, Theorem 8.1], [10, Theorem 5]). We have $\beta_t \subset \beta_\tau \subset \beta_\infty \subset \beta_\sigma \subset \mathscr{T}_{\|\cdot\|}$ and $\beta_t \subset \beta_p \subset \beta_\sigma$. For a net (f_α) in $C_b(X, E)$, $f_\alpha \to 0$ for β_z if and only if $\tilde{f}_\alpha \to 0$ for β_z in $C_b(X)$ (see [6, 10]).

Let $C_b(X) \otimes E$ stand for the algebraic tensor product of $C_b(X)$ and E; that is, $C_b(X) \otimes E$ is the space of all functions $\sum_{i=1}^n (u_i \otimes x_i)$, where $u_i \in C_b(X)$, $x_i \in E$ for $i = 1, \dots, n$, and $(u_i \otimes x_i)(t) = u_i(t) x_i$ for $t \in X$. Then $C_b(X) \otimes E$ is dense in $(C_b(X, E), \beta_z)$ for $z = \infty, \tau, t$ (see [6, 8]). Moreover, $C_b(X) \otimes E$ is dense in $(C_b(X, E), \beta_\sigma)$ if X or E is a D-space (see [6, Theorem 5.2], [13]) and in $(C_b(X, E), \beta_p)$ if X is real-compact (see [10, Theorem 7]).

Let $C_{rc}(X, E)$ denote the Banach space of all continuous functions $h : X \to E$ such that $h(X)$ is a relatively compact set in E, provided with the uniform norm $\|\cdot\|$. Then $C_b(X) \otimes E \subset C_{rc}(X, E) \subset B(\mathscr{B}, E)$.

Linear operators from the spaces $C_{rc}(X,E)$ and $C_b(X,E)$, equipped with the strict topologies $\beta_z(z = \sigma,\infty,\tau)$ to a locally convex space (F,ξ), were studied by Katsaras and Liu [14], Aguayo-Garrido, Nova-Yanéz and Sanchez [15, 16], and Khurana [17]. In particular, Katsaras and Liu found an integral representation of weakly compact operators $S : C_{rc}(X,E) \to F$ and characterizations of (β_z,ξ)-continuous and weakly compact operators $S : C_{rc}(X,E) \to F$ for $z = \sigma,\tau$ (see [14, Theorems 3, 4, 5]). Aguayo-Arrido and Nova-Yanéz derived a Riesz representation theorem for (β_z,ξ)-continuous and weakly compact operators $T : C_b(X,E) \to F$ for $z = \infty,\tau$ in terms of their representing operator measures (see [15, Theorems 5 and 6]). If X is a locally compact space, continuous operators on $C_o(X,E)$ were studied by Dobrakov (see [18]) and Mitter and Young (see [19]).

In this paper we develop the theory of continuous linear operators from $C_b(X,E)$, equipped with the strict topologies β_z $(z = \sigma,\infty,p,\tau,t)$ to a Banach space $(F,\|\cdot\|_F)$. In particular, we extend to "the completely regular setting" some classical results of Brooks and Lewis (see [20, Theorem 5], [21, Theorem 5.2], [22, Theorem 2.1]) concerning operators on the spaces $C(X,E)$ and $C_o(X,E)$, where X is a compact or a locally compact space, respectively. In Section 2, using the device of embedding the space $B(\mathscr{B},E)$ into $C_{rc}(X,E)''$ (the Banach bidual of $C_{rc}(X,E)$), we state the integral representation of bounded linear operators from $C_{rc}(X,E)$ to F. In Section 3 we derive general Riesz representation theorems for $(\beta_z,\|\cdot\|_F)$-continuous linear operators $T : C_b(X,E) \to F$ $(z = \sigma,\infty,p,\tau,t)$ with respect to the corresponding measures $m : \mathscr{B} \to \mathscr{L}(E,F'')$ (see Theorems 9 and 14 below). Section 4 is devoted to the study of $(\beta_\sigma,\|\cdot\|_F)$-continuous and strongly bounded operators $T : C_b(X,E) \to F$.

2. Integral Representation of Bounded Linear Operators on $C_{rc}(X,E)$

Let $M(X)$ stand for the Banach lattice of all Baire measures on $\mathscr{B}$, provided with the norm $\|\nu\| = |\nu|(X)$ (= the total variation of ν). Due to the Alexandrov representation theorem $C_b(X)'$ can be identified with $M(X)$ through the lattice isomorphism $M(X) \ni \nu \mapsto \varphi_\nu \in C_b(X)'$, where $\varphi_\nu(u) = \int_X u\, d\nu$ for $u \in C_b(X)$ and $\|\varphi_\nu\| = \|\nu\|$ (see [4, Theorem 5.1]).

By $M(X,E')$ we denote the set of all finitely additive measures $\mu : \mathscr{B} \to E'$ with the following properties:

(i) for each $x \in E$, the function $\mu_x : \mathscr{B} \to \mathbb{R}$ defined by $\mu_x(A) = \mu(A)(x)$ belongs to $M(X)$,

(ii) $|\mu|(X) < \infty$, where $|\mu|(A)$ stands for the variation of μ on $A \in \mathscr{B}$.

In view of [23, Theorem 2.5] $C_{rc}(X,E)'$ can be identified with $M(X,E')$ through the linear mapping $M(X,E') \ni \mu \mapsto \Phi_\mu \in C_{rc}(X,E)'$, where $\Phi_\mu(h) = \int_X h d\mu$ for $h \in C_{rc}(X,E)$ and $\|\Phi_\mu\| = |\mu|(X)$. Then one can embed $B(\mathscr{B},E)$ into $C_{rc}(X,E)''$ by the mapping $\pi : B(\mathscr{B},E) \to C_{rc}(X,E)''$, where for $g \in B(\mathscr{B},E)$,

$$\pi(g)(\Phi_\mu) := \int_X g d\mu \quad \text{for } \mu \in M(X,E'). \tag{1}$$

Let $i_F : F \to F''$ denote the canonical embedding; that is, $i_F(y)(y') = y'(y)$ for $y \in F$, $y' \in F'$. Moreover, let $j_F : i_F(F) \to F$ stand for the left inverse of i_F; that is, $j_F \circ i_F = id_F$.

Assume that $S : C_{rc}(X,E) \to F$ is a bounded linear operator. Let

$$\widehat{S} := S'' \circ \pi : B(\mathscr{B},E) \longrightarrow F'', \tag{2}$$

where $S' : F' \to C_{rc}(X,E)'$ and $S'' : C_{rc}(X,E)'' \to F''$ denote the conjugate and biconjugate operators of S, respectively. Then we can define a measure $m : \mathscr{B} \to \mathscr{L}(E,F'')$ (called a *representing measure* of S) by

$$m(A)(x) := \widehat{S}(\mathbb{1}_A \otimes x) = (S'' \circ \otimes \pi)(\mathbb{1}_A \otimes x) \quad \text{for } A \in \mathscr{B},\ x \in E. \tag{3}$$

Then $\widetilde{m}(X) < \infty$, where the semivariation $\widetilde{m}(A)$ of m on $A \in \mathscr{B}$ is defined by $\widetilde{m}(A) := \sup \|\sum m(A_i)(x_i)\|_{F''}$, where the supremum is taken over all finite $\mathscr{B}$-partitions (A_i) of A and $x_i \in B_E$ for each i. For $y' \in F'$ let us put

$$m_{y'}(A)(x) := (m(A)(x))(y') \quad \text{for } A \in \mathscr{B}, x \in E. \tag{4}$$

Let $|m_{y'}|(A)$ stand for the variation of $m_{y'}$ on A. Then (see [1, Section 4, Proposition 5])

$$\widetilde{m}(A) = \sup\{|m_{y'}|(A) : y' \in B_{F'}\}. \tag{5}$$

The following general properties of the operator $\widehat{S} : B(\mathscr{B},E) \to F''$ are well known (see [1, Section 6], [2, Section 1], [13, 24]):

$$\widehat{S}(g) = \int_X g dm \quad \text{for } g \in B(\mathscr{B},E), \quad \|\widehat{S}\| = \widetilde{m}(X), \tag{6}$$

and for each $y' \in F'$,

$$\widehat{S}(g)(y') = \int_X g dm_{y'} \quad \text{for } g \in B(\mathscr{B},E). \tag{7}$$

For $A \in \mathscr{B}$ let

$$\int_A g dm := \int_X \mathbb{1}_A g dm \quad \text{for } g \in B(\mathscr{B},E). \tag{8}$$

From the general properties of $\widehat{S}$ it follows that

$$\widehat{S}(C_{rc}(X,E)) \subset i_F(F), \qquad S(h) = j_F\left(\int_X h dm\right) \quad \text{for } h \in C_{rc}(X,E). \tag{9}$$

Hence for each $y' \in F'$ we get

$$y'(S(h)) = \int_X h dm_{y'} \quad \text{for } h \in C_{rc}(X,E), \tag{10}$$

and hence $m_{y'} \in M(X, E')$. Moreover, we have

$$\begin{aligned} \|S\| &= \|S'\| \\ &= \sup\{\|S'(y')\| : y' \in B_{F'}\} \\ &= \sup\{\|y' \circ S\| : y' \in B_{F'}\} \\ &= \sup\{\|\Phi_{m_{y'}}\| : y' \in B_{F'}\} \\ &= \sup\{|m_{y'}|(X) : y' \in B_{F'}\}, \end{aligned} \tag{11}$$

and using (5) we get

$$\|S\| = \widetilde{m}(X). \tag{12}$$

By $M(X, \mathscr{L}(E, F''))$ we will denote the space of all measures $m : \mathscr{B} \to \mathscr{L}(E, F'')$ such that $\widetilde{m}(X) < \infty$ and $m_{y'} \in M(X, E')$ for each $y' \in F'$. Thus the representing measure m of S belongs to $M(X, \mathscr{L}(E, F''))$.

For any $x \in E$ define

$$\begin{aligned} S_x(u) &:= S(u \otimes x) \quad \text{for } u \in C_b(X), \\ m_x(A) &:= m(A)(x) \quad \text{for } A \in \mathscr{B}. \end{aligned} \tag{13}$$

Then $S_x : C_b(X) \to F$ is a bounded linear operator. Let $\chi : B(\mathscr{B}) \to C_b(X)''$ stand for the canonical embedding; that is, for $u \in B(\mathscr{B})$,

$$\chi(u)(\varphi_\nu) = \int_X u\, d\nu \quad \text{for } \nu \in M(X). \tag{14}$$

Let

$$\widehat{S}_x := (S_x)'' \circ \chi : B(\mathscr{B}) \longrightarrow F''. \tag{15}$$

Then

$$\begin{aligned} \widehat{S}_x(C_b(X)) &\subset i_F(F), \\ S_x(u) &= j_F(\widehat{S}_x(u)) \quad \text{for } u \in C_b(X). \end{aligned} \tag{16}$$

The following lemma will be useful.

Lemma 1. *Let $S : C_{rc}(X, E) \to F$ be a bounded linear operator. Then $S''(\pi(\mathbb{1}_A \otimes x)) = (S_x)''(\chi(\mathbb{1}_A))$ for any $x \in E$ and $A \in \mathscr{B}$.*

Proof. Let $y' \in F'$. Then for each $u \in C_b(X)$,

$$\begin{aligned} (y' \circ S_x)(u) &= y'(S(u \otimes x)) \\ &= \int_X (u \otimes x)\, dm_{y'} = \int_X u\, dm_{x,y'} \\ &= \varphi_{m_{x,y'}}(u). \end{aligned} \tag{17}$$

Hence we have

$$\begin{aligned} &(S_x)''(\chi(\mathbb{1}_A))(y') \\ &\quad = \chi(\mathbb{1}_A)(S_x'(y')) \\ &\quad = \chi(\mathbb{1}_A)(y' \circ S_x) = \chi(\mathbb{1}_A)(\varphi_{m_{x,y'}}) \\ &\quad = \int_X \mathbb{1}_A dm_{x,y'} = m_{x,y'}(\mathbb{1}_A) = m_x(\mathbb{1}_A)(y'). \end{aligned} \tag{18}$$

On the other hand, for each $h \in C_{rc}(X, E)$, $(y' \circ S)(h) = \int_X h\, dm_{y'} = \Phi_{m_{y'}}(h)$, and hence

$$\begin{aligned} &S''(\pi(\mathbb{1}_A \otimes x)) \\ &\quad = (\mathbb{1}_A \otimes x)(S'(y')) = \pi(\mathbb{1}_A \otimes x)(y' \circ S) \\ &\quad = \pi(\mathbb{1}_A \otimes x)(\Phi_{m_{y'}}) = \Phi_{m_{y'}}(\mathbb{1}_A \otimes x) \\ &\quad = \int_X (\mathbb{1}_A \otimes x)\; dm_{y'} = m_{y'}(A)(x) = m_x(\mathbb{1}_A)(y'). \end{aligned} \tag{19}$$

It follows that $S''(\pi(\mathbb{1}_A \otimes x)) = (S_x)''(\chi(\mathbb{1}_A))$, as desired. □

From Lemma 1 for $A \in \mathscr{B}$ and $x \in E$ we get

$$m_x(A) := \widehat{S}(\mathbb{1}_A \otimes x) = S''(\pi(\mathbb{1}_A \otimes x)) = (S_x)''(\chi(\mathbb{1}_A)); \tag{20}$$

that is,

$$m_x(A) = \widehat{S}_x(\mathbb{1}_A), \qquad \widehat{S}_x(u) = \int_X u\, dm_x \quad \text{for } u \in B(\mathscr{B}). \tag{21}$$

Now we are ready to prove the following Bartle-Dunford-Schwartz type theorem (see [25, Theorem 5, pages 153-154]).

Theorem 2. *Let $S : C_{rc}(X, E) \to F$ be a bounded linear operator and let $M(X, \mathscr{L}(E, F''))$ be its representing measure. Then for each $x \in E$ the following statements are equivalent.*

(i) *$S_x : C_b(X) \to F$ is weakly compact.*

(ii) *$m(A)(x) \in i_F(F)$ for each $A \in \mathscr{B}$ and $\{j_F(m(A)(x)) : A \in \mathscr{B}\}$ is a relatively weakly compact set in F.*

(iii) *$m_x : \mathscr{B} \to F''$ is strongly bounded.*

Proof. (i)⇒(ii) Assume that S_x is weakly compact. Then by the Gantmacher theorem $(S_x)''(C_b(X)'') \subset i_F(F)$ and $(S_x)'' : C_b(X)'' \to F''$ is weakly compact (see [26, Theorem 17.2]). Hence $\widehat{S}_x(B(\mathscr{B})) \subset i_F(F)$ and $\widehat{S}_x : B(\mathscr{B}) \to F''$ is weakly compact. In view of (21) for each $x \in E$, $m_x(A) \in i_F(F)$ for $A \in \mathscr{B}$ and $m_x : \mathscr{B} \to F''$ is strongly bounded (see [25, Theorem 1, page 148]). It follows that $\{j_F(m(A)(x)) : A \in \mathscr{B}\}$ is a relatively weakly compact subset of F (see [24, Theorem 7]).

(ii)⇒(iii) It follows from [24, Theorem 7].

(iii)⇒(i) Assume that $m_x : \mathscr{B} \to F''$ is strongly bounded. Then by (21) $\widehat{S}_x : B(\mathscr{B}) \to F''$ is weakly compact and in view of (16) we derive that S_x is weakly compact. □

3. Integral Representation of Continuous Linear Operators on $C_b(X,E)$

The spaces of all σ-additive, u-additive, perfect, τ-additive, and tight members of $M(X)$ will be denoted by $M_\sigma(X)$, $M_\infty(X)$, $M_p(X)$, $M_\tau(X)$, and $M_t(X)$, respectively (see [3, 4]). Then $(C_b(X), \beta_z)' = \{\varphi_\nu : \nu \in M_z(X)\}$ for $z = \sigma, \infty, p, \tau, t$.

For the integration theory of functions $f \in C_b(X,E)$ with respect to $\mu \in M_z(X,E')$ we refer the reader to [6, page 197], [5, Definition 3.10], [27, page 375]. For $z = \sigma, \infty, p, \tau, t$ let

$$M_z\left(X,E'\right) := \left\{\mu \in M\left(X,E'\right) : \mu_x \in M_z(X) \text{ for each } x \in E\right\}. \tag{22}$$

Then $|\mu| \in M_z(X)$ if $\mu \in M_z(X,E')$ (see [5, Proposition 3.9], [6, Theorem 3.1], [10, Theorem 1]). For $\Phi \in C_b(X,E)'$ let us put, for $u \in C_b(X)^+$,

$$|\Phi|(u) := \sup\left\{\left|\Phi(f)\right| : f \in C_b(X,E), \tilde{f} \le u\right\}. \tag{23}$$

It is known that $|\Phi| : C_b(X)^+ \to \mathbb{R}^+$ is additive and positively homogeneous and can be extended to a linear functional on $C_b(X)$ (denoted by $|\Phi|$ again) by $|\Phi|(u) = |\Phi|(u^+) - |\Phi|(u^-)$ for $u \in C_b(X)$.

Theorem 3. *Assume that $z = \sigma$ and $C_b(X) \otimes E$ is dense in $(C_b(X,E), \beta_\sigma)$ (resp., $z = \infty$; $z = p$ and $C_b(X) \otimes E$ is dense in $(C_b(X,E), \beta_p)$; $z = \tau$; $z = t$). Then the following statements hold.*

(i) *For a linear functional Φ on $C_b(X,E)$ the following conditions are equivalent.*

(a) *Φ is β_z-continuous.*

(b) *There exists a unique $\mu \in M_z(X,E')$ such that*

$$\Phi(f) = \Phi_\mu(f) = \int_X f\,d\mu \quad \text{for } f \in C_b(X,E). \tag{24}$$

(ii) *For $\mu \in M_z(X,E')$, $|\Phi_\mu|(u) = \int_X u\,d|\mu| = \varphi_{|\mu|}(u)$ for $u \in C_b(X)$.*

Proof. (i) See [6, Theorems 5.3 and 4.2, Corollary 3.9], [5, Theorem 3.13], and [10, Theorem 8].

(ii) See [6, Theorem 2.1]. □

Assume that $\mathscr{M}$ is a subset of $M_z(X,E')$ and $\sup_{\mu\in\mathscr{M}}|\mu|(X) < \infty$, where $z = \sigma, \infty, p, \tau, t$. Then we say that $\mathscr{M}$ satisfies the condition (C_z) if we have the following:

(1) for $z = \sigma$: $\sup\{|\mu|(Z_n) : \mu \in \mathscr{M}\} \to 0$ whenever $Z_n \downarrow \emptyset$, $(Z_n) \subset \mathscr{Z}$;

(2) for $z = \infty$: for every partition of unity $(u_\alpha)_{\alpha\in\mathscr{A}}$ for X and every $\varepsilon > 0$ there exists a finite set $\mathscr{A}_\varepsilon$ in $\mathscr{A}$ such that $\sup_{\mu\in\mathscr{M}} \int_X (1 - \sum_{\alpha\in\mathscr{A}_\varepsilon} u_\alpha)\,d|\mu| < \varepsilon$;

(3) for $z = p$: for every continuous function f from X onto a separable metric space Y and every $\varepsilon > 0$, there is a compact subset K of Y such that $\sup_{\mu\in\mathscr{M}}|\mu|(X \setminus \overset{-1}{f}(K)) \le \varepsilon$;

(4) for $z = \tau$: $\sup\{|\mu|(Z_\alpha) : \mu \in \mathscr{M}\} \to 0$ whenever $Z_\alpha \downarrow \emptyset$, $(Z_\alpha) \subset \mathscr{Z}$;

(5) for $z = t$: for every $\varepsilon > 0$ there exists a compact subset K of X such that $\sup\{|\mu|(Z) : Z \in \mathscr{Z}, Z \subset X \setminus K\} \le \varepsilon$ for each $\mu \in \mathscr{M}$.

The following lemmas will be useful.

Lemma 4. *Assume that $\mathscr{M}$ is a subset of $M_z(X,E')$ and $\sup_{\mu\in\mathscr{M}}|\mu|(X) < \infty$, where $z = \sigma$ and $C_b(X) \otimes E$ is β_σ-dense in $C_b(X,E)$ (resp., $z = \infty$; $z = p$ and $C_b(X) \otimes E$ is β_p-dense in $C_b(X,E)$; $z = \tau$; $z = t$). Then the following statements are equivalent.*

(i) *$\{\Phi_\mu : \mu \in \mathscr{M}\}$ is β_z-equicontinuous.*

(ii) *$\{|\Phi_\mu| : \mu \in \mathscr{M}\}$ is β_z-equicontinuous.*

(iii) *$\{\varphi_{|\mu|} : \mu \in \mathscr{M}\}$ is β_z-equicontinuous.*

(iv) *The condition (C_z) holds.*

Proof. (i)⇔(ii) See [9, Lemma 2].

(ii)⇔(iii) It follows from Theorem 3.

(iii)⇔(iv) See [4, Theorem 11.14] for $z = \sigma$; [28, Proposition 3.6] for $z = \infty$; [28, Proposition 2.6] for $z = p$; [4, Theorem 11.24] for $z = \tau$; and [28, Proposition 1.1] for $z = t$. □

Lemma 5. *Assume that $z = \sigma$ and $C_b(X) \otimes E$ is β_σ-dense in $C_b(X,E)$ (resp., $z = \infty$; $z = p$, and $C_b(X) \otimes E$ is β_p-dense in $C_b(X,E)$; $z = \tau$; $z = t$). Let $\mu \in M_z(X,E')$. Then for $A \in \mathscr{B}$ the following statements hold.*

(i) *A functional $\Phi_A : C_{rc}(X,E) \to \mathbb{R}$ defined by $\Phi_A(h) = \int_A h\,d\mu$ is $\beta_z|_{C_{rc}(X,E)}$-continuous and can by uniquely extended to a β_z-continuous linear functional $\overline{\Phi_A} : C_b(X,E) \to \mathbb{R}$, and one will write the following:*

$$\int_A f\,d\mu := \overline{\Phi_A}(f) \quad \text{for } f \in C_b(X,E). \tag{25}$$

(ii) *$|\int_A f\,d\mu| \le \int_A \tilde{f}\,d|\mu|$ for $f \in C_b(X,E)$.*

Proof. (i) Assume that (h_α) is a net in $C_{rc}(X,E)$ such that $h_\alpha \to 0$ for β_z. Then

$$\left|\Phi_A(h_\alpha)\right| = \left|\int_A h_\alpha\,d\mu\right| \le \int_A \tilde{h}_\alpha\,d|\mu| \le \int_X \tilde{h}_\alpha\,d|\mu|. \tag{26}$$

Since $\tilde{h}_\alpha \to 0$ for β_z in $C_b(X)$ and $|\mu| \in M_z(X)$, we obtain that $\Phi_A(h_\alpha) \to 0$; that is, Φ_A is $\beta_z|_{C_{rc}(X,E)}$-continuous. Since $C_{rc}(X,E)$ is dense in $(C_b(X,E), \beta_z)$, Φ_A can be uniquely extended to a β_z-continuous linear functional $\overline{\Phi_A} : C_b(X,E) \to \mathbb{R}$ (see [29, Theorem 2.6]).

(ii) Assume that $f \in C_b(X,E)$. Choose a net (h_α) in $C_{rc}(X,E)$ such that $h_\alpha \to f$ for β_z. Then $\widetilde{h}_\alpha \to \widetilde{f}$ for β_z in $C_b(X)$. Then

$$\begin{aligned}\left|\int_A \widetilde{h}_\alpha d|\mu| - \int_A \widetilde{f} d|\mu|\right| &\le \int_A \left|\widetilde{h}_\alpha - \widetilde{f}\right| d|\mu| \\ &\le \int_X \left|\widetilde{h}_\alpha - \widetilde{f}\right| d|\mu|,\end{aligned} \tag{27}$$

and hence $\int_A \widetilde{f} d|\mu| = \lim_\alpha \int_A \widetilde{h}_\alpha d|\mu|$. Since $\int_A f d\mu = \overline{\Phi_A}(f) = \lim_\alpha \int_A h_\alpha d\mu$, we get

$$\begin{aligned}\left|\int_A f d\mu\right| &= \lim_\alpha \left|\int_A h_\alpha d\mu\right| \\ &\le \lim_\alpha \int_A \widetilde{h}_\alpha d|\mu| = \int_A \widetilde{f} d|\mu|.\end{aligned} \tag{28}$$

□

For $z = \sigma, \infty, p, \tau, t$ let us put

$$\begin{aligned}&M_z\left(X, \mathscr{L}\left(E, F''\right)\right) \\ &\quad := \left\{m \in M\left(X, \mathscr{L}\left(E, F''\right)\right) : m_{y'} \in M_z\left(X, E'\right)\right. \\ &\qquad \left.\text{for each } y' \in F'\right\}.\end{aligned} \tag{29}$$

Lemma 6. *Assume that $z = \sigma$ and $C_b(X) \otimes E$ is β_σ-dense in $C_b(X,E)$ (resp., $z = \infty$; $z = p$, and $C_b(X) \otimes E$ is β_p-dense in $C_b(X,E)$; $z = \tau$; $z = t$). Assume that $m \in M_z(X, \mathscr{L}(E, F''))$ and the set $\{m_{y'} : y' \in F'\}$ satisfies the condition (C_z). Then for $A \in \mathscr{B}$ the following statements hold.*

(i) *An operator $S_A : C_{rc}(X,E) \to F''$ defined by $S_A(h) = \int_A h dm$ is $(\beta_z|_{C_{rc}(X,E)}, \|\cdot\|_{F''})$-continuous and can be uniquely extended to a $(\beta_z, \|\cdot\|_{F''})$-continuous linear operator $\overline{S_A} : C_b(X,E) \to F''$, and one will write the following.*

$$\int_A f dm := \overline{S_A}(f) \quad \text{for } f \in C_b(X,E). \tag{30}$$

(ii) *For each $y' \in F'$, $(\int_A f dm)(y') = \int_A f dm_{y'}$ for $f \in C_b(X,E)$.*

Proof. (i) In view of Lemma 5 the set $\{\varphi_{|m_{y'}|} : y' \in B_{F'}\}$ is β_z-equicontinuous in $C_b(X)'_{\beta_z}$. Assume that (h_α) is a net in $C_{rc}(X,E)$ such that $h_\alpha \to 0$ for β_z. Let $\varepsilon > 0$ be given. Then there exists a neighborhood V_ε of 0 for β_z in $C_b(X)$ such that $\sup_{y' \in B_{F'}} |\int_X u d|m_{y'}|| \le \varepsilon$ for $u \in V_\varepsilon$. Since $\widetilde{h}_\alpha \to 0$ for β_z in $C_b(X)$, choose α_ε such that $h_\alpha \in V_\varepsilon$ for $\alpha \ge \alpha_\varepsilon$. Hence $\sup_{y' \in B_{F'}} \int_X \widetilde{h}_\alpha \, d|m_{y'}| \le \varepsilon$ for $\alpha \ge \alpha_\varepsilon$. It follows that, for $\alpha \ge \alpha_\varepsilon$ and each $y' \in B_{F'}$,

$$\begin{aligned}\left|\left(\int_A h_\alpha dm\right)(y')\right| &= \left|\int_A h_\alpha dm_{y'}\right| \\ &\le \int_A \widetilde{h}_\alpha d\left|m_{y'}\right| \le \int_X \widetilde{h}_\alpha d\left|m_{y'}\right| \le \varepsilon,\end{aligned} \tag{31}$$

and hence,

$$\|S_A(h_\alpha)\|_{F''} = \sup\left\{\left|S_A(h_\alpha)(y')\right| : y' \in B_{F'}\right\} \le \varepsilon. \tag{32}$$

This means that $S_A : C_{rc}(X,E) \to F''$ is $(\beta_z|_{C_{rc}(X,E)}, \|\cdot\|_{F''})$-continuous. Since $C_{rc}(X,E)$ is β_z-dense in $(C_b(X,E), \beta_z)$, S_A possesses a unique $(\beta_z, \|\cdot\|_{F''})$-continuous extension $\overline{S_A} : C_b(X,E) \to F''$ (see [29, Theorem 2.6]). Let

$$\int_A f dm := \overline{S_A}(f) \quad \text{for } f \in C_b(X,E). \tag{33}$$

(ii) Let $f \in C_b(X,E)$. Choose a net (h_α) in $C_{rc}(X,E)$ such that $h_\alpha \to f$ for β_z. By Lemma 5 and (7) for $y' \in F'$ we have

$$\begin{aligned}\left(\int_A f dm\right)(y') &= \left(\lim_\alpha \left(\int_A h_\alpha dm\right)\right)(y') \\ &= \lim_\alpha \left(\int_A h_\alpha dm_{y'}\right)(y') \\ &= \lim_\alpha \int_A h_\alpha dm_{y'} = \int_A f dm_{y'}.\end{aligned} \tag{34}$$

□

Corollary 7. *Assume that $z = \sigma$ and $C_b(X) \otimes E$ is β_σ-dense in $C_b(X,E)$ (resp., $z = \infty$; $z = p$ and $C_b(X) \otimes E$ is β_p-dense in $C_b(X,E)$; $z = \tau$; $z = t$). Assume that $m \in M_z(X, \mathscr{L}(E, F''))$ and the set $\{m_{y'} : y' \in B_{F'}\}$ satisfies the condition (C_z). Then for $A \in \mathscr{B}$ the following statements hold:*

$$\begin{aligned}&\text{(a) } \left|m_{y'}\right|(A) \\ &\quad = \sup\left\{\left|\int_A h dm_{y'}\right| : h \in C_b(X) \otimes E, \|h\| \le 1\right\} \\ &\quad = \sup\left\{\left|\int_A f dm_{y'}\right| : f \in C_b(X,E), \|f\| \le 1\right\}. \\ &\text{(b) } \widetilde{m}(A) \\ &\quad = \sup\left\{\left\|\int_A h dm\right\|_{F''} : h \in C_b(X) \otimes E, \|h\| \le 1\right\} \\ &\quad = \sup\left\{\left\|\int_A f dm\right\|_{F''} : f \in C_b(X,E), \|f\| \le 1\right\}.\end{aligned} \tag{35}$$

In particular, if $U \in \mathscr{P}$, then

$$\begin{aligned}\text{(c) } \left|m_{y'}\right|(U) &= \sup\left\{\left|\int_U h dm_{y'}\right| : h \in C_b(X) \otimes E,\right. \\ &\qquad \left.\|h\| \le 1, \ \operatorname{supp} h \subset U\right\} \\ &= \sup\left|\sum_{i=1}^n \int_X u_i dm_{x_i, y'}\right|,\end{aligned} \tag{36}$$

where the supremum is taken over all finite disjoint supported collections $\{u_1,\dots,u_n\}\subset C_b(X)$ *with* $\|u_i\|\le 1$ *and* $\operatorname{supp} u_i\subset U$ *and* $\{x_1,\dots,x_n\}\subset B_E$. *One has*

$$\text{(d)}\ \widetilde{m}(U)=\sup\left\{\left\|\int_U h\,dm\right\|_{F''}: h\in C_b(X)\otimes E,\ \|h\|\le 1, \operatorname{supp} h\subset U\right\}=\sup\left\{\left\|\int_U f\,dm\right\|_{F''}: f\in C_b(X,E),\ \|f\|\le 1, \operatorname{supp} f\subset U\right\}. \tag{37}$$

Proof. Let $A\in\mathscr{B}$ and $y'\in F'$. Then by Lemma 5 for $f\in C_b(X,E)$ with $\|f\|\le 1$ we have

$$\left|\int_A f\,dm_{y'}\right|\le\int_A\tilde{f}\,d\left|m_{y'}\right|\le\left|m_{y'}\right|(A). \tag{38}$$

On the other hand, let $\varepsilon>0$ be given. Then there exist a finite $\mathscr{B}$-partition $(A_i)_{i=1}^n$ of A and $x_i\in B_E$, $i=1,\dots,n$, such that

$$\left|m_{y'}\right|(A)-\frac{\varepsilon}{3}\le\left|\sum_{i=1}^n\left(m(A_i)(x_i)\right)(y')\right|=\left|\sum_{i=1}^n m_{x_i,y'}(A_i)\right|. \tag{39}$$

By the regularity of $m_{x_i,y'}\in M_z(X)$ for $i=1,\dots,n$, we can choose $Z_i\in\mathscr{Z}$, $Z_i\subset A_i$ such that $|m_{x_i,y'}|(A_i\setminus Z_i)\le\varepsilon/3n$ for $i=1,\dots,n$. Choose pairwise disjoint $V_i\in\mathscr{P}$ with $Z_i\subset V_i$ for $i=1,\dots,n$ such that $|m_{x_i,y'}|(V_i\setminus Z_i)\le\varepsilon/3n$. Then for $i=1,\dots,n$ we can choose $v_i\in C_b(X)$ with $0\le v_i\le\mathbb{1}_X$, $v_i|_{Z_i}\equiv 1$, and $v_i|_{X\setminus V_i}\equiv 0$ (see [4, page 115]). Define $h_o=\sum_{i=1}^n(v_i\otimes x_i)$. Then $\|h_o\|\le 1$ and $\int_A h_o\,dm_{y'}=\sum_{i=1}^n\int_A v_i\,dm_{x_i,y'}=\sum_{i=1}^n\int_{V_i\cap A}v_i\,dm_{x_i,y'}$. Hence we get

$$\begin{aligned}\left|m_{y'}\right|(A)-\frac{\varepsilon}{3}&\le\left|\sum_{i=1}^n m_{x_i,y'}(A_i)-\sum_{i=1}^n m_{x_i,y'}(Z_i)\right|\\&\quad+\left|\sum_{i=1}^n\int_{Z_i}v_i\,dm_{x_i,y'}-\sum_{i=1}^n\int_{V_i\cap A}v_i\,dm_{x_i,y'}\right|\\&\quad+\left|\int_A h_o\,dm_{y'}\right|\\&\le\sum_{i=1}^n\left|m_{x_i,y'}\right|(A_i\setminus Z_i)+\sum_{i=1}^n\left|m_{x_i,y'}\right|(V_i\setminus Z_i)\\&\quad+\left|\int_A h_o\,dm_{y'}\right|\\&\le\frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\left|\int_A h_o\,dm_{y'}\right|\end{aligned} \tag{40}$$

and hence $|m_{y'}|(A)\le|\int_A h_o\,d\,m_{y'}|+\varepsilon$. Thus the proof of (a) is complete.

In view of (5), (a), and Lemma 6 we get

$$\begin{aligned}\widetilde{m}(A)&=\sup\left\{\left|m_{y'}\right|(A): y'\in B_{F'}\right\}\\&=\sup\left\{\left|\left(\int_A h\,dm\right)(y')\right|: h\in C_b(X)\otimes E,\ \|h\|\le 1,\ y'\in B_{F'}\right\}\\&=\sup\left\{\left|\left(\int_A f\,dm\right)(y')\right|: f\in C_b(X,E),\ \|f\|\le 1,\ y'\in B_{F'}\right\}\\&=\sup\left\{\left\|\left(\int_A h\,dm\right)\right\|_{F''}: h\in C_b(X)\otimes E,\|h\|\le 1\right\}\\&=\sup\left\{\left\|\left(\int_A f\,dm\right)\right\|_{F''}: f\in C_b(X,E),\|f\|\le 1\right\};\end{aligned} \tag{41}$$

that is, (b) holds.

Assume now that $U\in\mathscr{P}$. Let $U_i=V_i\cap U\in\mathscr{P}$ for $i=1,\dots,n$. Then $|m_{x_i,y'}|(U_i\setminus Z_i)\le|m_{x_i,y'}|(V_i\setminus Z_i)\le\varepsilon/3n$ for $i=1,\dots,n$. For $i=1,\dots,n$ choose $u_i\in C_b(X)$ with $0\le u_i\le\mathbb{1}_X$, $u_i|_{Z_i}\equiv 1$, and $u_i|_{X\setminus U_i}\equiv 0$. Let $h_o=\sum_{i=1}^n(u_i\otimes x_i)$. Then $\|h_o\|\le 1$ and $\operatorname{supp} h_o\subset U$; and hence by (a), $|m_{y'}|(U)\le|\int_U h_o\,dm_{y'}|+\varepsilon$. Note that $\int_U h_o\,dm_{y'}=\sum_{i=1}^n\int_X u_i\,dm_{x_i,y'}$, where $\operatorname{supp} u_i$ are pairwise disjoint and $\operatorname{supp} u_i\subset U$ for $i=1,\dots,n$. Thus (c) holds.

Using (c) we easily show that (d) holds. Thus the proof is complete. □

Definition 8. Let $T: C_b(X,E)\to F$ be a bounded linear operator. Then the measure $m\in M(X,\mathscr{L}(E,F''))$ defined by

$$m(A)(x):=\left(\left(T|_{C_{rc}(X,E)}\right)''\circ\pi\right)(\mathbb{1}_A\otimes x)\quad\text{for } A\in\mathscr{B}, x\in E \tag{42}$$

will be called a representing measure of T.

Now we state general Riesz representation theorems for continuous linear operators on $C_b(X,E)$, provided with the strict topologies β_z, where $z=\sigma,\infty,p,\tau,t$.

Theorem 9. *Assume that* $z=\sigma$ *and* $C_b(X)\otimes E$ *is* β_σ*-dense in* $C_b(X,E)$ *(resp.,* $z=\infty$*;* $z=p$*, and* $C_b(X)\otimes E$ *is* β_p*-dense in* $C_b(X,E)$*;* $z=\tau$*;* $z=t$*).*

(I) *Let* $T: C_b(X,E)\to F$ *be a* $(\beta_z,\|\cdot\|_F)$*-continuous linear operator and let* $m\in M(X,\mathscr{L}(E,F''))$ *be its representing measure. Then the following statements hold.*

(i) $m\in M_z(X,\mathscr{L}(E,F''))$ *and* $\{m_{y'}: y'\in B_{F'}\}$ *satisfies the condition* (C_z).

(ii) *For each* $y'\in F'$, $y'(T(f))=\int_X f\,dm_{y'}$ *for* $f\in C_b(X,E)$.

(iii) *For each* $f \in C_b(X,E)$ *and* $A \in \mathscr{B}$ *there exists a unique vector in* F'', *denoted by* $\int_A f dm$, *such that* $(\int_A f dm)(y') = \int_A f dm_{y'}$ *for each* $y' \in F'$.

(iv) *For each* $A \in \mathscr{B}$, *the mapping* $C_b(X,E) \ni f \mapsto \int_A f dm \in F''$ *is a* $(\beta_z, \|\cdot\|_{F''})$*-continuous linear operator.*

(v) *For* $f \in C_b(X,E)$, $\int_X f dm \in i_F(F)$ *and* $T(f) = j_F(\int_X f dm)$.

(vi) $\|T\| = \widetilde{m}(X)$.

(II) *Let* $m \in M_z(X, \mathscr{L}(E,F''))$ *and let the set* $\{m_{y'} : y' \in B_{F'}\}$ *satisfy the condition* (C_z). *Then the statements (iii) and (iv) hold and for* $f \in C_b(X,E)$, $\int_X f dm \in i_F(F)$ *and the mapping* $T : C_b(X,E) \to F$ *defined by* $T(f) := j_F(\int_X f dm)$ *is a* $(\beta_z, \|\cdot\|_F)$*-continuous linear operator. Moreover, m coincides with the representing measure of T and the statements (ii) and (vi) hold.*

Proof. (I) In view of (10) for each $y' \in F'$, $y'(T(h)) = \int_X h dm_{y'}$ for $h \in C_{rc}(X,E)$. By Theorem 3 for each $y' \in F'$ there exists a unique $\mu_{y'\circ T} \in M_z(X,E')$ such that $(y'\circ T)(f) = \int_X f d\mu_{y'\circ T}$ for $f \in C_b(X,E)$. It follows that, for each $y' \in F'$, $m_{y'} = \mu_{y'\circ T}$ (see [23, Theorem 2.5]) and this means that $m \in M_z(X,\mathscr{L}(E,F''))$. Hence

$$y'(T(f)) = \int_X f dm_{y'} \quad \text{for } f \in C_b(X,E). \tag{43}$$

Since $\{y' \circ T : y' \in B_{F'}\}$ is β_z-equicontinuous in $C_b(X,E)'_{\beta_z}$, by Lemma 4 the set $\{m_{y'} : y' \in B_{F'}\}$ satisfies the condition (C_z). Thus (i) and (ii) hold. In view of Lemma 6, (iii) and (iv) are satisfied.

According to (9) for each $h \in C_{rc}(X,E)$, $\int_X h dm \in i_F(F)$ and $T(h) = j_F(\int_X h dm)$. Hence by Lemma 6, $\int_X f dm \in i_F(F)$. Let $f \in C_b(X,E)$. Choose a net (h_α) in $C_{rc}(X,E)$ such that $h_\alpha \to f$ for β_z. Hence

$$\begin{aligned} T(f) &= \lim_\alpha T(h_\alpha) = \lim_\alpha j_F\left(\int_X h_\alpha dm\right) \\ &= j_F\left(\lim_\alpha \int_X h_\alpha dm\right) = j_F\left(\int_X f dm\right). \end{aligned} \tag{44}$$

Thus (v) holds. Using (v) and Corollary 7 we get $\|T\| = \widetilde{m}(X)$.

(II) By Lemma 6 the statements (iii) and (iv) are satisfied.

Now let $f \in C_b(X,E)$. Choose a net (h_α) in $C_{rc}(X,E)$ such that $h_\alpha \to f$ for β_z. Then by Lemma 6, $\int_X f dm = \overline{S_X}(f) = \lim_\alpha \int_X h_\alpha dm \in i_F(F)$ because $\int_X h_\alpha dm \in i_F(F)$, and it follows that $T(= j_F \circ \overline{S_X})$ is $(\beta_z, \|\cdot\|_F)$-continuous.

Let $m_o \in M(X,\mathscr{L}(E,F''))$ stand for the representing measure of T. Note that, for $A \in \mathscr{B}$, $x \in E$, and $y' \in F'$ we have

$$\begin{aligned} (m_o(A)(x))(y') &= \left(\left(\left(T|_{C_{rc}(X,E)}\right)'' \circ \pi\right)(\mathbb{1}_A \otimes x)\right)(y') \\ &= \pi(\mathbb{1}_A \otimes x)\left(\left(T|_{C_{rc}(X,E)}\right)'(y')\right) \\ &= \pi(\mathbb{1}_A \otimes x)\left(y' \circ \left(T|_{C_{rc}(X,E)}\right)\right) \\ &= \int_X (\mathbb{1}_A \otimes x) dm_{y'} = \int_X \mathbb{1}_A dm_{x,y'} \\ &= (m(A)(x))(y'); \end{aligned} \tag{45}$$

that is, $m_o = m$. By the first part of the proof (ii) and (vi) hold. Thus the proof is complete. □

Following [14, 27] by $M_\sigma(\mathscr{Ba})$ we denote the space of all bounded countably additive, real-valued, regular (with respect to zero sets) measures on $\mathscr{Ba}$.

We define $M_\sigma(\mathscr{Ba},E')$ to be the set of all measures $\mu : \mathscr{Ba} \to E'$ such that the following two conditions are satisfied.

(i) For each $x \in E$, the function $\mu_x : \mathscr{Ba} \to \mathbb{R}$, defined by $\mu_x(A) = \mu(A)(x)$ for $A \in \mathscr{Ba}$, belongs to $M_\sigma(\mathscr{Ba})$.

(ii) $|\mu|(X) < \infty$, where for each $A \in \mathscr{Ba}$, we define $|\mu|(A) = \sup|\sum \mu(A_i)(x_i)|$, where the supremum is taken over all finite $\mathscr{Ba}$-partitions (A_i) of A and all finite collections $x_i \in B_E$.

It is known that if $\mu \in M_\sigma(\mathscr{Ba},E')$, then $|\mu| \in M_\sigma(\mathscr{Ba})$ (see [27, Lemma 2.1]).

The following result will be of importance (see [27, Theorem 2.5]).

Theorem 10. *Let* $\mu \in M_\sigma(X,E')$. *Then* μ *possesses a unique extension* $\overline{\mu} \in M_\sigma(\mathscr{Ba},E')$ *and* $|\overline{\mu}|(X) = |\mu|(X)$.

Arguing as in the proof of Lemma 6 we can obtain the following lemma.

Lemma 11. *Assume that* $C_b(X) \otimes E$ *is* β_σ*-dense in* $C_b(X,E)$ *and* $\mu \in M_\sigma(X,E')$. *Then for* $A \in \mathscr{Ba}$ *the following statements hold.*

(i) *A functional* $\Phi_A : C_{rc}(X,E) \to \mathbb{R}$ *defined by* $\Phi_A(h) = \int_A h d\overline{\mu}$ *is* $\beta_\sigma|_{C_{rc}(X,E)}$*-continuous and can be uniquely extended to a* β_σ*-continuous linear functional* $\overline{\Phi_A} : C_b(X,E) \to \mathbb{R}$, *and one will write the following:*

$$\int_A f d\overline{\mu} := \overline{\Phi_A}(f) \quad \text{for } f \in C_b(X,E). \tag{46}$$

(ii) *For* $f \in C_b(X,E)$, $|\int_A f d\overline{\mu}| \leq \int_A \widetilde{f} d|\overline{\mu}|$.

By $M_\sigma(X,\mathscr{L}(E,F))$ we will denote the space of all operator measures $m : \mathscr{B} \to \mathscr{L}(E,F)$ such that $\widetilde{m}(X) < \infty$ and

$m_{y'} \in M_\sigma(X, E')$ for each $y' \in F'$. By $M_\sigma(\mathscr{B}a, \mathscr{L}(E,F))$ we will denote the space of all operator measures $m : \mathscr{B}a \to \mathscr{L}(E,F)$ with $\widetilde{m}(X) < \infty$ such that $m_{y'} \in M_\sigma(\mathscr{B}a, E')$ for each $y' \in F'$.

Remark 12. Note that in view of the Orlicz-Pettis theorem every $m \in M_\sigma(\mathscr{B}a, \mathscr{L}(E,F))$ is countably additive in the strong operator topology; that is, for each $x \in E$, the measure $m_x : \mathscr{B}a \to F$ defined by $m_x(A) := m(A)(x)$ for $A \in \mathscr{B}a$ is countably additive. Moreover, in view of [30, Theorem 2] for each $x \in E$, m_x is inner regular by zero sets and outer regular by cozero sets; that is, for each $A \in \mathscr{B}a$ and $\varepsilon > 0$ there exist $Z \in \mathscr{Z}$ with $Z \subset A$ and $P \in \mathscr{P}$ with $A \subset \mathscr{P}$ such that $\|m_x\|(A \setminus Z) \le \varepsilon$ and $\|m_x\|(P \setminus A) \le \varepsilon$, ($\|m_x\|(A)$ denotes the semivariation of m_x on $A \in \mathscr{B}a$).

According to [14, Theorem 7] we have the following theorem.

Theorem 13. *Assume that $m \in M_\sigma(X, \mathscr{L}(E,F))$ and $\{m(A)(x) : A \in \mathscr{B}\}$ is a relatively weakly compact subset of F for each $x \in E$. Then m possesses a unique extension $\overline{m} \in M_\sigma(\mathscr{B}a, \mathscr{L}(E,F))$ such that $\widetilde{\overline{m}}(X) = \widetilde{m}(X)$.*

For a linear operator $T : C_b(X,E) \to F$ and $x \in E$ let $T_x(u) := T(u \otimes x)$ for $u \in C_b(X)$. For $m \in M_\sigma(\mathscr{B}, \mathscr{L}(E, F''))$ and $x \in E$ let $m_x(A) := m(A)(x)$ for $A \in \mathscr{B}$.

Theorem 14. *Assume that $C_b(X) \otimes E$ is β_σ-dense in $C_b(X,E)$.*

(I) *Let $T : C_b(X,E) \to F$ be a $(\beta_\sigma, \|\cdot\|_F)$-continuous linear operator such that $T_x : C_b(X) \to F$ is weakly compact for each $x \in E$, and let $m \in M(X, \mathscr{L}(E, F''))$ be the representing measure of T. Then the following statements hold.*

  (i) *$m \in M_\sigma(X, \mathscr{L}(E, F''))$ and $\widetilde{m}(Z_n) \to 0$ whenever $Z_n \downarrow \emptyset$, $(Z_n) \subset \mathscr{Z}$.*

  (ii) *$m(A)(x) \in i_F(F)$, for each $A \in \mathscr{B}$, $x \in E$, and the measure $m_F : \mathscr{B} \to \mathscr{L}(E,F)$, defined by $m_F(A)(x) := j_F(m(A)(x))$ for $A \in \mathscr{B}$, $x \in E$, belongs to $M_\sigma(X, \mathscr{L}(E,F))$ and possesses a unique extension $\overline{m} \in M_\sigma(\mathscr{B}a, \mathscr{L}(E,F))$ with $\widetilde{\overline{m}}(X) = \widetilde{m}(X)$ which is countably additive both in the strong operator topology and in the weak star operator topology. Moreover, $\overline{m}_{y'} = \overline{m_{y'}}$ for $y' \in F'$.*

  (iii) *For every $f \in C_b(X,E)$ and $A \in \mathscr{B}a$ there exists a unique vector in F, denoted by $\int_A f d\overline{m}$, such that, for each $y' \in F'$, $y'(\int_A f d\overline{m}) = \int_A f d\overline{m}_{y'}$.*

  (iv) *For each $A \in \mathscr{B}a$, the mapping $T_A : C_b(X,E) \to F$ defined by $T_A(f) = \int_A f d\overline{m}$ is a $(\beta_\sigma, \|\cdot\|_F)$-continuous linear operator.*

  (v) *$T(f) = T_X(f) = \int_X f d\overline{m}$ for $f \in C_b(X,E)$.*

(II) *Let $m \in M_\sigma(X, \mathscr{L}(E, F''))$ be such that $\widetilde{m}(Z_n) \to 0$ whenever $Z_n \downarrow \emptyset, (Z_n) \subset \mathscr{Z}$ and for each $x \in E$, let $m_x : \mathscr{B} \to F''$ be strongly bounded. Then the operator $T : C_b(X,E) \to F$ defined by $T(f) = j_F(\int_X f dm)$ is $(\beta_\sigma, \|\cdot\|_F)$-continuous and $T_x : C_b(X) \to F$ is weakly compact for each $x \in E$, and the statements (ii)–(v) hold.*

Proof. (I) (i) It follows from Theorem 9.

(ii) In view of Theorem 2 $m(A)(x) \in i_F(F)$ for $A \in \mathscr{B}$, $x \in E$, and $\{m_F(A)(x) : A \in \mathscr{B}\}$ is a relatively weakly compact in F for each $x \in E$. Since $m_F \in M_\sigma(X, \mathscr{L}(E,F))$, by Theorem 13 m_F possesses a unique extension $\overline{m} \in M_\sigma(\mathscr{B}a, \mathscr{L}(E,F))$ with $\widetilde{\overline{m}}(X) = \widetilde{m}(X)$. By the Orlicz-Pettis theorem $\overline{m}$ is countably additive in the strong operator topology. Moreover, since, for each $y' \in F'$, $|\overline{m}_{y'}| \in M_\sigma(\mathscr{B}a) = ca(\mathscr{B}a)$, we obtain that $\overline{m}_{y'} \in ca(\mathscr{B}a, E')$. This means that $\overline{m} : \mathscr{B}a \to \mathscr{L}(E,F)$ is countably additive in the weak star operator topology.

Let $y' \in F'$. Then for $A \in \mathscr{B}$ and $x \in E$ we have $\overline{m}_{y'}(A)(x) = m_{y'}(A)(x)$, and by Theorem 10, $\overline{m}_{y'} = \overline{m_{y'}}$.

(iii) For $A \in \mathscr{B}a$ let $S_A(h) := \int_A f d\overline{m}$ for $h \in C_{rc}(X,E)$. Proceeding as in the proof of Lemma 6 we can show that $S_A : C_{rc}(X,E) \to F$ is a $(\beta_\sigma|_{C_{rc}(X,E)}, \|\cdot\|_F)$-continuous linear operator, and hence S_A possesses a unique $(\beta_\sigma, \|\cdot\|_F)$-continuous linear extension $T_A : C_b(X,E) \to F$ (see [29, Theorem 2.6]). Let us write the following:

$$\int_A f d\overline{m} := T_A(f) \quad \text{for } f \in C_b(X,E). \tag{47}$$

Let $f \in C_b(X,E)$. Choose a net (h_α) in $C_{rc}(X,E)$ such that $h_\alpha \to f$ for β_σ. For each $y' \in F'$, $\overline{m}_{y'} = \overline{m_{y'}}$ (see (i)) and by Lemma 11 we have

$$\begin{aligned} y'\left(\int_A f d\overline{m}\right) &= y'\left(\lim_\alpha \int_A h_\alpha d\overline{m}\right) = \lim_\alpha \left(y'\left(\int_A h_\alpha d\overline{m}\right)\right) \\ &= \lim_\alpha \int_A h_\alpha d\overline{m}_{y'} = \lim_\alpha \int_A h_\alpha d\overline{m_{y'}} \\ &= \int_A f d\overline{m_{y'}} = \int_A f d\overline{m}_{y'}. \end{aligned} \tag{48}$$

(iv) It follows from the proof of (iii).

(v) Let $f \in C_b(X,E)$. In view of Theorem 9, for each $y' \in F'$, $y'(T(f)) = \int_X f dm_{y'}$. On the other hand by (ii) for $y' \in F'$ we have $y'(\int_X f d\overline{m}) = \int_X f d\overline{m}_{y'} = \int_X f dm_{y'}$. It follows that $T(f) = \int_X f d\overline{m}$.

(II) Since $\{m_{y'} : y' \in B_{F'}\}$ satisfies the condition (C_σ), by Theorem 9 for $f \in C_b(X,E)$, $\int_X f dm \in i_F(F)$ and the mapping $T : C_b(X,E) \to F$ defined by $T(f) := j_F(\int_X f dm)$ is a $(\beta_\sigma, \|\cdot\|_F)$-continuous linear operator, and m coincides with the representing measure of T. Hence in view of Theorem 2 $T_x : C_b(X) \to F$ is a weakly compact operator. Thus by the first part of the proof the statements (ii)–(v) are satisfied. □

4. Strongly Bounded Operators on $C_b(X,E)$

Definition 15. A bounded linear operator $T : C_b(X,E) \to F$ is said to be *strongly bounded* if its representing measure

$m \in M(X, \mathscr{L}(E, F''))$ is strongly bounded; that is, $\widetilde{m}(A_n) \to 0$ whenever (A_n) is a pairwise disjoint sequence in $\mathscr{B}$.

Note that $m \in M(X, \mathscr{L}(E, F''))$ is strongly bounded if and only if the family $\{|m_{y'}| : y' \in B_{F'}\}$ is uniformly strongly additive.

Now we are ready to state our main results that extend some classical results of Lewis (see [20, Theorem 5], [31, Lemma 1]) and Brooks and Lewis (see [22, Theorem 2.1], [21, Theorem 5.2]) concerning operators on the spaces $C(X, E)$ and $C_o(X, E)$, where X is a compact or a locally compact space, respectively.

Theorem 16. *Assume that $C_b(X) \otimes E$ is β_σ-dense in $C_b(X, E)$. Let $T : C_b(X, E) \to F$ be a $(\beta_\sigma, \|\cdot\|_F)$-continuous linear operator and let $m \in M(X, \mathscr{L}(E, F''))$ be its representing measure. Then $m \in M_\sigma(X, \mathscr{L}(E, F''))$ and the following statements are equivalent.*

(i) *T is strongly bounded.*

(ii) $\sup\{|\overline{m_{y'}}|(A_n) : y' \in B_{F'}\} \to 0$ *whenever $A_n \downarrow \emptyset$, $(A_n) \subset \mathscr{B}a$ (here $\overline{m_{y'}} \in M_\sigma(\mathscr{B}a, E')$ denotes the unique extension of $m_{y'} \in M_\sigma(X, E')$).*

(iii) *If (A_n) is a sequence in $\mathscr{B}a$ such that $A_n \downarrow \emptyset$, then there exists a nested sequence (U_n) in $\mathscr{P}$ such that $A_n \subset U_n$ for $n \in \mathbb{N}$ and* $\sup\{\|T(f)\|_F : f \in C_b(X, E), \|f\| \le 1$ *and* $\operatorname{supp} f \subset U_n\} \to 0$.

Proof. In view of Theorem 9 $m \in M_\sigma(X, \mathscr{L}(E, F''))$.

(i)⇒(ii) Assume that T is strongly bounded. Since the family $\{|m_{y'}| : y' \in B_{F'}\}$ is uniformly strongly additive, according to [25, Lemma 1, page 26] the family $\{|\overline{m_{y'}}| : y' \in B_{F'}\}$ is uniformly countably additive (see Theorem 16).

(ii)⇒(i) It follows from [25, Lemma 1, page 26].

(ii)⇒(iii) Assume that (ii) holds and (A_n) is a sequence in $\mathscr{B}a$ such that $A_n \downarrow \emptyset$. Then there exists $\lambda \in ca(\mathscr{B}a)^+$ such that $\{|\overline{m_{y'}}| : y' \in B_{F'}\}$ is uniformly λ-continuous (see [25, Theorem 4, pages 11-12]). Let $\varepsilon > 0$ be given. Hence there exists $\delta > 0$ such that $\sup\{|\overline{m_{y'}}|(A) : y' \in B_{F'}\} \le \varepsilon/2$ whenever $\lambda(A) \le \delta$ and $A \in \mathscr{B}a$. Since λ is zero-set regular, there exists a nested sequence (U_n) in $\mathscr{P}$ so that $A_n \subset U_n$ and $\lambda(U_n \setminus A_n) \le \delta$ for $n \in \mathbb{N}$. Hence $\sup\{|\overline{m_{y'}}|(U_n \setminus A_n) : y' \in B_{F'}\} \le \varepsilon/2$ for $n \in \mathbb{N}$. In view of (ii) there exists $n_\varepsilon \in \mathbb{N}$ such that $\sup\{|\overline{m_{y'}}|(A_n) : y' \in B_{F'}\} \le \varepsilon/2$ for $n \ge n_\varepsilon$. Hence $\sup\{|m_{y'}|(U_n) : y' \in B_{F'}\} \le \varepsilon$ for $n \ge n_\varepsilon$; that is, $\sup\{|m_{y'}|(U_n) : y' \in B_{F'}\} \to 0$.

Let $f \in C_b(X, E)$, $\|f\| \le 1$, and $\operatorname{supp} f \subset U_n$. Then by Theorem 9 we have

$$\begin{aligned} \|T(f)\|_F &= \sup\left\{\left|\int_X f\,dm_{y'}\right| : y' \in B_{F'}\right\} \\ &\le \sup\left\{\int_X \widetilde{f}\,d\left|m_{y'}\right| : y' \in B_{F'}\right\} \\ &\le \sup\left\{\left|m_{y'}\right|(U_n) : y' \in B_{F'}\right\}. \end{aligned} \tag{49}$$

It follows that $\sup\{\|T(f)\|_F : f \in C_b(X, E), \|f\| \le 1, \operatorname{supp} f \subset U_n\} \to 0$.

(iii)⇒(ii) Assume that (iii) holds and $A_n \downarrow \emptyset$, $(A_n) \subset \mathscr{B}a$. Then there exists a nested sequence (U_n) in $\mathscr{P}$ such that $A_n \subset U_n$ for $n \in \mathbb{N}$ and

$$\sup\left\{\|T(f)\|_F : f \in C_b(X, E), \|f\| \le 1, \operatorname{supp} f_n \subset U_n\right\} \longrightarrow 0. \tag{50}$$

Assume that (ii) does not hold. Then there exist $\varepsilon > 0$ and $n_\varepsilon \in \mathbb{N}$ such that $\sup\{|\overline{m_{y'}}|(A_{n_\varepsilon}) : y' \in B_{F'}\} \ge \varepsilon$ and $\|T(f)\|_F \le (1/8)\varepsilon$ whenever $f \in C_b(X, E)$, $\|f\| \le 1$, and $\operatorname{supp} f \subset U_{n_\varepsilon}$. It follows that there exists $y'_o \in B_{F'}$ such that $|\overline{m_{y'}}|(A_{n_\varepsilon}) \ge \varepsilon$. Hence there exist a finite $\mathscr{B}a$-partition $(B_i)_{i=1}^k$ of A_{n_ε} and $x_i \in B_E$, $i = 1, \dots, k$, such that

$$\left|\overline{m_{y'_o}}\right|(A_{n_\varepsilon}) - \frac{\varepsilon}{4} \le \left|\sum_{i=1}^k \overline{m_{y'_o}}(B_i)(x_i)\right| = \left|\sum_{i=1}^k \left(\overline{m_{y'_o}}\right)_{x_i}(B_i)\right|. \tag{51}$$

Since $|(\overline{m_{y'_o}})_{x_i}| \in M_\sigma(\mathscr{B}a)$ is zero-set regular (see [4, page 118]), we can choose $Z_i \in \mathscr{Z}$, $Z_i \subset B_i$, such that $|(\overline{m_{y'_o}})_{x_i}|(B_i \setminus Z_i) \le \varepsilon/4k$ for $i = 1, \dots, k$. Choose pairwise disjoint $V_i \in \mathscr{P}$ with $Z_i \subset V_i$ for $i = 1, \dots, k$ such that $|m_{x_i,y'_o}|(V_i \setminus Z_i) \le \varepsilon/4k$. Let $U_i = V_i \cap U_{n_\varepsilon}$ for $i = 1, \dots, k$. Then $U_i \in \mathscr{P}$ and $|m_{x_i,y'_o}|(U_i \setminus Z_i) \le \varepsilon/4k$ for $i = 1, \dots, k$. For $i = 1, \dots, k$ choose $u_i \in C_b(X)$ such that $0 \le u_i \le \mathbb{1}_X$, $u_i|_{Z_i} \equiv 0$, and $u_i|_{X \setminus U_i} \equiv 0$ (see [4, page 115]). Let $h_o = \sum_{i=1}^k (u_i \otimes x_i)$. Then $\|h_o\| \le 1$, $\operatorname{supp} h_o \subset U_{n_\varepsilon}$, and

$$\int_{U_{n_\varepsilon}} h_o\,dm_{y'_o} = \sum_{i=1}^k \int_{U_i} u_i\,dm_{x_i,y'_o}. \tag{52}$$

Hence we get

$$\begin{aligned} &\left|\overline{m_{y'_o}}\right|(A_{n_\varepsilon}) - \frac{\varepsilon}{4} \\ &\quad \le \left|\sum_{i=1}^k \left(\overline{m_{y'_o}}\right)_{x_i}(B_i) - \sum_{i=1}^k \left(\overline{m_{y'_o}}\right)_{x_i}(Z_i)\right| \\ &\qquad + \left|\sum_{i=1}^k \int_{Z_i} u_i\,dm_{x_i,y'_o} - \sum_{i=1}^k \int_{U_i} u_i\,dm_{x_i,y'_o}\right| \\ &\qquad + \left|\int_{U_{n_\varepsilon}} h_o\,dm_{y'_o}\right| \\ &\quad \le \sum_{i=1}^k \left|\left(\overline{m_{y'_o}}\right)_{x_i}\right|(B_i \setminus Z_i) + \sum_{k=1}^k \left|m_{x_i,y'_o}\right|(U_i \setminus Z_i) \\ &\qquad + \left|\int_{U_{n_\varepsilon}} h_o\,dm_{y'_o}\right| \\ &\quad \le \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \left|\int_{U_{n_\varepsilon}} h_o\,dm_{y'_o}\right|. \end{aligned} \tag{53}$$

Hence

$$\begin{aligned}\left|\int_{U_{n_\varepsilon}} h_o dm_{y'_o}\right| &\geq \left|\overline{m_{y'_o}}\right|\left(A_{n_\varepsilon}\right) - \frac{3}{4}\varepsilon \geq \frac{1}{4}\varepsilon,\\ \|T(h_o)\|_F &\geq \left|y'_o\left(T\left(h_o\right)\right)\right| = \left|\int_X h_o dm_{y'_o}\right| \\ &= \left|\int_{U_{n_\varepsilon}} h_o dm_{y'_o}\right| \geq \frac{1}{4}\varepsilon.\end{aligned} \tag{54}$$

Thus we get a contradiction to $\|T(h_o)\|_F \leq (1/8)\varepsilon$.

Thus the proof is complete. □

Theorem 17. *Assume that $C_b(X)\otimes E$ is β_σ-dense in $C_b(X,E)$. Let $T : C_b(X,E) \to F$ be a $(\beta_\sigma, \|\cdot\|_F)$-continuous and strongly bounded operator and let $m \in M(X, \mathscr{L}(E,F''))$ be its representing measure. Then the following statements hold.*

(i) *$m \in M_\sigma(X,\mathscr{L}(E,F''))$ and $m(A)(x) \in i_F(F)$ for $A \in \mathscr{B}$, $x \in E$, and the measure $m_F : \mathscr{B} \to \mathscr{L}(E,F)$, defined by $m_F(A)(x) := j_F(m(A)(x))$ for $A \in \mathscr{B}$, $x \in E$, belongs to $M_\sigma(X,\mathscr{L}(E,F))$ and possesses a unique extension $\overline{m} \in M_\sigma(\mathscr{B}a, \mathscr{L}(E,F))$ with $\widetilde{\overline{m}}(X) = \widetilde{m}_F(X) = \widetilde{m}(X)$ which is variationally semiregular; that is, $\widetilde{\overline{m}}(A_n) \to 0$ whenever $A_n \downarrow \emptyset$, $(A_n) \subset \mathscr{B}a$.*

(ii) *For every $f \in C_b(X,E)$ and $A \in \mathscr{B}a$ there exists a unique vector in F, denoted by $\int_A f d\overline{m}$, such that, for each $y' \in F'$, $y'(\int_A f d\overline{m}) = \int_A f d\overline{m}_{y'}$.*

(iii) *For each $A \in \mathscr{B}a$, $\int_A f_n d\overline{m} \to 0$ whenever (f_n) is a uniformly bounded sequence in $C_b(X,E)$ such that $f_n(t) \to 0$ for $t \in X$.*

(iv) *$T(f) = \int_X f d\overline{m}$ for $f \in C_b(X,E)$.*

(v) *$T(f_n) \to 0$ whenever (f_n) is a uniformly bounded sequence in $C_b(X,E)$ such that $f_n(t) \to 0$ for $t \in X$.*

Proof. (i) Note that, for $x \in E$, $\|m_x(A)\|_{F''} \leq \widetilde{m}(A)\|x\|_E$ for $A \in \mathscr{B}$. Hence $m_x : \mathscr{B} \to F''$ is strongly bounded, and by Theorems 2 and 14 $m(A)(x) \in i_F(F)$ and m_F possesses a unique extension $\overline{m} \in M_\sigma(\mathscr{B}a, \mathscr{L}(E,F))$ with $\widetilde{\overline{m}}(X) = \widetilde{m}_F(X) = \widetilde{m}(X)$. Since $\overline{m}_{y'} = \overline{m_{y'}}$ for $y' \in F'$, by Theorem 16 we have $\widetilde{\overline{m}}(A_n) = \sup\{|\overline{m}_{y'}|(A_n) : y' \in B_{F'}\} \to 0$ whenever $A_n \downarrow \emptyset, (A_n) \subset \mathscr{B}a$.

(ii) It follows from Theorem 14 because for each $x \in E$, $T_x : C_c(X) \to F$ is weakly compact (see Theorem 2).

(iii) In view of (i) there exists $\lambda \in ca(\mathscr{B}a)^+$ such that $\{|\overline{m}_{y'}| : y' \in B_{F'}\}$ is λ-continuous (see [25, Theorem 4, pages 11-12]). Let (f_n) be a sequence in $C_b(X,E)$ such that $\sup_n\|f_n\| = M < \infty$ and $f_n(t) \to 0$ for every $t \in X$. Let $\varepsilon > 0$ be given. Then there exists $\delta > 0$ such that $\sup\{|\overline{m}_{y'}|(A) : y' \in B_{F'}\} \leq \varepsilon/2M$ whenever $\lambda(A) \leq \delta$, $A \in \mathscr{B}a$. Since $\widetilde{f}_n \in B(\mathscr{B})$ for $n \in \mathbb{N}$, by the Egoroff theorem there exists $A_\delta \in \mathscr{B}a$ with $\lambda(X \setminus A_\delta) \leq \delta$ and $\sup_{t\in A_\delta}\widetilde{f}_n(t) \to 0$. Choose $n_\varepsilon \in \mathbb{N}$ such that $\sup_{t\in A_\delta}\widetilde{f}_n(t) \leq \varepsilon/2\widetilde{m}(X)$ for $n \geq n_\varepsilon$.

Let $A \in \mathscr{B}a$. Note that $\overline{m}_{y'} = \overline{m_{y'}}$ for $y' \in F'$. Then by Lemma 11 and (ii), for $n \geq n_\varepsilon$ and $y' \in B_{F'}$ we get

$$\begin{aligned}&\left|y'\left(\int_A f_n d\overline{m}\right)\right| \\ &= \left|\int_A f_n d\overline{m}_{y'}\right| \\ &\leq \int_A \widetilde{f}_n d\left|\overline{m}_{y'}\right| \leq \int_X \widetilde{f}_n d\left|\overline{m}_{y'}\right| \\ &= \int_{A_\delta} \widetilde{f}_n d\left|\overline{m}_{y'}\right| + \int_{X\setminus A_\delta} \widetilde{f}_n d\left|\overline{m}_{y'}\right| \\ &\leq \frac{\varepsilon}{2\widetilde{m}(X)}\left|\overline{m}_{y'}\right|(A_\delta) + M\cdot\left|\overline{m}_{y'}\right|(X\setminus A_\delta) \\ &\leq \frac{\varepsilon}{2\widetilde{m}(X)}\left|m_{y'}\right|(X) + M\cdot\frac{\varepsilon}{2M} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned} \tag{55}$$

Hence $\|\int_A f_n d\overline{m}\|_F \leq \varepsilon$ for $n \geq n_\varepsilon$, as desired.

(iv) It follows from Theorem 14.

(v) It follows from (iii) and (iv). □

Let $\mathscr{L}^\infty(\mathscr{B}a, E)$ stand for the Banach space of all bounded strongly $\mathscr{B}a$-measurable functions $g : X \to E$, equipped with the uniform norm $\|\cdot\|$. Assume that $m : \mathscr{B} \to \mathscr{L}(E,F)$ with $\widetilde{m}(X) < \infty$ is variationally semiregular. Then every $g \in \mathscr{L}^\infty(\mathscr{B}a, E)$ is m-integrable (see [32, Definition 2, page 523 and Theorem 5, page 524]) and $\int_X g_n dm \to 0$ whenever (g_n) is a uniformly bounded sequence in $\mathscr{L}^\infty(\mathscr{B}a, E)$ converging pointwise to 0 (see [33, Proposition 2.2]).

Recall that a series $\sum_{i=1}^\infty z_i$ in a Banach space G is called *weakly unconditionally Cauchy* (wuc) if, for each $z' \in G'$, $\sum_{i=1}^\infty |z'(z_i)| < \infty$. We say that a linear operator $T : G \to F$ is *unconditionally converging* if for every weakly unconditionally Cauchy series $\sum_{i=1}^\infty z_i$ in G, the series $\sum_{i=1}^\infty T(z_i)$ converges unconditionally in a Banach space F.

As an application of Theorem 17 we have the following result.

Corollary 18. *Assume that $C_b(X)\otimes E$ is β_σ-dense in $C_b(X,E)$, where E is a separable Banach space which contains no isomorphic copy of c_o. Let $T : C_b(X,E) \to F$ be a $(\beta_\sigma, \|\cdot\|_F)$-continuous and strongly bounded operator. Then T is unconditionally converging.*

Proof. Assume that $\sum_{i=1}^\infty f_i$ is a wuc series in the Banach space $C_b(X,E)$. Hence $\sum_{i=1}^\infty |x'(f_i(t))| < \infty$ for each $t \in X$ and $x' \in E'$ because $\delta_{t,x'} \in C_b(X,E)'$, where $\delta_{t,x'}(f) = x'(f(t))$ for $f \in C_b(X,E)$. It follows that $\sum_{i=1}^\infty f_i(t)$ is an unconditionally convergent series in E for each $t \in X$ because E contains no isomorphic copy of c_o (see [34]). Let $g_o(t) = \lim_n S_n(t)$ for $t \in X$, where $S_n(t) = \sum_{i=1}^n f_i(t)$ for $t \in X, n \in \mathbb{N}$. Then $\sup_n\|S_n\| < \infty$ because $\sum_{i=1}^\infty f_i$ is wuc (see [34]) and $S_n \in \mathscr{L}^\infty(\mathscr{B}a, E)$ because E is assumed to be separable (see [2, Theorem 21, page 9]). Hence $g_o \in \mathscr{L}^\infty(\mathscr{B}a, E)$ (see [2, Theorem 10, page 6]).

Let $m \in M_\sigma(X, \mathscr{L}(E,F''))$ be the representing measure of T and let $\overline{m} \in M_\sigma(\mathscr{B}a, \mathscr{L}(E,F))$ be a unique extension of $m_F \in M_\sigma(\mathscr{B}, \mathscr{L}(E,F))$ (see Theorem 17). Since $\overline{m}$ is variationally semiregular, in view of [33, Proposition 2.2] we have

$$\lim_n \sum_{i=1}^{n} T(f_i) = \lim_n \int_X S_n d\overline{m} = \int_X g_o d\overline{m} \in E. \tag{56}$$

Hence $\sum_{i=1}^{\infty} T(f_i) = \int_X g_o d\overline{m}$. Finally, if (n_j) is any permutation of $\mathbb{N}$, then $\lim_n \sum_{j=1}^{n} f_{n_j}(t) = g_o(t)$ for $t \in X$. Then $\sum_{j=1}^{\infty} T(f_{n_j}) = \int_X g_o d\overline{m}$, as desired. □

Remark 19. A related result to Corollary 18 for strongly bounded operators on the space $C_o(X, E)$ of E-valued continuous functions vanishing at infinity defined on a locally compact space X was obtained by Brooks and Lewis (see [21, Theorem 5.2]).

Recall that a Banach space E is said to be a Schur space if every weakly convergent sequence in E is norm convergent.

As a consequence of Theorem 17 we derive the following Dunford-Pettis type theorem for operators on $C_b(X, E)$.

Theorem 20. *Assume that $C_b(X) \otimes E$ is β_σ-dense in $C_b(X, E)$, where E is a Schur space. Let $T : C_b(X, E) \rightarrow F$ be a $(\beta_\sigma, \|\cdot\|_F)$-continuous and strongly bounded operator. Then $T(f_n) \rightarrow 0$ in F whenever (f_n) is a $\sigma(C_b(X, E), M_\sigma(X, E'))$ convergent to 0 sequence in $C_b(X, E)$.*

Proof. Assume that $f_n \rightarrow 0$ for $\sigma(C_b(X, E), M_\sigma(X, E'))$. Then according to [11, Corollary 5], we obtain that $\sup_n \|f_n\| < \infty$ and $f_n(t) \rightarrow 0$ in $\sigma(E, E')$ for each $t \in X$. It follows that $\|f_n(t)\|_E \rightarrow 0$ for $t \in X$ because E is supposed to be a Schur space. Using Theorem 17 we derive that $T(f_n) \rightarrow 0$ in F, as desired. □

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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