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Research Article

Tricorns and Multicorns of S-Iteration Scheme

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Complex graphics of dynamical system have striking features of fractals and become a wide area of research due to their beauty and complexity of their nature. The aim of this paper is to study dynamics of relative superior tricorns and multicorns using S-iteration schemes. Several examples are presented to explore the geometry of relative superior tricorns and multicorns for antipolynomial $z \rightarrow \bar{z}^n + c$ of complex polynomial $z^n + c$ for $n \geq 2$.

1. Introduction

In 1918, French mathematician Julia [1] investigated the iteration process of complex function and attained a Julia set. On the other hand, the object Mandelbrot set was given by Mandelbrot [2]. In 1989, Crowe et al. [3] considered formal analogy with Mandelbrot set and named it “Mandelbrot sets” and showed its feature bifurcations along arcs rather than at points. The word “tricorn” was coined by Milnor for the connectedness locus for antiholomorphic polynomials $\bar{z}^2 + c$, which plays an intermediate role between quadratic and cubic polynomials. Tricorn has many similarities with the Mandelbrot set due to a compact subset of $\mathbb{C}$.

Milnor [4] found it in a real slice of the cubic connectedness locus. Winters [5] explained that boundary of the tricorn contains a smooth arc. The symmetries of tricorns and multicorns have been analyzed by Lau and Schleicher [6]. Nakane and Schleicher [7] presented various properties of tricorns and multicorns and quoted that the multicorns are the generalized tricorns or the tricorns of higher order. They also investigate that the Julia set of a polynomial of the form $p_c(z) = \bar{z}^n + c$ for $n \geq 2$ is either connected or disconnected. The set of parameters c such that the Julia set of p_c is connected is called the multicorn. Tricorn prints, such as tricorn mugs and tricorn shirts, are being used for commercial purpose.

The dynamics of antiholomorphic complex polynomials $z \rightarrow \bar{z}^n + c$ for $n \geq 2$ was studied and explored using Mann iteration by Rani [8, 9]. Relative superior tricorns and relative superior multicorns were introduced using Ishikawa iterates by Chauhan et al. [10]. Also, they studied their corresponding relative superior Julia sets.

In this paper we introduce and visualize a new class of relative superior tricorns and relative superior multicorns using S-iteration scheme.

This paper is organized as follows. In Section 2, some basic definitions are presented. Section 3 contains the escape criterion for relative superior tricorns and multicorns. In Section 4, we generate relative superior tricorns and multicorns of S-iteration scheme for quadratic, cubic, and biquadratic functions. At last, paper has been concluded in Section 5.

2. Preliminaries

Definition 1 (see [11], multicorn). The multicorn A_c for the quadratic function $A_c(z) = \bar{z}^n + c$ is defined as the collection of all $c \in \mathbb{C}$ for which the orbit of the point 0 is bounded; that is,

$$A_c = \{c \in \mathbb{C} : A_c^n(0) \text{ does not tend to } \infty\}, \tag{1}$$

where $\mathbb{C}$ is a complex space and A_c^n is the nth iterate of the function $A_c(z)$. An equivalent formulation is that the connectedness of loci for higher degree antiholomorphic polynomials $A_c(z) = \overline{z}^n + c$ is called multicorns.

Notice that, at $n = 2$, multicorns reduce to tricorn. Moreover, the tricorn naturally lives in the real slice $d = \overline{c}$ in the two-dimensional parameter space of maps $z \to (z^2 + d)^2 + c$. They have $(n+1)$-fold rotational symmetries. Also, by dividing these symmetries, the resulting multicorns are called unicorns [7].

Definition 2 (see [12], S-iteration scheme for relative superior tricorns and multicorns). Let X be a subset of real or complex numbers and $f : X \to X$. For $x_0 \in X$, one constructs the sequences $\{x_n\}$ and $\{y_n\}$ in X in the following manner:

$$x_n = (1 - s_{n-1}) f(x_{n-1}) + s_{n-1} f(y_{n-1}),$$
$$y_n = (1 - s_n') x_n + s_n' f(x_n), \tag{2}$$

where $0 < s_n < 1$, $0 < s_n' < 1$, and s_n, s_n' both are convergent to nonzero number.

The sequences $\{x_n\}$ and $\{y_n\}$ constructed above are called S-iteration scheme sequences of iterations or relative superior sequences of iterates. We denote it by $\mathrm{RSO}(x_0, s_n, s_n', t)$.

Definition 3 (Mandelbrot set). The Mandelbrot set M consists of all parameters c for which the filled Julia set of Q_c is connected; that is

$$M = \{c \in C : K(Q_c) \text{ is connected}\}. \tag{3}$$

In fact, M contains an enormous amount of information about the structure of Julia sets. The Mandelbrot set M for the quadratic $Q_c(z) = z^2 + c$ is defined as the collection of all $c \in C$ for which the orbit of the point 0 is bounded; that is,

$$M = \{c \in C : \{Q_c^n(0)\} \ (n = 0, 1, 2, \ldots) \text{ is bounded}\}. \tag{4}$$

We choose the initial point 0, as 0 is the only critical point of Q_c [11].

3. Escape Criterion for Relative Superior Tricorns and Multicorns

The escape criterion plays an important role in the generation and analysis of relative superior tricorns and multicorns. We now obtain a general escape criterion for polynomials of the form $G_c(z) = z^n + c$.

Theorem 4. *For general function $G_c(z) = z^n + c$, $n = 1, 2, 3, \ldots$, suppose that $|z| \geq |c| > (2/s)^{1/n-1}$ and $|z| \geq |c| > (2/s')^{1/n-1}$, where $0 < s, s' < 1$, and c is a complex number. Define*

$$\begin{aligned} z_1 &= (1 - s)(z^n + c) + sG_c(z), \\ &\vdots \\ z_n &= (1 - s)(z_{n-1}^n + c) + sG_c(z_{n-1}). \end{aligned} \tag{5}$$

Then $|z_n| \to \infty$ as $n \to \infty$. Thus the general escape criterion is $\max\{|c|, (2/s)^{1/n-1}, (2/s')^{1/n-1}\}$.

Proof. We will use induction. For $n = 1$, we get $G_c(z) = z + c$, so the escape criterion is $|c|$, which is obvious; that is, $|z| > \max\{|c|, 0, 0\}$. For $n = 2$, we get $G_c(z) = z^2 + c$ so the escape criterion is $|z| > \max\{|c|, 2/s, 2/s'\}$. For $n = 3$, we get $G_c(z) = z^3 + c$ so the escape criterion is $|z| > \max\{|c|, (2/s)^{1/2}, (2/s')^{1/2}\}$.

Now suppose that theorem is true for any n. Let $G_c(z) = z^{n+1} + c$ and $|z| \geq |c| > (2/s)^{1/n}$ and $|z| \geq |c| > (2/s')^{1/n}$ exist. Then for $G_c'(z) = z^{n+1} + c$, consider

$$\begin{aligned} |G_c(z)| &= \left|(1 - s')z + s'G_c'(z)\right| \\ &= \left|(1 - s')z + s'(z^{n+1} + c)\right| \\ &= \left|(1 - s')z + s'z^{n+1} + s'c\right| \\ &\geq \left|s'z^{n+1} + (1 - s')z\right| - |s'c| \\ &\geq \left|s'z^{n+1} + (1 - s')z\right| - |s'z| \\ &\geq \left|s'z^{n+1}\right| - \left|(1 - s')z\right| - s'|z| \\ &= \left|s'z^{n+1}\right| - |z| + s'|z| - s'|z| \\ &\geq |z|\left(\left|s'z^n\right| - 1\right). \end{aligned} \tag{6}$$

Also, for

$$z_n = (1 - s) f(z_{n-1}) + sG_c(z), \tag{7}$$

we obtain

$$\begin{aligned} |z_1| &= \left|(1 - s)(z^{n+1} + c) + s|z|\left(\left|s'z^n\right| - 1\right)\right| \\ &= \left|(1 - s)z^{n+1} + (1 - s)c + s|z|\left|s'z^n\right| - s|z|\right| \\ &= \left|ss'|z| \cdot |z^n| - s|z| + (1 - s)z^{n+1} + (1 - s)c\right| \\ &\geq \left|ss'|z| \cdot |z^n| - s|z| - (s - 1)z^{n+1}\right| - |(1 - s)c| \\ &\geq \left|ss'\left|z^{n+1}\right| - s|z|\right| - \left|(s - 1)z^{n+1}\right| - (1 - s)|z| \\ &\geq ss'\left|z^{n+1}\right| - s|z| - (s - 1)\left|z^{n+1}\right| - |z| + s|z| \\ &\geq (ss' - s + 1)\left|z^{n+1}\right| - |z| \\ &\geq |z|\left((ss' - s + 1)|z^n| - 1\right). \end{aligned} \tag{8}$$

Since $|z| > (2/s)^{1/n}$ and $|z| > (2/s')^{1/n}$ it follows that $|z^n| > 2/s$ and $|z^n| > 2/s'$. It can be easily seen that

$$|z^n| > \frac{2}{(ss')} > \frac{2}{(ss' - s + 1)}, \tag{9}$$

which implies that

$$(ss' - s + 1)|z^n| - 1 > 1. \tag{10}$$

Hence there exists $\lambda > 0$ such that $(ss' - s + 1)|z^n| - 1 > 1 + \lambda$. Consequently

$$\begin{aligned} |z_1| &> (1+\lambda)|z|, \\ &\vdots \\ |z_n| &> (1+\lambda)^n |z|. \end{aligned} \tag{11}$$

Hence $|z_n| \to \infty$ as $n \to \infty$. So $|z| > \max\{|c|, (2/s)^{1/n}, (2/s')^{1/n}\}$ is the escape criterion. This completes the proof. □

Corollary 5. *Suppose that $|c| > (2/s)^{1/n-1}$ and $|c| > (2/s')^{1/n-1}$ exist. Then the relative superior orbit of S-iteration scheme $RSO(G_c, 0, s, s')$ escapes to infinity.*

Corollary 6. *Assume that $|z_k| > \max\{|c|, (2/s)^{1/k-1}, (2/s')^{1/k-1}\}$ for some $k \geq 0$. Then $|z_{k+1}| > \gamma^n |z_k|$ and $|z_n| \to \infty$ as $n \to \infty$.*

This corollary provides an algorithm for computing the relative superior Mandelbrot sets for the functions of the form $G_c(z) = z^n + c$ and $n = 2, 3, \ldots$ also gives escape criterion to generate relative superior tricorns and multicorns.

4. Generation of Relative Superior Tricorns and Multicorns

We generate relative superior tricorns and multicorns of S-iteration scheme for quadratic, cubic, and biquadratic functions using software MAPLE.

4.1. Relative Superior Tricorns for Quadratic Functions. In case of quadratic antipolynomial, relative superior tricorns maintain the symmetry along x-axis (Figures 1–6).

4.2. Relative Superior Multicorns for Cubic Functions. In case of cubic antipolynomial, relative superior multicorns maintain the symmetry along x-axis and y-axis (Figures 7–12).

4.3. Relative Superior Multicorns for Biquadratic Functions. In case of biquadratic antipolynomial, relative superior multicorns maintain the symmetry along x-axis (Figures 13–18).

4.4. Generalization of Relative Superior Multicorns. See Figures 19–24.

5. Conclusions

In this paper relative superior antifractal has been visualized with respect to relative superior orbit and analyzed the pattern of symmetry among them. In the dynamics of antipolynomials $z \to \bar{z}^n + c$ for $n \geq 2$, we obtained many relative superior tricorns and multicorns for the same value of n by using different values of s and s' in S-iteration scheme. We found that the number of branches and main ovoids attached to the branches of the relative superior tricorns and

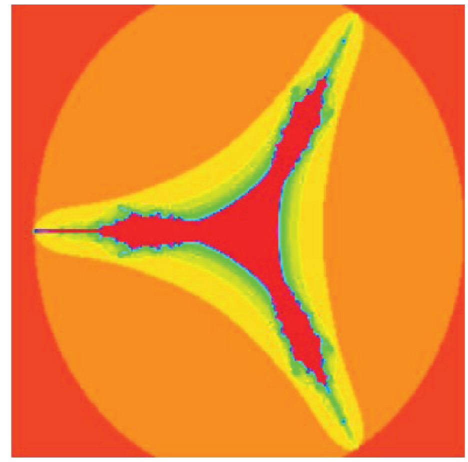

Figure 1: Relative superior tricorn for $s = 1.0$ and $s' = 1.0$.

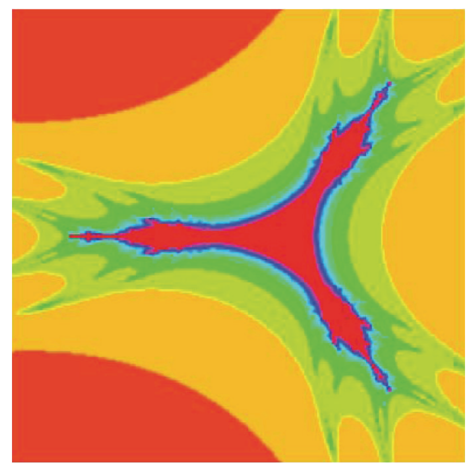

Figure 2: Relative superior tricorn for $s = 0.2$ and $s' = 0.4$.

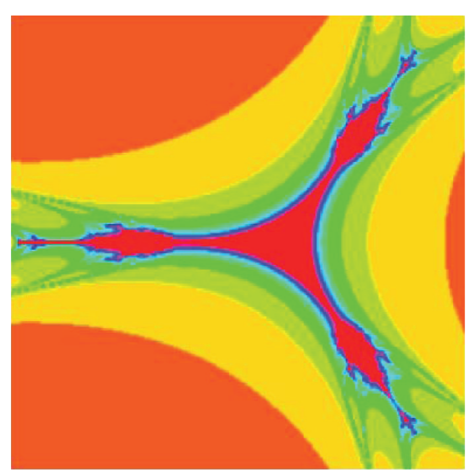

Figure 3: Relative superior tricorn for $s = 0.4$ and $s' = 0.3$.

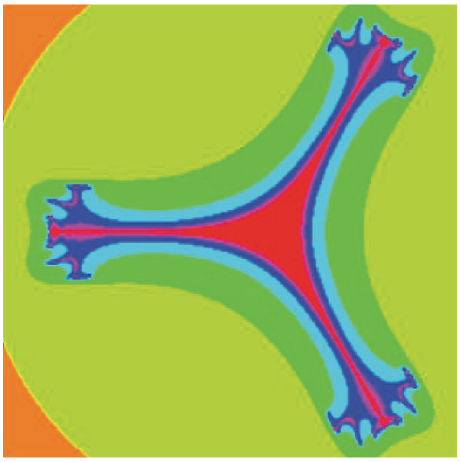

Figure 4: Relative superior tricorn for $s = 0.6$ and $s' = 0.5$.

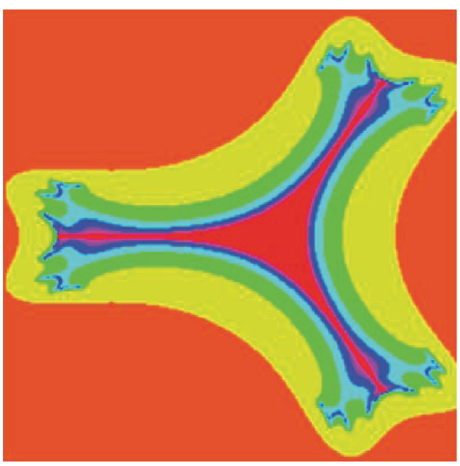

Figure 5: Relative superior tricorn for $s = 0.4$ and $s' = 0.7$.

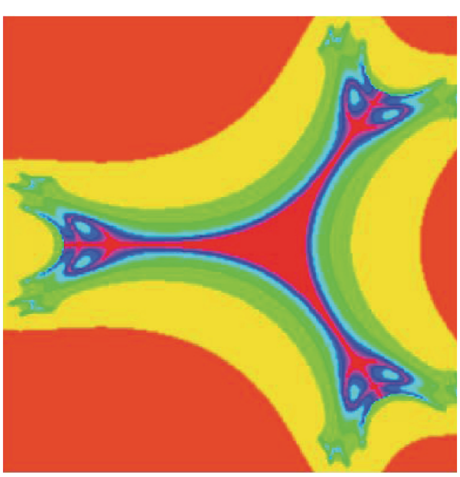

Figure 6: Relative superior tricorn for $s = 0.3$ and $s' = 0.7$.

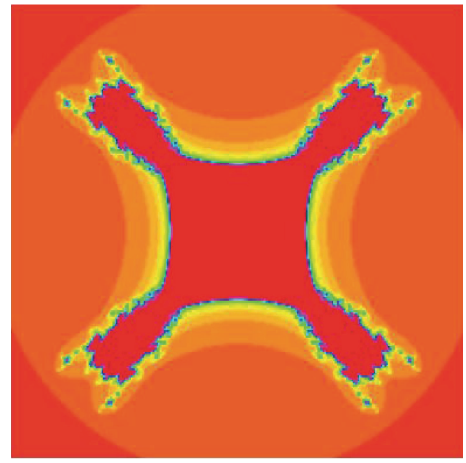

Figure 7: Relative superior multicorn for $s = 1.0$ and $s' = 1.0$.

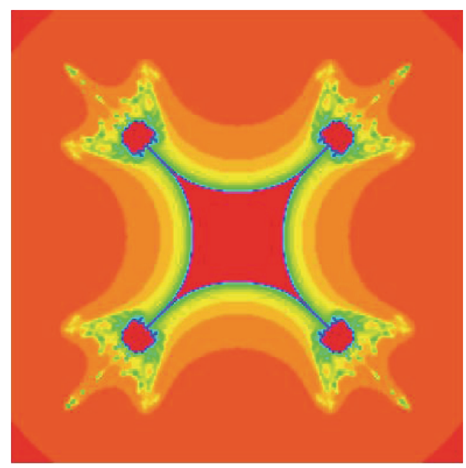

Figure 8: Relative superior multicorn for $s = 0.3$ and $s' = 0.5$.

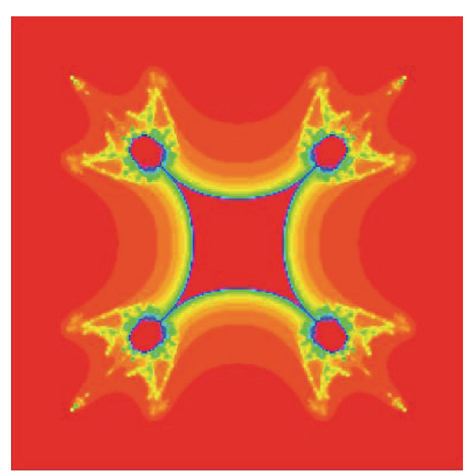

Figure 9: Relative superior multicorn for $s = 0.2$ and $s' = 0.6$.

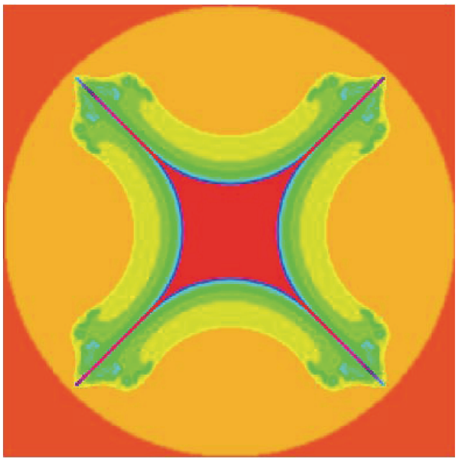

Figure 10: Relative superior multicorn for $s = 0.6$ and $s' = 0.5$.

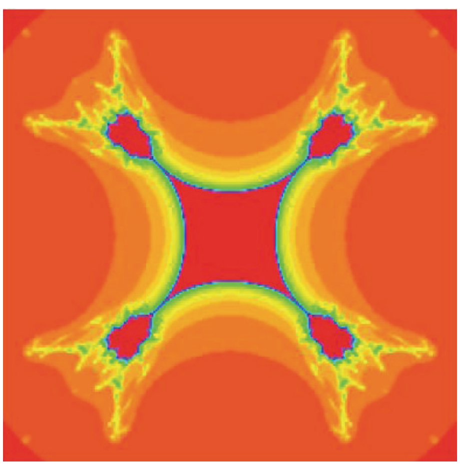

Figure 11: Relative superior multicorn for $s = 0.4$ and $s' = 0.3$.

Figure 12: Relative superior multicorn for $s = 0.3$ and $s' = 0.8$.

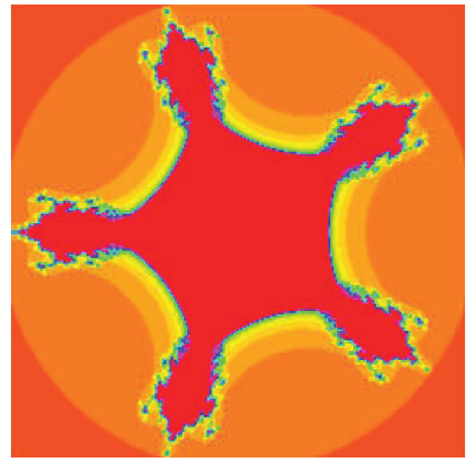

Figure 13: Relative superior multicorn for $s = 1.0$ and $s' = 1.0$.

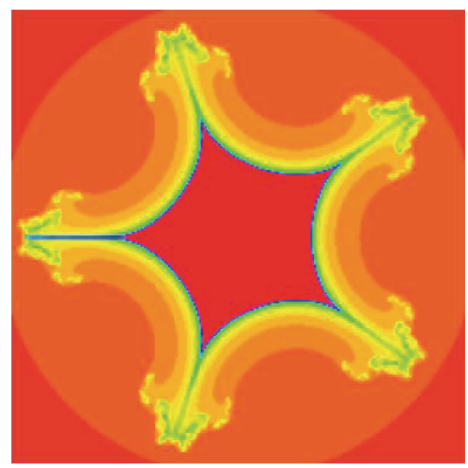

Figure 14: Relative superior multicorn for $s = 0.5$ and $s' = 0.6$.

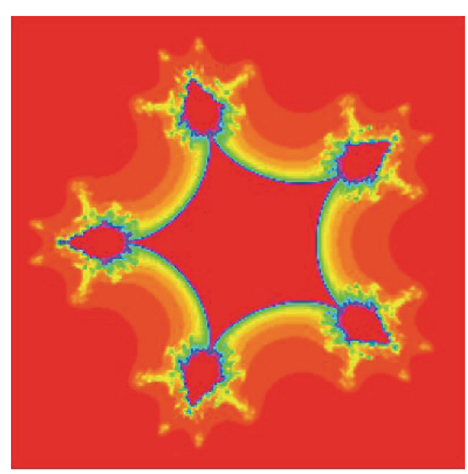

Figure 15: Relative superior multicorn for $s = 0.1$ and $s' = 0.8$.

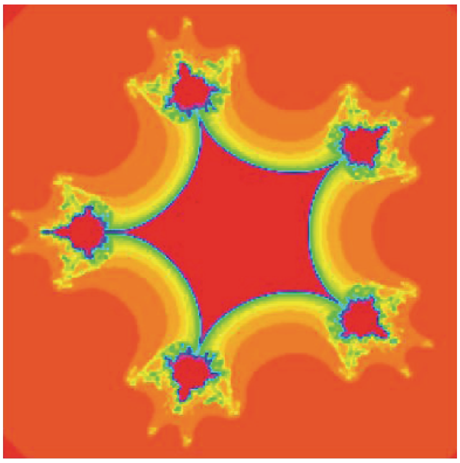

FIGURE 16: Relative superior multicorn for $s = 0.2$ and $s' = 0.6$.

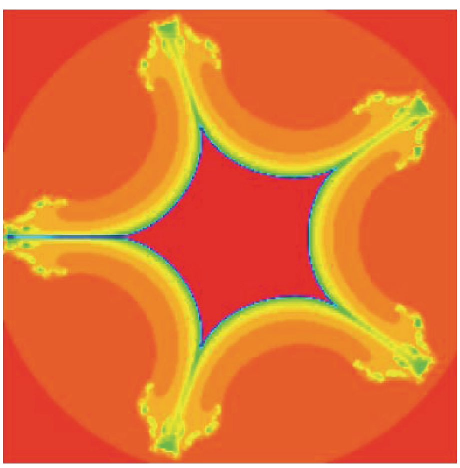

FIGURE 17: Relative superior multicorn for $s = 0.7$ and $s' = 0.4$.

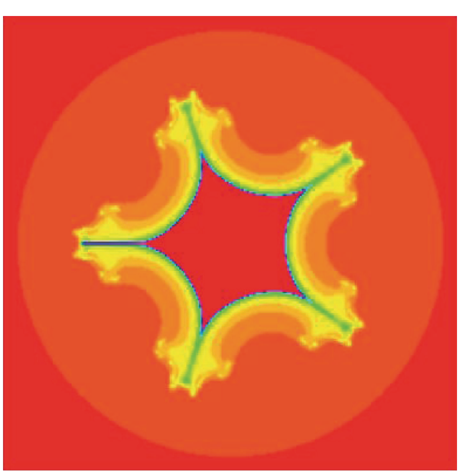

FIGURE 18: Relative superior multicorn for $s = 0.3$ and $s' = 0.8$.

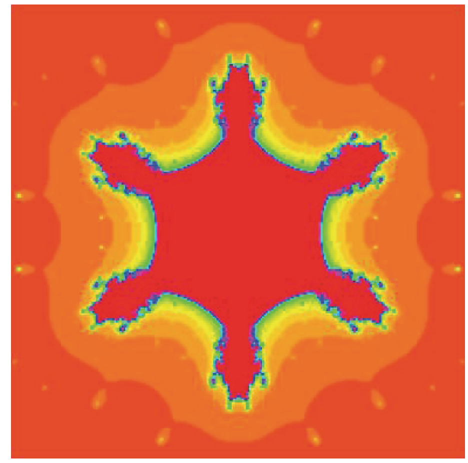

FIGURE 19: Relative superior multicorn for $s = 0.01$, $s' = 0.4$, and $n = 5$.

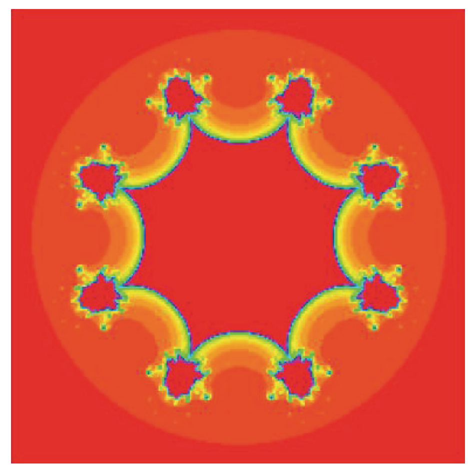

FIGURE 20: Relative superior multicorn for $s = 0.3$, $s' = 0.4$, and $n = 7$.

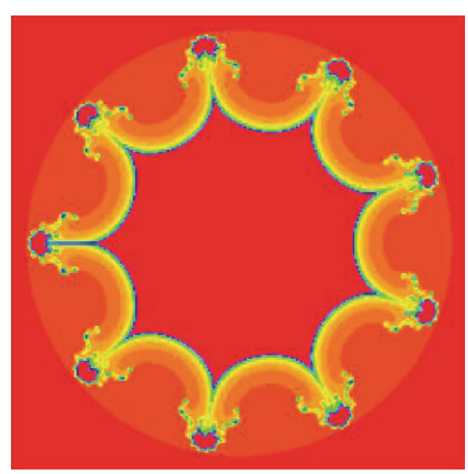

FIGURE 21: Relative superior multicorn for $s = 0.6$, $s' = 0.5$, and $n = 8$.

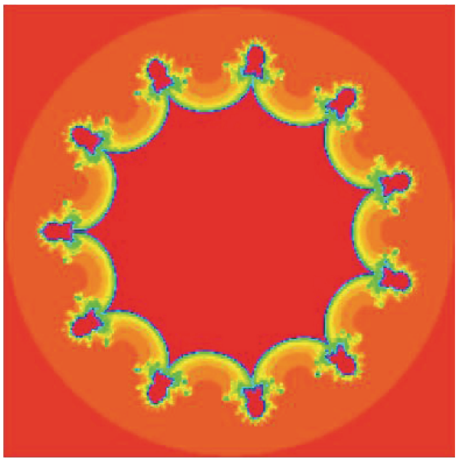

Figure 22: Relative superior multicorn for $s = 0.2$, $s' = 0.7$, and $n = 10$.

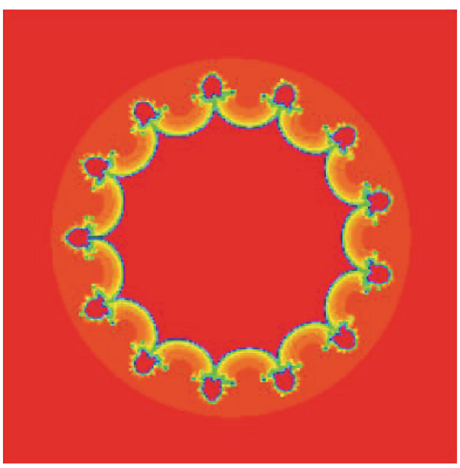

Figure 23: Relative superior multicorn for $s = 0.3$, $s' = 0.6$, and $n = 12$.

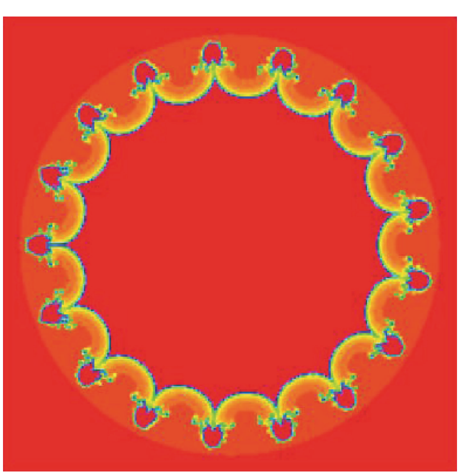

Figure 24: Relative superior multicorn for $s = 0.4$, $s' = 0.6$, and $n = 16$.

multicorns had been $n + 1$, where n is the power of $\overline{z}$. We also found that for n is odd the symmetry of relative superior multicorn is about both x-axis and y-axis but for n is even the symmetry is maintained only along x-axis. We believe that results of this paper will inspire those who are interested in generating automatically nicely looking graphics.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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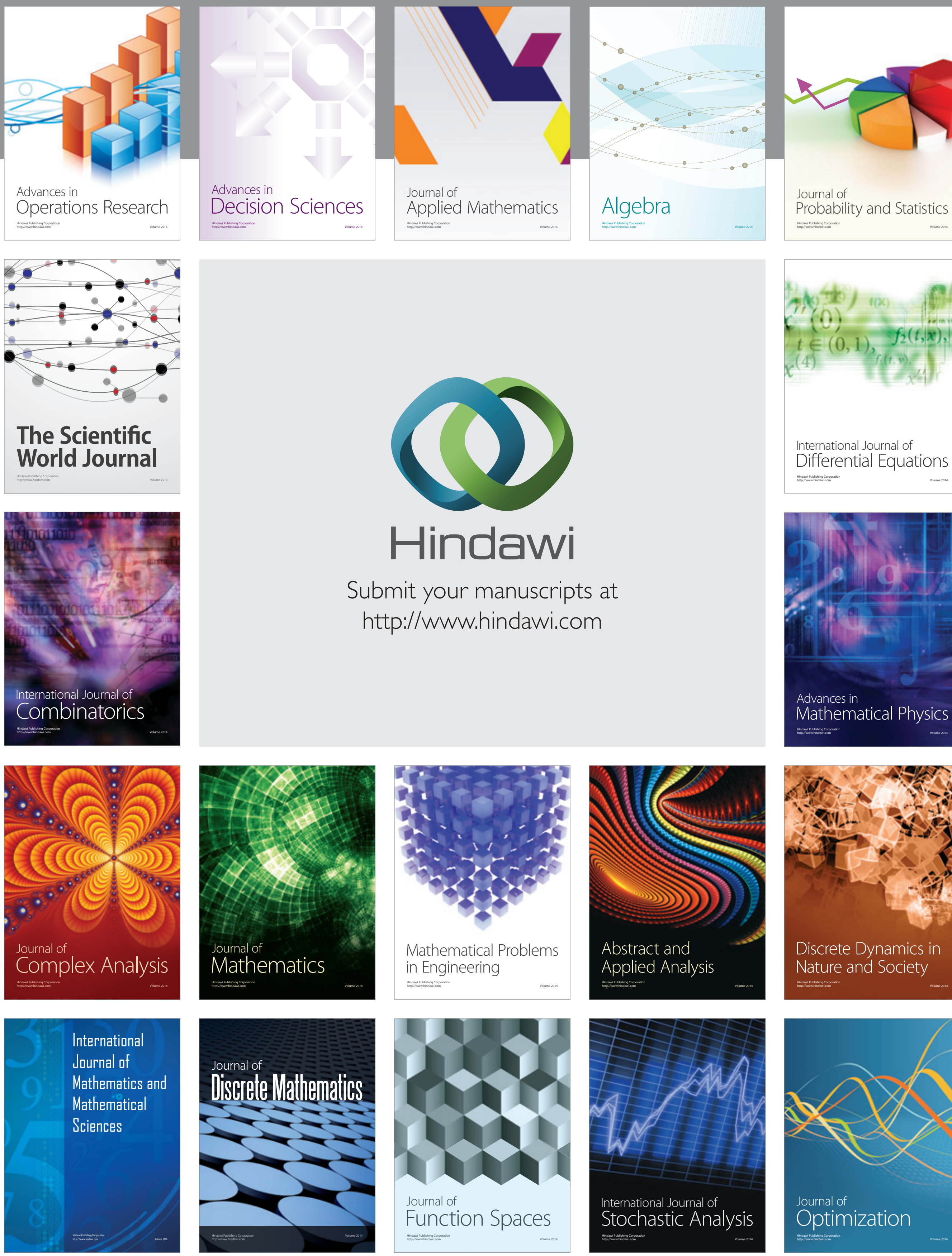
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