Hindawi Publishing Corporation
Journal of Function Spaces
Volume 2015, Article ID 409215, 11 pages
http://dx.doi.org/10.1155/2015/409215

Research Article

Boundedness of Singular Integrals on Hardy Type Spaces Associated with Schrödinger Operators

Jianfeng Dong,[1] Jizheng Huang,[2] and Heping Liu[3]

[1] *Department of Mathematics, Shanghai University, Shanghai 200444, China*
[2] *College of Sciences, North China University of Technology, Beijing 100144, China*
[3] *LMAM, School of Mathematical Sciences, Peking University, Beijing 100871, China*

Correspondence should be addressed to Jizheng Huang; hjzheng@163.com

Received 19 May 2014; Accepted 21 August 2014

Academic Editor: Jose Luis Sanchez

Let $L = -\Delta + V$ be a Schrödinger operator on $\mathbb{R}^n$, $n \geq 3$, where $V \not\equiv 0$ is a nonnegative potential belonging to the reverse Hölder class $B_{n/2}$. The Hardy type spaces H_L^p, $n/(n+\delta) < p \leq 1$, for some $\delta > 0$, are defined in terms of the maximal function with respect to the semigroup $\{e^{-tL}\}_{t>0}$. In this paper, we investigate the bounded properties of some singular integral operators related to L, such as $L^{i\gamma}$ and $\nabla L^{-1/2}$, on spaces H_L^p. We give the molecular characterization of H_L^p, which is used to establish the H_L^p-boundedness of singular integrals.

1. Introduction

Let $L = -\Delta + V$ be a Schrödinger operator on $\mathbb{R}^n$, $n \geq 3$, where $V \not\equiv 0$ is a nonnegative potential belonging to the reverse Hölder class B_q for some $q \geq n/2$; that is, there exists a constant $C > 0$ such that the reverse Hölder inequality

$$\left(\frac{1}{|B|}\int_B V^q(x)\,dx\right)^{1/q} \leq C\left(\frac{1}{|B|}\int_B V(x)\,dx\right) \tag{1}$$

holds for every ball B in $\mathbb{R}^n$. It is well known that if $V \in B_q$ then $V \in B_{q+\varepsilon}$ for some $\varepsilon > 0$. Also obviously, $B_{q_1} \subset B_{q_2}$ when $q_1 > q_2$.

Some singular integral operators related to L, such as the imaginary power $L^{i\gamma}$, and the Riesz transform $\nabla L^{-1/2}$ have been studied by Shen [1]. Some of his results are following. The operator $L^{i\gamma}$ is a Calderón-Zygmund operator for any $\gamma \in \mathbb{R}$. $\nabla L^{-1/2}$ is a Calderón-Zygmund operator if $q \geq n$. When $n/2 \leq q < n$, $\nabla L^{-1/2}$ is bounded on L^p for $1 < p \leq p_0$, where $1/p_0 = 1/q - 1/n$. The above range of p is optimal. Earlier results were given by Fefferman [2] and Zhong [3].

The Hardy type spaces H_L^p, $n/(n+\delta) < p \leq 1$ for some $\delta > 0$, associated with L, are studied by Dziubański and Zienkiewicz [4, 5]. They establish the $H_L^{p,\infty}$ atomic decomposition theorem and the Riesz transform characterization of H_L^1. Specifically, $\nabla L^{-1/2}$ is bounded from H_L^1 to L^1. We will investigate the bounded properties of the operators $L^{i\gamma}$ and $\nabla L^{-1/2}$ on spaces H_L^p. To do this, we give the molecular characterization of H_L^p.

Without loss of generalization, we assume that $V \in B_{q_0}$ for some $q_0 > n/2$ and set $\delta = \min(2 - n/q_0, 1)$. When $q_0 > n$, we set $\eta = 1 - n/q_0$. Throughout the paper, we will use A and C to denote the positive constants, which are independent of main parameters and may be different at each occurrence. By $B_1 \sim B_2$, we mean that there exists a constant $C > 1$ such that $1/C \leq B_1/B_2 \leq C$.

Let $\{T_t^L\}_{t>0} = \{e^{-tL}\}_{t>0}$ be the semigroup of linear operators generated by $-L$ and $K_t^L(x,y)$ their kernels. Since V is nonnegative, the Feynman-Kac formula implies that

$$0 \leq K_t^L(x,y) \leq K_t(x-y) = (4\pi t)^{-n/2} e^{-(4t)^{-1}|x-y|^2}, \tag{2}$$

where $K_t(x)$ is the convolution kernels of the heat semigroup $\{T_t\}_{t>0} = \{e^{t\Delta}\}_{t>0}$. The estimate (2) can be improved as follows. We introduce the auxiliary function $\rho(x,V)=\rho(x)$ defined by

$$\rho(x)=\sup\left\{r>0:\frac{1}{r^{n-2}}\int_{B(x,r)}V(y)\,dy\le 1\right\}. \tag{3}$$

It is known that $0<\rho(x)<\infty$. For every $N>0$,

$$K_t^L(x,y)\le C_N t^{-n/2}e^{-(5t)^{-1}|x-y|^2}\left(1+\frac{\sqrt{t}}{\rho(x)}+\frac{\sqrt{t}}{\rho(y)}\right)^{-N}, \tag{4}$$

(cf. [6, Theorem 4.10]). Let $0<\delta'<\delta$; for every $N>0$ and all $|h|\le\sqrt{t}$,

$$\begin{aligned}&\left|K_t^L(x+h,y)-K_t^L(x,y)\right|\\&\quad\le C_N\left(\frac{|h|}{\sqrt{t}}\right)^{\delta'}t^{-n/2}e^{-At^{-1}|x-y|^2}\left(1+\frac{\sqrt{t}}{\rho(x)}+\frac{\sqrt{t}}{\rho(y)}\right)^{-N}\end{aligned} \tag{5}$$

(cf. [6, Proposition 4.11]).

We define the Hardy type spaces H_L^p, $n/(n+\delta)<p\le 1$, in terms of the maximal function with respect to the semigroup $\{T_t^L\}_{t>0}$.

For $p=1$, the Hardy space H_L^1 is defined, according to Dziubański and Zienkiewicz [4], by

$$H_L^1=\left\{f\in L^1:M^Lf\in L^1\right\}, \tag{6}$$

where

$$M^Lf(x)=\sup_{t>0}\left|T_t^Lf(x)\right|. \tag{7}$$

The norm of a function $f\in H_L^1$ is defined to be $\|f\|_{H_L^1}=\|M^Lf\|_{L^1}$.

The Hardy spaces, H_L^p, $n/(n+\delta)<p<1$, consist of some kind of distributions. But $M^Lf(x)$ may have no meaning for a tempered distribution f because $K_t^L(x,y)$ are not smooth. Let f be a locally integrable function. $B=B(x,r)$ is the ball of radius r centered at x. Set

$$\begin{aligned}f_B&=\frac{1}{|B|}\int_B f(y)\,dy,\\ f(B,V)&=\begin{cases}f_B, & \text{if } r<\rho(x),\\ 0, & \text{if } r\ge\rho(x).\end{cases}\end{aligned} \tag{8}$$

Let $n/(n+\delta)<p<1, 1\le q'\le\infty$. A locally integrable function f is said to be in the Campanato type space $\Lambda^L_{1/p-1,q'}$ if

$$\begin{aligned}\|f\|_{\Lambda^L_{1/p-1,q'}}&=\sup_{B\subset\mathbb{R}^d}\left\{|B|^{1-1/p}\left(\int_B|f-f(B,V)|^{q'}\frac{dx}{|B|}\right)^{1/q'}\right\}\\&<\infty.\end{aligned} \tag{9}$$

All spaces $\Lambda^L_{1/p-1,q'}$ are mutually coincident with equivalent norms and will be simply denoted by $\Lambda^L_{1/p-1}$ (cf. [7]). Due to (4) and (5), for every $t>0$,

$$\sup_{x\in\mathbb{R}^d}\left\|K_t^L(x,\cdot)\right\|_{\Lambda^L_{1/p-1}}<Ct^{-d/2p} \tag{10}$$

(cf. [7, Lemma 1]). Thus the semigroup maximal function Mf is well defined for distributions in $(\Lambda^L_{1/p-1})^*$. We define the Hardy space, H_L^p, $n/(n+\delta)<p<1$, by

$$H_L^p=\left\{f\in\left(\Lambda^L_{1/p-1}\right)^*:M^Lf\in L^p\right\}, \tag{11}$$

and set $\|f\|_{H_L^p}=\|M^Lf\|_{L^p}$.

Similar to the classical case, the Hardy space H_L^p admits an atomic decomposition. Let $n/(n+\delta)<p\le 1\le q\le\infty$, $p\ne q$. A function a is called an $H_L^{p,q}$-atom associated with a ball $B(x_0,r)$ if

(1) supp $a\subset B(x_0,r)$,

(2) $\|a\|_{L^q}\le|B(x_0,r)|^{1/q-1/p}$,

(3) *if* $r<\rho(x_0)$, then $\int a(x)dx=0$.

Proposition 1 (see [7, Theorem 1]). *Given p,q as above, then $f\in H_L^p$ if and only if f can be written as $f=\sum_j\lambda_ja_j$, where a_j are $H_L^{p,q}$-atoms and $\sum_j|\lambda_j|^p<\infty$. The sum converges in H_L^p norm and also in $(\Lambda^L_{1/p-1})^*$ when $p<1$. Moreover,*

$$\|f\|_{H_L^p}\sim\|f\|_{H_L^{p,q,a}}=\inf\left\{\left(\sum_j\left|\lambda_j\right|^p\right)^{1/p}\right\}, \tag{12}$$

where the infimum is taken over all decompositions of f into $H_L^{p,q}$-atoms.

Now we state the main results in this paper.

Theorem 2. *For any $\gamma\in\mathbb{R}$, the imaginary power $L^{i\gamma}$ is bounded on H_L^p for $n/(n+\delta)<p\le 1$. When $q_0>n$, the Riesz transform $\nabla L^{-1/2}$ is bounded on H_L^p for $n/(n+\eta)<p\le 1$. Moreover, $\nabla L^{-1/2}$ is bounded on H_L^1 whenever $q_0>n/2$.*

Remark 3. When $n/2<q_0<n$, the kernel of Riesz transform $\nabla L^{-1/2}$ only satisfies the Hörmander condition with respect to the second variable, which is weaker than the smoothness condition of standard kernels. Thus we cannot expect, in general consideration, to deal with the boundedness of $\nabla L^{-1/2}$ for the case of $p<1$.

In order to prove Theorem 2, we give the molecular characterization of H_L^p.

Let $n/(n+\delta)<p\le 1\le q\le\infty$, $p\ne q$ and $\epsilon>1/p-1$. Set $a=1-1/p+\epsilon$, $b=1-1/q+\epsilon$. A function $M\in L^q$ is called an $H_L^{p,q,\epsilon}$-molecule with the center x_0 if

(1) $|x|^{nb}M(x)\in L^q$,

(2) $\mathcal{N}(M) = \mu_1^{b-a}\|M\|_{L^q}^{a/b} \quad \||\cdot - x_0|^{nb}M\|_{L^q}^{1-a/b} \le 1$,

(3) *if* $\|M\|_{L^q}^{1/(a-b)} < \mu_1\rho(x_0)^n$, then $\int M(x)dx = 0$,

where μ_1 is the volume of the unit ball.

Theorem 4. *Given p, q, ϵ as above, then $f \in H_L^p$ if and only if f can be written as $f = \sum_j \lambda_j M_j$, where M_j are $H_L^{p,q,\epsilon}$-molecules and $\sum_j |\lambda_j|^p < \infty$. The sum converges in H_L^p norm and also in $(\Lambda_{1/p-1}^L)^*$ when $p < 1$, where M_j are H_L^p-molecules. Moreover,*

$$\|f\|_{H_L^p} \sim \|f\|_{H_L^{p,q,\epsilon,M}} = \inf\left\{\left(\sum_j |\lambda_j|^p\right)^{1/p}\right\}, \tag{13}$$

where the infimum is taken over all decompositions of f into $H_L^{p,q,\epsilon}$-molecules.

Remark 5. It is easy to verify that any $H_L^{p,q}$-atom is an $H_L^{p,q,\epsilon}$-molecule with a constant factor less than or equal to 1. We will see that the image of an $H_L^{p,q}$-atom under the action of a singular integral operator may not be an $H_L^{p,q,\epsilon}$-molecule but is a sum of two $H_L^{p,q,\epsilon}$-molecules up to constant factors. This is different from the classical case.

This paper is organized as follows. In Section 2, we collect some useful facts and results about the potential V, the auxiliary function $\rho(x)$ and the kernels of operators $L^{i\gamma}$, and $\nabla L^{-1/2}$, which will be used in the sequel. Most of these results are already known. In Section 3, we prove Theorem 4. The proof of Theorem 2 is given in the last two sections. The H_L^p-boundedness for $p < 1$ is proved in Section 4 while H_L^1-boundedness is proved in Section 5.

2. Preliminaries

First we list some known facts and results about the potential V, the auxiliary function $\rho(x)$, and the kernels of operators $L^{i\gamma}$ and $\nabla L^{-1/2}$.

Lemma 6. *$V(x)dx$ is a doubling measure; that is, there exists a constant $C_0 > 0$ such that*

$$\int_{B(x,2r)} V(y)\,dy \le C_0 \int_{B(x,r)} V(y)\,dy. \tag{14}$$

Lemma 7. *Consider*

$$\frac{1}{r^{n-2}}\int_{B(x,r)} V(y)\,dy \le C\left(\frac{R}{r}\right)^{n/q_0-2}\frac{1}{R^{n-2}}\int_{B(x,R)} V(y)\,dy, \quad 0 < r < R < \infty. \tag{15}$$

Lemma 8. *There exists $m_0 > 0$ such that*

$$\frac{1}{R^{n-2}}\int_{B(x,R)} V(y)\,dy \le C\left(1 + \frac{R}{\rho(x)}\right)^{m_0}. \tag{16}$$

Lemma 9. *There exists $k_0 > 0$ such that*

$$\frac{1}{C}\left(1 + \frac{|x-y|}{\rho(x)}\right)^{-k_0} \le \frac{\rho(y)}{\rho(x)} \le C\left(1 + \frac{|x-y|}{\rho(x)}\right)^{k_0/(k_0+1)}. \tag{17}$$

In particular, $\rho(y) \sim \rho(x)$ if $|x-y| < C\rho(x)$.

Let $F_\gamma^L(x,y)$ and $F_\gamma(x,y)$ be the kernels of $L^{i\gamma}$ and $(-\Delta)^{i\gamma}$, respectively, and $R^L(x,y)$ and $R(x,y)$ the kernels of $\nabla L^{-1/2}$ and $\nabla(-\Delta)^{-1/2}$, respectively. Set $\widetilde{F}_\gamma(x,y) = F_\gamma^L(x,y) - F_\gamma(x,y)$, $\widetilde{R}(x,y) = R^L(x,y) - R(x,y)$.

Lemma 10. *$L^{i\gamma}$ is a Calderón-Zygmund operator. It does not matter to assume that $n/2 < q_0 < n$. The kernel $F_\gamma^L(x,y)$ satisfies*

$$\left|F_\gamma^L(x,y+h) - F_\gamma^L(x,y)\right| \le \frac{Ce^{\pi|\gamma|/2}|h|^\delta}{|x-y|^{n+\delta}}, \quad |h| \le \frac{|x-y|}{2}, \tag{18}$$

and, for any $N > 0$,

$$\left|F_\gamma^L(x,y)\right| \le \frac{C_N e^{\pi|\gamma|/2}}{|x-y|^n}\left(1 + \frac{|x-y|}{\rho(y)}\right)^{-N}. \tag{19}$$

In addition,

$$\left|\widetilde{F}_\gamma(x,y)\right| \le \frac{Ce^{\pi|\gamma|/2}}{|x-y|^n}\left(\frac{|x-y|}{\rho(y)}\right)^\delta. \tag{20}$$

Lemma 11. *When $n/2 < q_0 < n$, $\nabla L^{-1/2}$ is bounded on L^p for $1 < p \le p_0$, where $1/p_0 = 1/q_0 - 1/n$. The kernel $R^L(x,y)$ satisfies, for any $N > 0$,*

$$\left|R^L(x,y)\right| \le \frac{C_N}{|x-y|^{n-1}}\left(\int_{B(x,|x-y|/4)} \frac{V(z)\,dz}{|z-x|^{n-1}} + \frac{1}{|x-y|}\right) \times \left(1 + \frac{|x-y|}{\rho(y)}\right)^{-N}. \tag{21}$$

In addition,

$$\left|\widetilde{R}(x,y)\right| \le \frac{C}{|x-y|^{n-1}} \times \left(\int_{B(x,|x-y|/4)} \frac{V(z)\,dz}{|z-x|^{n-1}} + \frac{1}{|x-y|}\left(\frac{|x-y|}{\rho(y)}\right)^\delta\right). \tag{22}$$

Lemma 12. *When $q_0 > n$, $\nabla L^{-1/2}$ is a Calderón-Zygmund operator. The kernel $R^L(x, y)$ satisfies*

$$\left|R^L(x, y+h) - R^L(x, y)\right| \le \frac{C|h|^\eta}{|x-y|^{n+\eta}}, \quad |h| \le \frac{|x-y|}{2}, \tag{23}$$

and, for any $N > 0$,

$$\left|R^L(x, y)\right| \le \frac{C_N}{|x-y|^n}\left(1 + \frac{|x-y|}{\rho(y)}\right)^{-N}. \tag{24}$$

In addition, for any $\delta' < 1$,

$$\left|\widetilde{R}(x, y)\right| \le \frac{C}{|x-y|^n}\left(\frac{|x-y|}{\rho(y)}\right)^{\delta'}. \tag{25}$$

For Lemmas 6–12, we refer readers to [1]. We also need the following estimates about $\widetilde{F}_\gamma(x, y)$ and $\widetilde{R}(x, y)$.

Lemma 13. *When $n/2 < q_0 < n$,*

$$\left|\widetilde{F}_\gamma(x, y+h) - \widetilde{F}_\gamma(x, y)\right| \le \frac{Ce^{\pi|\gamma|/2}}{|x-y|^n}\left(\frac{|h|}{\rho(y)}\right)^\delta, \quad |h| \le \frac{|x-y|}{2}. \tag{26}$$

When $q_0 > n$, for any $\delta' < 1$,

$$\left|\widetilde{R}(x, y+h) - \widetilde{R}(x, y)\right| \le \frac{C}{|x-y|^n}\left(\frac{|h|}{\rho(y)}\right)^{\delta'}, \quad |h| \le \frac{|x-y|}{2}. \tag{27}$$

Proof. It is well known that

$$\begin{gathered}\left|F_\gamma(x, y+h) - F_\gamma(x, y)\right| \le \frac{C|h|}{|x-y|^{n+1}}, \quad |h| \le \frac{|x-y|}{2}, \\ \left|R(x, y+h) - R(x, y)\right| \le \frac{C|h|}{|x-y|^{n+1}}, \quad |h| \le \frac{|x-y|}{2}.\end{gathered} \tag{28}$$

Therefore, we also have the estimates

$$\left|\widetilde{F}_\gamma(x, y+h) - \widetilde{F}_\gamma(x, y)\right| \le \frac{Ce^{\pi|\gamma|/2}|h|^\delta}{|x-y|^{n+\delta}}, \quad |h| \le \frac{|x-y|}{2}, \tag{29}$$

$$\left|\widetilde{R}(x, y+h) - \widetilde{R}(x, y)\right| \le \frac{C|h|^\eta}{|x-y|^{n+\eta}}, \quad |h| \le \frac{|x-y|}{2}. \tag{30}$$

We may assume that $|x-y| < \rho(y)$. Otherwise, Lemma 13 is obvious.

We will use the following known facts (cf. [1]). Let $\Gamma^L(x, y, \tau)$ and $\Gamma(x, y, \tau)$ denote, respectively, the fundamental solutions for the operators $L + i\tau$ and $-\Delta + i\tau$ in $\mathbb{R}^n$, where $\tau \in \mathbb{R}$. They satisfy the following estimates. For any $k > 0$ and $|h| \le |x-y|/2$,

$$\begin{aligned}&\left|\Gamma(x, y, \tau)\right| \le \frac{C_k}{\left(1 + |\tau|^{1/2}|x-y|\right)^k}\frac{1}{|x-y|^{n-2}}, \\ &\left|\Gamma^L(x, y+h, \tau) - \Gamma^L(x, y, \tau)\right| \\ &\quad \le \frac{C_k}{\left(1 + |\tau|^{1/2}|x-y|\right)^k}\frac{|h|^\delta}{|x-y|^{n-2+\delta}}\left(1 + \frac{|x-y|}{\rho(y)}\right)^{-k}\end{aligned} \tag{31}$$

when $n/2 < q_0 < n$. Set $\widetilde{\Gamma}(x, y, \tau) = \Gamma^L(x, y, \tau) - \Gamma(x, y, \tau)$. Then $\widetilde{\Gamma}(x, y, \tau)$ is expressed as

$$\widetilde{\Gamma}(x, y, \tau) = -\int_{\mathbb{R}^n} \Gamma(x, z, \tau) V(z) \Gamma^L(z, y, \tau)\, dz. \tag{32}$$

Thus,

$$\begin{aligned}&\left|\widetilde{\Gamma}(x, y+h, \tau) - \widetilde{\Gamma}(x, y, \tau)\right| \\ &\quad \le \int_{\mathbb{R}^n} |\Gamma(x, z, \tau)| V(z) \left|\Gamma^L(z, y+h, \tau) - \Gamma^L(z, y, \tau)\right| dz \\ &\quad \le \int_{\mathbb{R}^n} \left(C_k|h|^\delta V(z)\left(1 + \rho(y)^{-1}|z-y|\right)^{-k} dz\right) \\ &\qquad \times \left(\left(1 + |\tau|^{1/2}|x-z|\right)^k\left(1 + |\tau|^{1/2}|z-y|\right)^k \right. \\ &\qquad \left. \times |x-z|^{n-2}|z-y|^{n-2+\delta}\right)^{-1} \\ &\quad = \int_{|z-x|<|x-y|/2}(\cdots) + \int_{|z-y|<|x-y|/2}(\cdots) \\ &\qquad + \int_{|z-x|\ge|x-y|/2,|z-y|\ge|x-y|/2}(\cdots) \\ &\quad = I_1 + I_2 + I_3.\end{aligned} \tag{33}$$

Note that $V \in B_{q_0+\varepsilon}$ for some $\varepsilon > 0$. Using Hölder inequality and $B_{q_0+\varepsilon}$ condition, it is easy to see that, for $0 \le \sigma \le \delta$,

$$\int_{B(x,R)} \frac{V(y)}{|x-y|^{n-2+\sigma}}\, dy \le \frac{C}{R^{n-2+\sigma}} \int_{B(x,R)} V(y)\, dy. \tag{34}$$

Note that $\rho(x) \sim \rho(y)$ when $|x-y| < \rho(y)$. Making use of (34), we get

$$\begin{aligned} I_1 &\le \frac{C_k |h|^{\delta}}{\left(1+|\tau|^{1/2}|x-y|\right)^k |x-y|^{n-2+\delta}} \int_{|z-x|<|x-y|/2} \frac{V(z)\,dz}{|x-z|^{n-2}} \\ &\le \frac{C_k |h|^{\delta}}{\left(1+|\tau|^{1/2}|x-y|\right)^k |x-y|^{n-2+\delta}} \frac{1}{|x-y|^{n-2}} \\ &\quad \times \int_{|z-x|<|x-y|/2} V(z)\,dz \\ &\le \frac{C_k}{\left(1+|\tau|^{1/2}|x-y|\right)^k |x-y|^{n-2}} \left(\frac{|h|}{\rho(y)}\right)^{\delta}, \end{aligned} \tag{35}$$

where we have used Lemma 7 in the last inequality. Similarly,

$$\begin{aligned} I_2 &\le \frac{C_k |h|^{\delta}}{\left(1+|\tau|^{1/2}|x-y|\right)^k |x-y|^{n-2}} \int_{|z-y|<|x-y|/2} \frac{V(z)\,dz}{|z-y|^{n-2+\delta}} \\ &\le \frac{C_k |h|^{\delta}}{\left(1+|\tau|^{1/2}|x-y|\right)^k |x-y|^{n-2+\delta}} \frac{1}{|x-y|^{n-2}} \\ &\quad \times \int_{|z-y|<|x-y|/2} V(z)\,dz \\ &\le \frac{C_k}{\left(1+|\tau|^{1/2}|x-y|\right)^k |x-y|^{n-2}} \left(\frac{|h|}{\rho(y)}\right)^{\delta}. \end{aligned} \tag{36}$$

To estimate I_3, we write

$$\begin{aligned} I_3 &\le \frac{C_k |h|^{\delta}}{\left(1+|\tau|^{1/2}|x-y|\right)^k} \\ &\quad \times \int_{|z-y|\ge|x-y|/2} \left(1+\frac{|x-y|}{\rho(y)}\right)^{-k} \frac{V(z)\,dz}{|z-y|^{2n-4+\delta}} \\ &\le \frac{C_k |h|^{\delta}}{\left(1+|\tau|^{1/2}|x-y|\right)^k} \\ &\quad \times \left(\int_{|x-y|/2\le|z-y|<\rho(y)} \frac{V(z)\,dz}{|z-y|^{2n-4+\delta}} \right. \\ &\qquad \left. + \rho(y)^k \int_{|z-y|\ge\rho(y)} \frac{V(z)\,dz}{|z-y|^{2n-4+\delta+k}} \right). \end{aligned} \tag{37}$$

Using Hölder inequality and B_{q_0} condition, we obtain

$$\begin{aligned} &\int_{|x-y|/2\le|z-y|<\rho(y)} \frac{V(z)\,dz}{|z-y|^{2n-4+\delta}} \\ &\quad \le \left(\int_{|z-y|<\rho(y)} V(z)^{q_0}\,dz \right)^{1/q_0} \\ &\qquad \times \left(\int_{|z-y|\ge|x-y|/2} \frac{dz}{|z-y|^{(2n-4+\delta)q_0'}} \right)^{1/q_0'} \\ &\quad \le \frac{C}{|x-y|^{n-2}\rho(y)^{\delta}}. \end{aligned} \tag{38}$$

Using Lemma 6 and taking k sufficiently large, we get

$$\begin{aligned} &\rho(y)^k \int_{|z-y|\ge\rho(y)} \frac{V(z)\,dz}{|z-y|^{2n-4+\delta+k}} \\ &\quad \le C\rho(y)^{4-2n-\delta} \sum_{j=1}^{\infty} 2^{-j(2n-4+\delta+k)} \int_{|z-y|\le 2^j\rho(y)} V(z)\,dz \\ &\quad \le C\rho(y)^{4-2n-\delta} \sum_{j=1}^{\infty} 2^{-j(2n-4+\delta+k)} C_0^j \int_{|z-y|\le\rho(y)} V(z)\,dz \\ &\quad \le C\rho(y)^{2-n-\delta} \\ &\quad \le \frac{C}{|x-y|^{n-2}\rho(y)^{\delta}}. \end{aligned} \tag{39}$$

Therefore,

$$\begin{aligned} &\left|\widetilde{\Gamma}(x,y+h,\tau) - \widetilde{\Gamma}(x,y,\tau)\right| \\ &\quad \le \frac{C_k}{\left(1+|\tau|^{1/2}|x-y|\right)^k |x-y|^{n-2}} \left(\frac{|h|}{\rho(y)}\right)^{\delta}. \end{aligned} \tag{40}$$

We also have

$$\nabla_x \widetilde{\Gamma}(x,y,\tau) = -\int_{\mathbb{R}^n} \nabla_x \Gamma(x,z,\tau) V(z) \Gamma^L(z,y,\tau)\,dz, \tag{41}$$

where $\nabla_x \Gamma(x,z,\tau)$ satisfies the estimate

$$\left|\nabla_x \Gamma(x,y,\tau)\right| \le \frac{C_k}{\left(1+|\tau|^{1/2}|x-y|\right)^k} \frac{1}{|x-y|^{n-1}}. \tag{42}$$

If $q_0 > n$, by the same argument as (40), for any $\delta' < 1$,

$$\begin{aligned} &\left|\nabla_x \widetilde{\Gamma}(x,y+h,\tau) - \nabla_x \widetilde{\Gamma}(x,y,\tau)\right| \\ &\quad \le \frac{C_k}{\left(1+|\tau|^{1/2}|x-y|\right)^k |x-y|^{n-1}} \left(\frac{|h|}{\rho(y)}\right)^{\delta'}. \end{aligned} \tag{43}$$

By the functional calculus and making use of (40), we obtain

$$\begin{aligned}&\left|\widetilde{F}_\gamma(x,y+h)-\widetilde{F}_\gamma(x,y)\right|\\&=\frac{1}{2\pi}\left|\int_{\mathbb{R}}(-i\tau)^{i\gamma}\left(\widetilde{\Gamma}(x,y+h,\tau)-\widetilde{\Gamma}(x,y,\tau)\right)d\tau\right|\\&\leq\frac{Ce^{\pi|\gamma|/2}}{|x-y|^n}\left(\frac{|h|}{\rho(y)}\right)^{\delta}.\end{aligned}\tag{44}$$

This proves (26).

Similarly, it follows from (43) that

$$\begin{aligned}&\left|\widetilde{R}(x,y+h)-\widetilde{R}(x,y)\right|\\&=\frac{1}{2\pi}\left|\int_{\mathbb{R}}(-i\tau)^{-1/2}\left(\nabla_x\widetilde{\Gamma}(x,y+h,\tau)-\nabla_x\widetilde{\Gamma}(x,y,\tau)\right)d\tau\right|\\&\leq\frac{C}{|x-y|^n}\left(\frac{|h|}{\rho(y)}\right)^{\delta'}.\end{aligned}\tag{45}$$

This proves (27). □

3. Molecular Characterization

Essentially, the proof of Theorem 4 is the same as the usual molecular theory.

Proof of Theorem 4. By Proposition 1, it is sufficient to prove that for any $H_L^{p,q,\epsilon}$-molecule $M(x)$ admits an atomic decomposition $M=\sum_j\lambda_j a_j$, where a_j are $H_L^{p,q}$-atoms and $\sum_j|\lambda_j|^p<C$.

We will give the proof in case $q=2$. The proof is similar in the case of $q\neq 2$. Suppose $M(x)$ is an $H_L^{p,2,\epsilon}$-molecule centered at x_0. Let $\sigma=\|M\|_{L^2}^{1/(a-b)}$, where $\epsilon>1/p-1$, $a=1-1/p+\epsilon$, $b=1/2+\epsilon$. If $\sigma<\mu_1\rho(x_0)^n$, we return the usual molecular theory (cf. [8]). Thus nothing needs to be proved. Suppose $\sigma\geq\mu_1\rho(x_0)^n$. Set

$$\begin{gathered}B_k=\left\{x:|x-x_0|\leq 2^k\mu_1^{-1/n}\sigma^{1/n}\right\},\quad k=0,1,2,\ldots,\\E_0=B_0,\qquad E_k=B_k\setminus B_{k-1},\quad k=1,2,\ldots.\end{gathered}\tag{46}$$

Then

$$M(x)=\sum_{k=0}^{\infty}M(x)\chi_{E_k}(x)=\sum_{k=0}^{\infty}M_k(x).\tag{47}$$

Note that supp $M_k\subset B_k$ and $2^k\mu_1^{-1/n}\sigma^{1/n}\geq\rho(x_0)$, $k=0,1,2,\ldots$. Also we have

$$\|M_0\|_{L^2}\leq\|M\|_{L^2}=\sigma^{a-b}=|B_0|^{1/2-1/p},\tag{48}$$

$$\begin{aligned}\|M_k\|_{L^2}&\leq 2^{-(k-1)nb}\mu_1^b\sigma^{-b}\left\||\cdot-x_0|^{nb}M\right\|_{L^2}\\&\leq 2^{-(k-1)nb}\sigma^{-b}\|M\|_{L^2}^{a/(a-b)}\\&=2^{-(k-1)nb}\sigma^{a-b}\\&=2^{nb-kna}|B_k|^{1/2-1/p}.\end{aligned}\tag{49}$$

Thus $M_k(x)=\lambda_k a_k(x)$, $k=0,1,2,\ldots$, where a_k are $H_L^{p,q}$-atoms and $\sum_{k=0}^{\infty}|\lambda_k|^p<C$.

Originally, the sum in (47) converges pointwise. When $p=1$, it is easy to see that the sum in (47) converges in L^1. If $n/(n+\delta)<p<1$, for any $g\in\Lambda_{1/p-1}^L$,

$$\begin{aligned}&\left\|\left(1+|x|^{nb}\right)^{-1}g\right\|_{L^2}\\&\leq\left(\int_{|x|<\rho(0)}|g(x)|^2dx\right)^{1/2}+\sum_{k=1}^{\infty}2^{-(k-1)nb}\rho(0)^{-nb}\\&\quad\times\left(\int_{2^{k-1}\rho(0)\leq|x|<2^k\rho(0)}|g(x)|^2dx\right)^{1/2}\\&\leq C\sum_{k=0}^{\infty}2^{-kna}\|g\|_{\Lambda_{1/p-1}^L}\\&\leq C\|g\|_{\Lambda_{1/p-1}^L}.\end{aligned}\tag{50}$$

Therefore,

$$\|Mg\|_{L^1}\leq\left\|\left(1+|x|^{nb}\right)M\right\|_{L^2}\left\|\left(1+|x|^{nb}\right)^{-1}g\right\|_{L^2}<\infty.\tag{51}$$

It follows that the sum in (47) converges in $(\Lambda_{1/p-1}^L)^*$. The proof of Theorem 4 is completed. □

4. H_L^p-Boundedness

In this section, we prove the boundedness of $L^{i\gamma}$ on H_L^p, $n/(n+\delta)<p\leq 1$. When $q_0>n$, the boundedness of $\nabla L^{-1/2}$ on H_L^p, $n/(n+\eta)<p\leq 1$, can be proved by the same method. In fact, their kernels satisfy similar estimates.

Let $a(x)$ be an $H_L^{p,q}$-atom associated with a ball $B(x_0,r)$ for some suitable q. If $r\geq\rho(x_0)$, we will prove that $L^{i\gamma}a(x)$ is an $H_L^{p,q,\epsilon}$-molecule up to a constant factor. If $r<\rho(x_0)$, $L^{i\gamma}a(x)$ may be not an $H_L^{p,q,\epsilon}$-molecule up to a constant factor but $(-\Delta)^{i\gamma}a(x)$is (cf. [9]). We will prove that $(L^{i\gamma}-(-\Delta)^{i\gamma})a(x)$ is an $H_L^{p,q,\epsilon}$-molecule up to a constant factor for some suitable ϵ. This means that $\|L^{i\gamma}a(x)\|_{H_L^p}\leq C$ uniformly. Because the semigroup maximal function M^Lf is subadditive, by Proposition 1, $L^{i\gamma}$ is bounded on H_L^p, $n/(n+\delta)<p\leq 1$.

First, let $r\geq\rho(x_0)$. Because

$$\left\|L^{i\gamma}a(x)\right\|_{L^q}\leq C\|a(x)\|_{L^q}\leq C\left|B(x_0,r)\right|^{a-b},\tag{52}$$

where $\epsilon>1/p-1$, $a=1-1/p+\epsilon$, $b=1-1/q+\epsilon$, we have

$$\left\|L^{i\gamma}a(x)\right\|_{L^q}^{1/(a-b)}\geq\frac{1}{C}\rho(x_0)^n.\tag{53}$$

Thus there needs no the cancelation condition. We only need to estimate $\mathcal{N}(L^{i\gamma}a)$. Write

$$\begin{aligned}\left(\int_{\mathbb{R}^n}|x-x_0|^{nbq}\left|L^{i\gamma}a(x)\right|^q dx\right)^{1/q} &\le \left(\int_{B(x_0,2r)}|x-x_0|^{nbq}\left|L^{i\gamma}a(x)\right|^q dx\right)^{1/q}\\ &+\left(\int_{|x-x_0|\ge 2r}|x-x_0|^{nbq}\left|L^{i\gamma}a(x)\right|^q dx\right)^{1/q}\\ &= I_1+I_2.\end{aligned} \tag{54}$$

It is obvious that

$$\begin{aligned}I_1 &\le r^{nb}\left\|L^{i\gamma}a(x)\right\|_{L^q}\le C\left|B(x_0,r)\right|^b\\ &\times\|a(x)\|_{L^q}\le C\left|B(x_0,r)\right|^a.\end{aligned} \tag{55}$$

For $y\in B(x_0,r)$, if $\rho(y)>r$, by Lemma 9, $\rho(y)\le C\rho(x_0)\le Cr$. Note that $|x-y|\sim|x-x_0|$ when $x\notin B(x_0,2r)$, $y\in B(x_0,r)$. Using Lemma 10, we get

$$\begin{aligned}I_2 &=\left(\int_{|x-x_0|\ge 2r}\left(\int_{B(x_0,r)}|x-x_0|^{nb}\times\left|F_\gamma^L(x,y)a(y)\right|dy\right)^q dx\right)^{1/q}\\ &\le\int_{B(x_0,r)}|a(y)|dy\left(\int_{|x-x_0|\ge 2r}|x-x_0|^{nbq}\times\left|F_\gamma^L(x,y)\right|^q dx\right)^{1/q}\\ &\le C\int_{B(x_0,r)}|a(y)|\;r^N dy\times\left(\int_{|x-x_0|\ge 2r}|x-x_0|^{(nb-n-N)q}\,dx\right)^{1/q}\\ &\le C\,r^{n-n/p+N}r^{nb-n/q'-N}\\ &\le C\left|B(x_0,r)\right|^a\end{aligned} \tag{56}$$

and provide $N>n\epsilon$. Therefore,

$$\left\||x-x_0|^{nb}L^{i\gamma}a(x)\right\|_{L^q}\le C\left|B(x_0,r)\right|^a. \tag{57}$$

It follows that

$$\mathcal{N}\left(L^{i\gamma}a\right)=\mu_1^{b-a}\left\|L^{i\gamma}a(x)\right\|_{L^q}^{a/b}\left\||x-x_0|^{nb}L^{i\gamma}a(x)\right\|_{L^q}^{1-a/b}\le C. \tag{58}$$

Next, suppose $r<\rho(x_0)$. Let us estimate $\|(L^{i\gamma}-(-\Delta)^{i\gamma})a(x)\|_{L^q}$. Consider

$$\begin{aligned}\left(\int_{\mathbb{R}^n}\left|\left(L^{i\gamma}-(-\Delta)^{i\gamma}\right)a(x)\right|^q dx\right)^{1/q} &\le\left(\int_{B(x_0,2r)}\left|\left(L^{i\gamma}-(-\Delta)^{i\gamma}\right)a(x)\right|^q dx\right)^{1/q}\\ &+\left(\int_{2r\le|x-x_0|<2\rho(x_0)}\left|\left(L^{i\gamma}-(-\Delta)^{i\gamma}\right)a(x)\right|^q dx\right)^{1/q}\\ &+\left(\int_{|x-x_0|\ge 2\rho(x_0)}\left|\left(L^{i\gamma}-(-\Delta)^{i\gamma}\right)a(x)\right|^q dx\right)^{1/q}\\ &=J_1+J_2+J_3.\end{aligned} \tag{59}$$

Note that $\rho(y)\sim\rho(x_0)$, when $y\in B(x_0,r)$ and by Lemma 10, we have

$$\begin{aligned}J_1 &=\left(\int_{B(x_0,2r}\left(\int_{B(x_0,r)}\left|\widetilde{F}_\gamma(x,y)a(y)\right|\,dy\right)^q dx\right)^{1/q}\\ &\le\int_{B(x_0,r)}|a(y)|\left(\int_{B(x_0,2r)}\left|\widetilde{F}_\gamma(x,y)\right|^q dx\right)^{1/q}dy\\ &\le C\int_{B(x_0,r)}|a(y)|\,\rho(x_0)^{-\delta}\left(\int_{B(x_0,2r)}\frac{dx}{|x-y|^{(n-\delta)q}}\right)^{1/q}dy\\ &\le C\rho(x_0)^{-\delta}r^{n-n/p}\left(\int_{B(0,3r)}\frac{dx}{|x|^{(n-\delta)q}}\right)^{1/q}\\ &\le C\rho(x_0)^{-\delta}r^{n/q-n/p+\delta}\\ &\le C\rho(x_0)^{n(a-b)}.\end{aligned} \tag{60}$$

Here we choose q such that $1<q<n/(n-\delta)$ and $n/q-n/p+\delta>0$ or, equivalently, $1<q<np/(n-p\delta)$. When $2r\le|x-x_0|<2\rho(x_0)$, using the cancelation condition of a and Lemma 13, we obtain

$$\begin{aligned}\left|\left(L^{i\gamma}-(-\Delta)^{i\gamma}\right)a(x)\right| &=\left|\int_{B(x_0,r)}\left(\widetilde{F}_\gamma(x,y)-\widetilde{F}_\gamma(x,x_0)\right)a(y)\,dy\right|\\ &\le\frac{C\rho(x_0)^{-\delta}}{|x-x_0|^n}\int_{B(x_0,r)}|a(y)|\,|y-x_0|^\delta dy\\ &\le\frac{C\rho(x_0)^{-\delta}r^{n-n/p+\delta}}{|x-x_0|^n}.\end{aligned} \tag{61}$$

It follows that

$$J_2 \leq C\rho(x_0)^{-\delta} r^{n-n/p+\delta}\left(\int_{2r\leq|x-x_0|<2\rho(x_0)} \frac{dx}{|x-x_0|^{nq}}\right)^{1/q}$$
$$\leq C\rho(x_0)^{-\delta} r^{n/q-n/p+\delta}$$
$$\leq C\rho(x_0)^{n(a-b)}. \tag{62}$$

When $|x-x_0| \geq 2\rho(x_0)$, by (29), we have

$$\left|\left(L^{i\gamma} - (-\Delta)^{i\gamma}\right) a(x)\right|$$
$$= \left|\int_{B(x_0,r)} \left(\widetilde{F}_\gamma(x,y) - \widetilde{F}_\gamma(x,x_0)\right) a(y)\, dy\right|$$
$$\leq \frac{C}{|x-x_0|^{n+\delta}} \int_{B(x_0,r)} |a(y)|\, |y-x_0|^\delta dy \tag{63}$$
$$\leq \frac{C\, r^{n-n/p+\delta}}{|x-x_0|^{n+\delta}}.$$

Then

$$J_3 \leq Cr^{n-n/p+\delta}\left(\int_{|x-x_0|\geq 2\rho(x_0)} \frac{dx}{|x-x_0|^{(n+\delta)q}}\right)^{1/q}$$
$$\leq Cr^{n-n/p+\delta}\rho(x_0)^{-n/q'-\delta} \tag{64}$$
$$\leq C\rho(x_0)^{n(a-b)}.$$

We have seen that

$$\left\|\left(L^{i\gamma} - (-\Delta)^{i\gamma}\right) a(x)\right\|_{L^q}^{1/(a-b)} \geq \frac{1}{C}\, \rho(x_0)^n. \tag{65}$$

As above, there needs no the cancelation condition . To finish the proof, we only need to prove $\mathcal{N}((L^{i\gamma} - (-\Delta)^{i\gamma})a) \leq C$ or, equivalently,

$$\left\||x-x_0|^{nb}\left(L^{i\gamma} - (-\Delta)^{i\gamma}\right) a(x)\right\|_{L^q} \leq C\, \rho(x_0)^{na}. \tag{66}$$

Write

$$\left(\int_{\mathbb{R}^n} |x-x_0|^{nbq}\, \left|L^{i\gamma} - (-\Delta)^{i\gamma}\, a(x)\right|^q dx\right)^{1/q}$$
$$\leq \left(\int_{B(x_0,2\rho(x_0))} |x-x_0|^{nbq}\, \left|L^{i\gamma} - (-\Delta)^{i\gamma} a(x)\right|^q dx\right)^{1/q}$$
$$+ \left(\int_{|x-x_0|\geq 2\rho(x_0)} |x-x_0|^{nbq}\, \left|L^{i\gamma} - (-\Delta)^{i\gamma} a(x)\right|^q dx\right)^{1/q}$$
$$= H_1 + H_2. \tag{67}$$

It is clear that

$$H_1 \leq C\rho(x_0)^{nb}\left\|L^{i\gamma} - (-\Delta)^{i\gamma} a(x)\right\|_{L^q} \leq C\rho(x_0)^{na}. \tag{68}$$

By (63),

$$H_2 \leq Cr^{n-n/p+\delta}\left(\int_{|x-x_0|\geq 2\rho(x_0)} |x-x_0|^{(nb-n-\delta)q} dx\right)^{1/q}$$
$$\leq Cr^{n-n/p+\delta}\rho(x_0)^{nb-\delta-n/q'} \tag{69}$$
$$\leq C\rho(x_0)^{na},$$

where we have taken ϵ such that $1/p - 1 < \epsilon < \delta/n$, which implies that $(nb-n-\delta)q + n < 0$. The proof is complete.

5. H_L^1-Boundedness

In this section we prove the boundedness of $\nabla L^{-1/2}$ on H_L^1 when $n/2 < q_0 < n$.

Let $a(x)$ be an $H_L^{1,q}$-atom associated with a ball $B(x_0, r)$ for some suitable q. As the above section, if $r \geq \rho(x_0)$, we will prove that $\nabla L^{-1/2} a(x)$ is an $H_L^{1,q,\epsilon}$-molecule up to a constant factor. If $r < \rho(x_0)$, we will prove that $(\nabla L^{-1/2} - \nabla(-\Delta)^{-1/2})a(x)$ is an $H_L^{1,q,\epsilon}$-molecule up to a constant factor for some suitable ϵ. In any case we have $\|\nabla L^{-1/2} a(x)\|_{H_L^p} \leq C$ uniformly.

Suppose $r \geq \rho(x_0)$. It follows from Lemma 11 that

$$\left\|\nabla L^{-1/2} a(x)\right\|_{L^q} \leq C\|a(x)\|_q \leq C\, |B(x_0,r)|^{a-b}, \tag{70}$$

provide $1 < q \leq p_0$, where $1/p_0 = 1/q_0 - 1/n$, $a = \epsilon > 0$, $b = 1 - 1/q + \epsilon$. Thus there needs no the cancelation condition. Write

$$\left(\int_{\mathbb{R}^n} |x-x_0|^{nbq}\left|\nabla L^{-1/2} a(x)\right|^q dx\right)^{1/q}$$
$$\leq \left(\int_{B(x_0,2r)} |x-x_0|^{nbq}\left|\nabla L^{-1/2} a(x)\right|^q dx\right)^{1/q}$$
$$+ \left(\int_{|x-x_0|\geq 2r} |x-x_0|^{nbq}\left|\nabla L^{-1/2} a(x)\right|^q dx\right)^{1/q} \tag{71}$$
$$= I_1 + I_2.$$

It is obvious that

$$I_1 \leq r^{nb}\left\|\nabla L^{-1/2} a(x)\right\|_{L^q}$$
$$\leq C\, |B(x_0,r)|^b\, \|a(x)\|_{L^q} \leq C\, |B(x_0,r)|^a. \tag{72}$$

On the other hand, we have

$$\begin{aligned} I_2 &\le \left(\int_{|x-x_0|\ge 2r}\left(\int_{B(x_0,r)}|x-x_0|^{nb}\right.\right.\\ &\qquad \left.\left.\times\left|R^L(x,y)\right||a(y)|\right)^q dx\right)^{1/q} dy\\ &\le \int_{B(x_0,r)}|a(y)|\left(\int_{|x-x_0|\ge 2r}|x-x_0|^{nbq}\right.\\ &\qquad \left.\times\left|R^L(x,y)\right|^q dx\right)^{1/q} dy\\ &= \int_{B(x_0,r)} G(y)|a(y)|\,dy. \end{aligned} \tag{73}$$

Note that $|x-y| \sim |x-x_0|$ when $|x-x_0| \ge 2r$ and $y \in B(x_0,r)$ and by Lemma 11,

$$\begin{aligned} G(y) &\le C\rho(y)^N\\ &\quad\times\left\{\left(\int_{|x-x_0|\ge 2r}\left(\int_{B(x,|x-y|/4)}\frac{V(z)\,dz}{|z-x|^{n-1}}\right)^q\right.\right.\\ &\qquad \left.\times\frac{dx}{|x-x_0|^{(N-nb+n-1)q}}\right)^{1/q}\\ &\qquad \left.+\left(\int_{|x-x_0|\ge 2r}\frac{dx}{|x-x_0|^{(N-nb+n)q}}\right)^{1/q}\right\}. \end{aligned} \tag{74}$$

Since $\rho(y) \le Cr$ for $y \in B(x_0,r)$, it is clear that

$$\begin{aligned} &\rho(y)^N\left(\int_{|x-x_0|\ge 2r}\frac{dx}{|x-x_0|^{(N-nb+n)q}}\right)^{1/q}\\ &\quad \le Cr^{nb-n/q'} \le C\left|B(x_0,r)\right|^a \end{aligned} \tag{75}$$

provide $N > na$. We have taken q such that $1 < q \le p_0$, where $1/p_0 = 1/q_0 - 1/n$. Let $1/q = 1/s - 1/n$. Then $s \le q_0$. Using the theorem on fractional integrals, B_s condition, and Lemma 8, we obtain

$$\begin{aligned} &\rho(y)^N\left(\int_{|x-x_0|\ge 2r}\left(\int_{B(x,|x-y|/4)}\frac{V(z)\,dz}{|z-x|^{n-1}}\right)^q\right.\\ &\qquad \left.\times\frac{dx}{|x-x_0|^{(N-nb+n-1)q}}\right)^{1/q}\\ &\le Cr^N\sum_{j=1}^{\infty}\left(\int_{2^jr\le|x-x_0|<2^{j+1}r}\frac{1}{|x-x_0|^{(n+N-nb-1)q}}\right.\\ &\qquad \left.\cdot\left(\int_{B(x,|x-y|/4)}\frac{V(z)\,dz}{|z-x|^{n-1}}\right)^q dx\right)^{1/q}\\ &\le C\sum_{j=1}^{\infty}2^{-jN}\left(2^jr\right)^{-n+nb+1}\\ &\qquad \times\left(\int_{B(x_0,2^{j+1}r)}\left(\int_{B(x,|x-y|/4)}\frac{V(z)\,dz}{|z-x|^{n-1}}\right)^q dx\right)^{1/q}\\ &\le C\sum_{j=1}^{\infty}2^{-jN}\left(2^jr\right)^{-n+nb+1}\left(\int_{B(x_0,2^{j+1}r)}V(x)^s dx\right)^{1/s}\\ &\le C\sum_{j=1}^{\infty}2^{-jN}\left(2^jr\right)^{-n+nb+1}\left|B(x_0,2^{j+1}r)\right|^{1/s-1}\\ &\qquad \times\int_{B(x_0,2^{j+1}r)}V(x)\,dx\\ &\le C\sum_{j=1}^{\infty}2^{-j(N-m_0)}\left(2^jr\right)^{-n+nb+n/s-1}\\ &= C\sum_{j=1}^{\infty}2^{-j(N-m_0-na)}r^{na}\\ &\le C\left|B(x_0,r)\right|^a \end{aligned} \tag{76}$$

provide N sufficiently large. Thus $G(y) \le C|B(x_0,r)|^a$. It follows that

$$\begin{aligned} I_2 &\le \int_{B(x_0,r)} G(y)|a(y)|\,dy\\ &\le C\left|B(x_0,r)\right|^a\|a\|_{L^1} \le C\left|B(x_0,r)\right|^a. \end{aligned} \tag{77}$$

Therefore, $\||\cdot-x_0|^{nb}\nabla L^{-1/2}a\|_{L^q} \le C|B(x_0,r)|^a$ and $\mathcal{N}(\nabla L^{-1/2}a) \le C$.

In case $r < \rho(x_0)$, we need to prove that $(\nabla L^{-1/2} - \nabla(-\Delta)^{-1/2})a(x)$ is an $H_L^{1,q,\epsilon}$-molecule up to a constant factor for some suitable ϵ. First we give the estimate of $\|(\nabla L^{-1/2} - \nabla(-\Delta)^{-1/2})a(x)\|_{L^q}$. Write

$$\begin{aligned} &\left(\int_{\mathbb{R}^n}\left|\left(\nabla L^{-1/2}-\nabla(-\Delta)^{-1/2}\right)a(x)\right|^q dx\right)^{1/q}\\ &\le\left(\int_{B(x_0,2\rho(x_0))}\left|\left(\nabla L^{-1/2}-\nabla(-\Delta)^{-1/2}\right)a(x)\right|^q dx\right)^{1/q}\\ &\quad+\left(\int_{|x-x_0|\ge 2\rho(x_0)}\left|\left(\nabla L^{-1/2}-\nabla(-\Delta)^{-1/2}\right)a(x)\right|^q dx\right)^{1/q}\\ &= J_1 + J_2. \end{aligned} \tag{78}$$

We have

$$\begin{aligned} J_1 &\le \int_{B(x_0,r)} |a(y)| \left(\int_{B(x_0,2\rho(x_0))} \left|\widetilde{R}(x,y)\right|^q dx \right)^{1/q} dy \\ &= \int_{B(x_0,r)} \widetilde{G}(y) |a(y)| \, dy. \end{aligned} \tag{79}$$

By Lemma 11,

$$\begin{aligned} \widetilde{G}(y) \le & \left(\int_{B(x_0,2\rho(x_0))} \frac{C}{|x-y|^{(n-1)q}} \right. \\ & \left. \times \left(\int_{B(x,|x-y|/4)} \frac{V(z)\,dz}{|z-x|^{n-1}} \right)^q dx \right)^{1/q} \\ & + C\rho(y)^{-\delta} \left(\int_{B(x_0,2\rho(x_0))} \frac{dx}{|x-y|^{(n-\delta)q}} \right)^{1/q} \\ = & \widetilde{G}_1(y) + \widetilde{G}_2(y). \end{aligned} \tag{80}$$

Note that $\rho(y) \sim \rho(x_0)$ when $y \in B(x_0, r)$. It is obvious that

$$\widetilde{G}_2(y) \le C\rho(x_0)^{-n/q'} = C\rho(x_0)^{n(a-b)}, \tag{81}$$

provide $1 < q < n/(n-\delta)$. On the other hand, using the theorem on fractional integrals and B_s condition with $s \le q_0$, $1/q = 1/s - 1/n$, and $1 < q < n/(n-\delta)$, we get

$$\begin{aligned} \widetilde{G}_1(y) \le & \sum_{j=0}^{\infty} \left(\int_{2^{-j+1}\rho(x_0) \le |x-y| < 2^{-j+2}\rho(x_0)} \frac{C}{|x-y|^{(n-1)q}} \right. \\ & \left. \cdot \left(\int_{B(x,|x-y|/4)} \frac{V(z)\,dz}{|z-x|^{n-1}} \right)^q dx \right)^{1/q} \\ \le & C \sum_{j=0}^{\infty} \left(2^{-j}\rho(x_0)\right)^{-n+1} \\ & \times \left(\int_{|x-y|<2^{-j+2}\rho(x_0)} \right. \\ & \left. \times \left(\int_{B(x,|x-y|/4)} \frac{V(z)\,dz}{|z-x|^{n-1}} \right)^q dx \right)^{1/q} \\ \le & C \sum_{j=0}^{\infty} \left(2^{-j}\rho(x_0)\right)^{-n+1} \left(\int_{|x-y|<2^{-j+3}\rho(x_0)} V(x)^s \, dx \right)^{1/s} \\ \le & C \sum_{j=0}^{\infty} \left(2^{-j}\rho(x_0)\right)^{-2n+n/s+1} \int_{|x-y|<2^{-j+3}\rho(x_0)} V(x)\,dx \\ \le & C \sum_{j=0}^{\infty} \left(2^{-j}\rho(x_0)\right)^{-n+n/s-1} 2^{-j\delta} \\ \le & C\rho(x_0)^{n(a-b)}, \end{aligned} \tag{82}$$

where we have used Lemma 7 in the last second inequality. Thus,

$$J_1 \le \int_{B(x_0,r)} \widetilde{G}(y) |a(y)| \, dy \le C\rho(x_0)^{n(a-b)}. \tag{83}$$

Since $|x-y| \sim |x-x_0|$ when $|x-x_0| \ge 2\rho(x_0)$ and $y \in B(x_0,r)$, $|\widetilde{R}(x,y)| \le C/|x-y|^n$, it is easy to see that

$$\begin{aligned} J_2 &\le \int_{B(x_0,r)} |a(y)| \left(\int_{|x-x_0| \ge 2\rho(x_0)} \left|\widetilde{R}(x,y)\right|^q dx \right)^{1/q} dy \\ &\le C \int_{B(x_0,r)} |a(y)| \left(\int_{|x-x_0| \ge 2\rho(x_0)} \frac{dx}{|x-y|^{nq}} \right)^{1/q} dy \\ &\le C \int_{B(x_0,r)} |a(y)| \rho(x_0)^{n(a-b)} dy \\ &\le C\rho(x_0)^{n(a-b)}. \end{aligned} \tag{84}$$

Therefore we have

$$\left\| \left(\nabla L^{-1/2} - \nabla(-\Delta)^{-1/2} \right) a(x) \right\|_{L^q} \le C\rho(x_0)^{n(a-b)}. \tag{85}$$

As above, there needs no the cancelation condition. Write

$$\begin{aligned} & \left(\int_{\mathbb{R}^n} |x-x_0|^{nbq} \left| \left(\nabla L^{-1/2} - \nabla(-\Delta)^{-1/2} \right) a(x) \right|^q dx \right)^{1/q} \\ & \le \left(\int_{B(x_0,2\rho(x_0))} |x-x_0|^{nbq} \right. \\ & \qquad \left. \times \left| \left(\nabla L^{-1/2} - \nabla(-\Delta)^{-1/2} \right) a(x) \right|^q dx \right)^{1/q} \\ & \quad + \left(\int_{|x-x_0| \ge 2\rho(x_0)} |x-x_0|^{nbq} \left| \nabla L^{-1/2} a(x) \right|^q dx \right)^{1/q} \\ & \quad + \left(\int_{|x-x_0| \ge 2\rho(x_0)} |x-x_0|^{nbq} \left| \nabla(-\Delta)^{-1/2} a(x) \right|^q dx \right)^{1/q} \\ & = H_1 + H_2 + H_3. \end{aligned} \tag{86}$$

It is obvious that

$$H_1 \le C\rho(x_0)^{nb} \left\| \left(\nabla L^{-1/2} - \nabla(-\Delta)^{-1/2} \right) a(x) \right\|_{L^q} \le C\rho(x_0)^{na}. \tag{87}$$

We have

$$\begin{aligned} H_2 \le & \int_{B(x_0,r)} |a(y)| \\ & \times \left(\int_{|x-x_0| \ge 2\rho(x_0)} |x-x_0|^{nbq} \right. \\ & \left. \times \left| R^L(x,y) \right|^q dx \right)^{1/q} dy \\ = & \int_{B(x_0,r)} G_0(y) |a(y)| \, dy. \end{aligned} \tag{88}$$

Since $|x-y| \sim |x-x_0|$ and $\rho(y) \sim \rho(x_0)$ when $|x-x_0| \geq 2\rho(x_0)$ and $y \in B(x_0, r)$, by Lemma 11,

$$\begin{aligned} G_0(y) &\leq C\rho(x_0)^N \\ &\times \left\{ \left(\int_{|x-x_0|\geq 2\rho(x_0)} \left(\int_{B(x,|x-y|/4)} \frac{V(z)\,dz}{|z-x|^{n-1}} \right)^q \right.\right. \\ &\left. \times \frac{dx}{|x-x_0|^{(N-nb+n-1)q}} \right)^{1/q} \\ &\left. + \left(\int_{|x-x_0|\geq 2\rho(x_0)} \frac{dx}{|x-x_0|^{(N-nb+n)q}} \right)^{1/q} \right\}. \end{aligned} \quad (89)$$

Similar to $G(y)$ in the proof of (77), we obtain $G_0(y) \leq C\rho(x_0)^{na}$ by the same argument. It follows that

$$H_2 \leq \int_{B(x_0,r)} G_0(y)\,|a(y)|\,dy \leq C\,\rho(x_0)^{na}. \quad (90)$$

Using the cancelation condition of a,

$$\begin{aligned} H_3 \leq& \int_{B(x_0,r)} |a(y)| \left(\int_{|x-x_0|\geq 2\rho(x_0)} |x-x_0|^{nbq} \right. \\ &\left. \times |R(x,y)-R(x,x_0)|^q dx \right)^{1/q} dy \\ \leq& \; C\int_{B(x_0,r)} |a(y)| \\ &\times \left(\int_{|x-x_0|\geq 2\rho(x_0)} \frac{|y-x_0|^q dx}{|x-x_0|^{(n+1-nb)q}} \right)^{1/q} dy \\ \leq& \; C\int_{B(x_0,r)} |a(y)|\,\rho(x_0)^{na} dy \\ \leq& \; C\rho(x_0)^{na}, \end{aligned} \quad (91)$$

where we have taken ϵ such that $0 < \epsilon < 1/n$. This proves that

$$\left\| |x-x_0|^{nb} \left(\nabla L^{-1/2} - \nabla(-\Delta)^{-1/2}\right) a(x) \right\|_{L^q} \leq C\rho(x_0)^{na}. \quad (92)$$

It follows that $\mathcal{N}((\nabla L^{-1/2} - \nabla(-\Delta)^{-1/2})a) \leq C$. The proof is completed.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

This work was supported by Research Fund for the Doctoral Program of Higher Education of China (20113108120001), the National Natural Science Foundation of China under Grants nos. 11471018 and 11371036, and the Beijing Natural Science Foundation under Grant no. 1142005.

References

[1] Z. W. Shen, “L^p estimates for L^p operators with certain potentials,” *Annales de l’Institut Fourier*, vol. 45, no. 2, pp. 513–546, 1995.

[2] C. L. Fefferman, “The uncertainty principle,” *American Mathematical Society*, vol. 9, no. 2, pp. 129–206, 1983.

[3] J. Zhong, *Harmornic analysis for some Schrödinger type operators [Ph.D. thesis]*, Princeton University, 1993.

[4] J. Dziubański and J. Zienkiewicz, “Hardy space H^1 associated to Schrödinger operator with potential satisfying reverse Hölder inequality,” *Revista Matemática Iberoamericana*, vol. 15, pp. 279–296, 1999.

[5] J. Dziubański and J. Zienkiewicz, “H^p spaces for Schrödinger operators, Fourier Analysis and Ralated Topics,” *Banach Center Publications*, vol. 56, pp. 45–53, 2002.

[6] J. Dziubański and J. Zienkiewicz, “H^P spaces associated with Schrödinger operators with potentials from reverse Hölder classes,” *Colloquium Mathematicum*, vol. 98, no. 1, pp. 5–38, 2003.

[7] J. Z. Huang and H. P. Liu, “Littlewood-Paley theory and Hardy spaces related to Schrödinger operators,” Submitted.

[8] M. Taibleson and G. Weiss, “The molecular characterization of certain Hardy spces,” *Asterisque*, vol. 77, pp. 67–151, 1980.

[9] J. Alvarez and M. Milman, “H^p continuty properties of Calderon-Zygmund operators,” *Journal of Mathematical Analysis and Applications*, vol. 118, pp. 65–79, 1986.