Hindawi Publishing Corporation
Journal of Function Spaces
Volume 2015, Article ID 420750, 10 pages
http://dx.doi.org/10.1155/2015/420750

Research Article

Strong Summability of Fourier Transforms at Lebesgue Points and Wiener Amalgam Spaces

Ferenc Weisz

Department of Numerical Analysis, Eötvös L. University, Pázmány P. Sétány 1/C, Budapest 1117, Hungary

Correspondence should be addressed to Ferenc Weisz; weisz@inf.elte.hu

Received 1 March 2015; Revised 8 May 2015; Accepted 8 May 2015

Academic Editor: Gelu Popescu

We characterize the set of functions for which strong summability holds at each Lebesgue point. More exactly, if f is in the Wiener amalgam space $W(L_1, \ell_q)(\mathbb{R})$ and f is almost everywhere locally bounded, or $f \in W(L_p, \ell_q)(\mathbb{R})$ $(1 < p < \infty, 1 \le q < \infty)$, then strong θ-summability holds at each Lebesgue point of f. The analogous results are given for Fourier series, too.

1. Introduction

It was proved by Lebesgue [1] that the Fejér means [2] of the trigonometric Fourier series of an integrable function converge almost everywhere to the function; that is,

$$\frac{1}{n+1}\sum_{k=0}^{n}\left(s_k f(x) - f(x)\right) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty \tag{1}$$

for almost every $x \in \mathbb{T}$, where $\mathbb{T}$ denotes the torus and $s_k f$ the kth partial sum of the Fourier series of the one-dimensional function f. The set of convergence is characterized as the Lebesgue points of f.

Hardy and Littlewood [3] considered the so-called strong summability and verified that the strong means

$$\frac{1}{n+1}\sum_{k=0}^{n}\left|s_k f(x) - f(x)\right|^q \tag{2}$$

tend to 0 at each Lebesgue point of f, as $n \to \infty$, whenever $f \in L_p(\mathbb{T})$ $(1 < p < \infty)$ (for Fourier transforms see Giang and Móricz [4]). This result does not hold for $p = 1$ (see Hardy and Littlewood [5]). However, the strong means tend to 0 almost everywhere for all $f \in L_1(\mathbb{T})$. This is because of Marcinkiewicz [6] for $q = 2$ and Zygmund [7] for all $q > 0$ (see also Bary [8]). Later Gabisoniya [9] characterized the set of convergence as the so-called Gabisoniya points. Strong summability with lacunary partial sums and Lebesgue points are investigated by Belinsky et al. [10–13].

In a general method of summation, the so-called θ-summation method, which is generated by a single function θ and which includes the well-known Fejér, Riesz, Weierstrass, and Abel summability methods, is studied intensively in the literature (see, e.g., Butzer and Nessel [14], Trigub and Belinsky [15–17], Liflyand [18], and Weisz [19, 20]). In this paper we generalize some of the above-mentioned results for strong θ-summability of Fourier transforms and for Wiener amalgam spaces. We characterize the set of functions for which strong summability holds at each Lebesgue point. More exactly, we will show that

$$\lim_{T\to\infty}\frac{-1}{T}\int_0^\infty \theta'\left(\frac{t}{T}\right)\left|s_t f(x) - f(x)\right|^r dt = 0 \tag{3}$$

at each Lebesgue point x of $f \in W(L_1, \ell_q)(\mathbb{R}) \supset L_q(\mathbb{R})$ $(1 \le q < \infty)$ when f is locally bounded at x, where $r > 0$. Moreover, the convergence holds at each Lebesgue point of f if $f \in W(L_p, \ell_q)(\mathbb{R}) \supset L_p(\mathbb{R})$ $(1 < p < \infty,\ 1 \le q < \infty)$. Here $W(L_p, \ell_q)(\mathbb{R})$ denotes the Wiener amalgam spaces. Gabisoniya's result was generalized in [21]. The analogous results are given for Fourier series, too.

2. Wiener Amalgam Spaces and Lebesgue Points

Let us fix $d \ge 1$, $d \in \mathbb{N}$. For a set $\mathbb{Y} \ne \emptyset$ let $\mathbb{Y}^d$ be its Cartesian product $\mathbb{Y} \times \cdots \times \mathbb{Y}$ taken with itself d-times. We

briefly write $L_p(\mathbb{R}^d)$ instead of the $L_p(\mathbb{R}^d, \lambda)$ space equipped with the norm

$$\|f\|_p := \left(\int_{\mathbb{R}^d} |f(x)|^p \, d\lambda(x)\right)^{1/p} \quad (1 \le p < \infty), \tag{4}$$

with the usual modification for $p = \infty$, where λ is the Lebesgue measure. $L_p^{\mathrm{loc}}(\mathbb{R}^d)$ $(1 \le p < \infty)$ denotes the space of measurable functions f for which $|f|^p$ is locally integrable. We say that f is locally bounded at x if there exists a neighborhood of x such that f is bounded on this neighborhood.

Now we generalize the L_p spaces. A measurable function f belongs to the *Wiener amalgam space* $W(L_p, \ell_q)(\mathbb{R}^d)$ $(1 \le p, q \le \infty)$ if

$$\|f\|_{W(L_p,\ell_q)} := \left(\sum_{k\in\mathbb{Z}^d} \|f(\cdot + k)\|^q_{L_p[0,1)^d}\right)^{1/q} < \infty, \tag{5}$$

with the obvious modification for $q = \infty$. It is easy to see that $W(L_p, \ell_p)(\mathbb{R}^d) = L_p(\mathbb{R}^d)$ and the following continuous embeddings hold true:

$$\begin{aligned} &W\left(L_{p_1}, \ell_q\right)\left(\mathbb{R}^d\right) \supset W\left(L_{p_2}, \ell_q\right)\left(\mathbb{R}^d\right) \quad (p_1 \le p_2), \\ &W\left(L_p, \ell_{q_1}\right)\left(\mathbb{R}^d\right) \subset W\left(L_p, \ell_{q_2}\right)\left(\mathbb{R}^d\right) \quad (q_1 \le q_2), \end{aligned} \tag{6}$$

$(1 \le p_1, p_2, q_1, q_2 \le \infty)$. Thus

$$W\left(L_\infty, \ell_1\right)\left(\mathbb{R}^d\right) \subset L_p\left(\mathbb{R}^d\right) \subset W\left(L_1, \ell_\infty\right)\left(\mathbb{R}^d\right) \quad (1 \le p \le \infty). \tag{7}$$

A point $x \in \mathbb{R}^d$ is called a *p-Lebesgue point* (or a Lebesgue point of order p) of $f \in L_p^{\mathrm{loc}}(\mathbb{R}^d)$ if

$$\lim_{h\to 0}\left(\frac{1}{(2h)^d}\int_{-h}^{h}\cdots\int_{-h}^{h}|f(x-s) - f(x)|^p \, ds\right)^{1/p} = 0. \tag{8}$$

It was proved by Feichtinger and Weisz [22, 23] that almost every point $x \in \mathbb{R}^d$ is a p-Lebesgue point of $f \in W(L_p, \ell_\infty)(\mathbb{R}^d)$ $(1 \le p < \infty)$. In context of Lebesgue points of L_p-functions we call also for the earlier papers of Belinsky et al. [12, 13].

In this paper the constants C and C_p may vary from line to line and the constants C_p are depending only on p.

3. The Kernel Functions

The *Fourier transform* of $f \in L_1(\mathbb{R})$ is given by

$$\widehat{f}(x) = \frac{1}{(2\pi)^{1/2}}\int_{\mathbb{R}} f(u)\, e^{-\imath xu} du \quad (x \in \mathbb{R}), \tag{9}$$

where $\imath = \sqrt{-1}$. Suppose first that $f \in L_p(\mathbb{R})$ for some $1 \le p \le 2$. The Fourier inversion formula

$$f(x) = \frac{1}{(2\pi)^{1/2}}\int_{\mathbb{R}} \widehat{f}(u)\, e^{\imath xu} du \quad \left(x \in \mathbb{R},\ \widehat{f} \in L_1(\mathbb{R})\right) \tag{10}$$

motivates the definition of the Dirichlet integral $s_t f$ $(t > 0)$ introduced by

$$\begin{aligned} s_t f(x) &:= \frac{1}{(2\pi)^{1/2}}\int_{-t}^{t} \widehat{f}(u)\, e^{\imath xu} du \\ &= \frac{1}{2\pi}\int_{\mathbb{R}} f(x-u)\, D_t(u)\, du, \end{aligned} \tag{11}$$

where the *Dirichlet kernel* is defined by

$$D_t(x) := \int_{-t}^{t} e^{\imath xu} du = 2\frac{\sin(tx)}{x}. \tag{12}$$

Obviously, $|D_t| \le Ct$.

It is easy to see that, with the help of the integral in (11), the definition of $s_t f$ can be extended to all $f \in W(L_1, \ell_q)(\mathbb{R})$ with $1 \le q < \infty$. Note that $W(L_1, \ell_p)(\mathbb{R}) \supset L_p(\mathbb{R})$, where $1 \le p < \infty$. It is known (see, e.g., Grafakos [24] or [20]) that, for $f \in L_p(\mathbb{R})$, $1 < p < \infty$,

$$\lim_{T\to\infty} s_T f = f \quad \text{in the } L_p(R)\text{-norm and a.e.} \tag{13}$$

This convergence does not hold for $p = 1$. However, using a summability method, we can generalize these results. We may take a general summability method, the so-called θ-summation defined by a function $\theta : \mathbb{R}_+ \to \mathbb{R}$. This summation contains all well-known summability methods, such as the Marcinkiewicz-Fejér, Riesz, Weierstrass, Abel, Picard, and Bessel summations.

Suppose that θ is continuous on $\mathbb{R}_+$; the support of θ is $[0, c]$ for some $0 < c \le \infty$ and θ is differentiable on $(0, c)$. Suppose further that

$$\begin{aligned} &\theta(0) = 1, \\ &\int_0^\infty (t \vee 1)^d \left|\theta'(t)\right| dt < \infty, \\ &\lim_{t\to\infty} t^d \theta(t) = 0, \end{aligned} \tag{14}$$

where $d \in \mathbb{N}$, $\vee$ denotes the maximum, and $\wedge$ denotes the minimum.

For $T > 0$ the *θ-means* of a function $f \in L_p(\mathbb{R})$ $(1 \le p \le 2)$ are defined by

$$\sigma_T^\theta f(x) := \frac{1}{(2\pi)^{1/2}}\int_{\mathbb{R}} \theta\left(\frac{|u|}{T}\right) \widehat{f}(u)\, e^{\imath xu} du. \tag{15}$$

It is easy to see that

$$\sigma_T^\theta f(x) = \frac{1}{(2\pi)^{1/2}}\int_{\mathbb{R}} f(x-u)\, K_T^\theta(u)\, du. \tag{16}$$

Note that this formula is well defined for all $f \in W(L_1, \ell_\infty)(\mathbb{R})$. Here

$$\begin{aligned} K_T^\theta(x) &:= \frac{1}{(2\pi)^{1/2}} \int_{\mathbb{R}} \theta\left(\frac{|u|}{T}\right) e^{\imath xu} du \\ &= \frac{-1}{(2\pi)^{1/2} T} \int_{\mathbb{R}} \int_{|u|}^{\infty} \theta'\left(\frac{t}{T}\right) dt e^{\imath xu} du \\ &= \frac{-1}{(2\pi)^{1/2} T} \int_0^\infty \theta'\left(\frac{t}{T}\right) \int_{-t}^{t} e^{\imath xu} du\, dt \\ &= \frac{-1}{(2\pi)^{1/2} T} \int_0^\infty \theta'\left(\frac{t}{T}\right) D_t(x)\, dt \end{aligned} \tag{17}$$

denotes the *θ-kernel*. Thus

$$\sigma_T^\theta f(x) = \frac{-1}{T} \int_0^\infty \theta'\left(\frac{t}{T}\right) s_t f(x)\, dt \tag{18}$$

for all $f \in W(L_1, \ell_q)(\mathbb{R})$ with $1 \le q < \infty$. Note that for the Fejér means (i.e., for $\theta(t) = \max((1-|t|), 0)$) we get the usual definition

$$\sigma_T^\theta f(x) = \frac{1}{T} \int_0^T s_t f(x)\, dt. \tag{19}$$

In Feichtinger and Weisz [22, 23] we have proved that, under conditions (14) and (25) with $d = 1$,

$$\lim_{T\to\infty} \sigma_T^\theta f(x) = f(x) \tag{20}$$

for all Lebesgue points of $f \in W(L_1, \ell_\infty)(\mathbb{R})$. In this paper, we investigate the problem of the strong summability, that is, whether the convergence

$$\lim_{T\to\infty} \frac{-1}{T} \int_0^\infty \theta'\left(\frac{t}{T}\right) |s_t f(x) - f(x)|^r dt = 0 \tag{21}$$

holds for Lebesgue points and some $r > 0$. Usually θ is increasing; then we can take the absolute value of θ' in the integral.

To this end we have to introduce some d-dimensional definitions. In the d-dimensional case we define the *Dirichlet kernel* by

$$D_t(x) := \prod_{i=1}^d D_t(x_i) = 2^d \prod_{i=1}^d \frac{\sin(tx_i)}{x_i} \quad (t > 0) \tag{22}$$

and the so called *Marcinkiewicz-θ-kernel* by

$$K_T^\theta(x) = \frac{-1}{(2\pi)^{d/2} T} \int_0^\infty \theta'\left(\frac{t}{T}\right) D_t(x)\, dt \quad (T > 0), \tag{23}$$

where $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$. In [21], we have seen that we may suppose that $x_1 > x_2 > \cdots > x_d > 0$ and $x_1 - \sum_{j=2}^d x_j > 0$ and we proved the next lemma. Denote by

$$\operatorname{soc} t := \begin{cases} \cos t, & \text{if } d \text{ is even;} \\ \sin t, & \text{if } d \text{ is odd.} \end{cases} \tag{24}$$

Lemma 1. *Let*

$$\left| \int_0^\infty \theta'(t)\, t^i (soc)^{(i)}(tu)\, dt \right| \le Cu^{-\alpha} \quad (i = 0, \ldots, d-1) \tag{25}$$

for some $0 < \alpha < \infty$. *Then*

$$\left|K_T^\theta(x)\right| \le CT^{d-j} x_1^{-1} \cdots x_j^{-1} \quad (j = 0, \ldots, d). \tag{26}$$

If in addition $x_1 - \sum_{j=2}^d x_j > 1/T$ *and* $x_{j+1} < 1/T$, *where* $x_{d+1} = 0$, *then*

$$\left|K_T^\theta(x)\right| \le CT^{-\alpha+d-j} x_1^{-1} \cdots x_j^{-1} \left(x_1 - \sum_{j=2}^d x_j\right)^{-\alpha} \quad (j = 1, \ldots, d). \tag{27}$$

The next lemma is due to the author [25].

Lemma 2. *If (14) and (25) are satisfied for some* $d \in \mathbb{N}$ *and* $0 < \alpha < \infty$, *then* $\int_{\mathbb{R}^d} |K_T^\theta| d\lambda \le C$ $(T \in \mathbb{R}_+)$.

4. Strong Summability of Fourier Transforms

In this section we characterize a wide set of functions for which strong summability holds at each Lebesgue point. For the convergence of $f \in W(L_p, \ell_q)(\mathbb{R})$ $(1 < p < \infty,\ 1 \le q < \infty)$ at p-Lebesgue points we proved the following result in [21]. Note that $W(L_p, \ell_p)(\mathbb{R}) = L_p(\mathbb{R})$.

Theorem 3. *Suppose that (14) and (25) hold for some* $d \in \mathbb{N}$ *and* $0 < \alpha < \infty$. *Let* $f \in W(L_p, \ell_q)(\mathbb{R})$ *for some* $1 < p < \infty$ *and* $1 \le q < \infty$. *If* x_j *is a p-Lebesgue point of f for all* $j = 1, \ldots, d$, *then*

$$\lim_{T\to\infty} \frac{-1}{T} \int_0^\infty \theta'\left(\frac{t}{T}\right) \prod_{j=1}^d \left(s_t f(x_j) - f(x_j)\right) dt = 0. \tag{28}$$

If all x_j $(j = 1, \ldots, d)$ are equal, then we obtain the following.

Corollary 4. *Suppose that (14) and (25) hold for some even* $d \in \mathbb{N}$ *and* $0 < \alpha < \infty$. *Let* $f \in W(L_p, \ell_q)(\mathbb{R})$ *for some* $1 < p < \infty$ *and* $1 \le q < \infty$. *If* $x \in \mathbb{R}$ *is a p-Lebesgue point of f, then*

$$\lim_{T\to\infty} \frac{-1}{T} \int_0^\infty \theta'\left(\frac{t}{T}\right) |s_t f(x) - f(x)|^d dt = 0. \tag{29}$$

Obviously, the convergence holds almost everywhere. Corollary 4 does not hold for $p = 1$ (see Hardy and Littlewood [5]). However, we [21] extended it for $p = 1$, but for much more specialized points than the Lebesgue points, for the so-called Gabisoniya points, which were introduced in [9]. In the next theorem we generalize Theorem 3 and Corollary 4 for $p = 1$ and for a subspace of $W(L_1, \ell_q)(\mathbb{R})$.

Theorem 5. *Suppose that (14) and (25) hold for some $d \in \mathbb{N}$ and $0 < \alpha < \infty$. Let $f \in W(L_1, \ell_q)(\mathbb{R})$ for some $1 \le q < \infty$. If x_j is a Lebesgue point of f and f is locally bounded at x_j for all $j = 1, \ldots, d$, then*

$$\lim_{T\to\infty} \frac{-1}{T} \int_0^\infty \theta'\left(\frac{t}{T}\right) \prod_{j=1}^{d} \left(s_t f\left(x_j\right) - f\left(x_j\right)\right) dt = 0. \quad (30)$$

Proof. It is easy to see that

$$\begin{aligned} &\frac{-1}{T} \int_0^\infty \theta'\left(\frac{t}{T}\right) \prod_{j=1}^{d} \left(s_t f\left(x_j\right) - f\left(x_j\right)\right) dt = \frac{-1}{T} \\ &\cdot \int_0^\infty \theta'\left(\frac{t}{T}\right) \prod_{j=1}^{d} \left(\frac{1}{2\pi} \int_{\mathbb{R}} f\left(x_j - s_j\right) D_t\left(s_j\right) ds_j \right. \\ &\left. - f\left(x_j\right)\right) dt = \frac{-1}{T} \int_0^\infty \theta'\left(\frac{t}{T}\right) \prod_{j=1}^{d} \lim_{n\to\infty} \frac{1}{2\pi} \\ &\cdot \int_{-n}^{n} \left(f\left(x_j - s_j\right) - f\left(x_j\right)\right) D_t\left(s_j\right) ds_j\, dt. \end{aligned} \quad (31)$$

Since

$$\begin{aligned} &\left|\int_{-n}^{n} f\left(x_j - s_j\right) D_t\left(s_j\right) ds_j\right| \le Ct \|f\|_{W(L_1,\ell_q)}, \\ &\left|\int_{-n}^{n} f\left(x_j\right) D_t\left(s_j\right) ds_j\right| = \left|f\left(x_j\right)\right| \left|\int_{-n}^{n} \frac{\sin\left(ts_j\right)}{s_j} ds_j\right| \\ &= \left|f\left(x_j\right)\right| \left|\int_{-nt}^{nt} \frac{\sin u}{u} du\right| \le C\left|f\left(x_j\right)\right|, \end{aligned} \quad (32)$$

we obtain

$$\begin{aligned} &\frac{-1}{T} \int_0^\infty \theta'\left(\frac{t}{T}\right) \prod_{j=1}^{d} \left(s_t f\left(x_j\right) - f\left(x_j\right)\right) dt = \lim_{n\to\infty} \frac{-1}{T} \\ &\cdot \int_0^\infty \theta'\left(\frac{t}{T}\right) \\ &\cdot \frac{1}{(2\pi)^d} \int_{[-n,n]^d} \prod_{j=1}^{d} \left(f\left(x_j - s_j\right) - f\left(x_j\right)\right) \\ &\cdot D_t(s)\, ds\, dt = \lim_{n\to\infty} \frac{1}{(2\pi)^d} \\ &\cdot \int_{[-n,n]^d} \frac{-1}{T} \int_0^\infty \theta'\left(\frac{t}{T}\right) \prod_{j=1}^{d} \left(f\left(x_j - s_j\right) - f\left(x_j\right)\right) \\ &\cdot D_t(s)\, dt\, ds = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \prod_{j=1}^{d} \left(f\left(x_j - s_j\right)\right. \\ &\left. - f\left(x_j\right)\right) K_T^\theta(s)\, ds. \end{aligned} \quad (33)$$

For simplicity we will prove the rest of the theorem for $d = 3$, only. It can be proved for higher dimensions similarly. As we mentioned earlier, we may suppose that $s_1 > s_2 > s_3 > 0$ and $s_1 - s_2 - s_3 > 0$. Let us fix a small $r > 0$ and denote the square $[0, r/2]^3$ by $S_{r/2}$. We can prove in the same way as we did in Theorem 4 of [21] that

$$\int_{S_{r/2}} \prod_{j=1}^{3} \left|f\left(x_j - s_j\right) - f\left(x_j\right)\right| \left|K_T^\theta(s)\right| ds < \epsilon \quad (34)$$

if r is small enough and T is large enough. The estimation of this integral on the set $S_{r/2}^c$ of that proof does not work now, because there we used a modified maximal function which is not necessarily bounded in our case. So we have to show here that

$$\lim_{T\to\infty} \int_{S_{r/2}^c} \prod_{j=1}^{3} \left|f\left(x_j - s_j\right) - f\left(x_j\right)\right| \left|K_T^\theta(s)\right| ds = 0. \quad (35)$$

To this end let us introduce the sets

$$\begin{aligned} A_1 &:= \left\{s : s_1 > \frac{r}{2},\ 0 < s_3 < s_2 < \frac{1}{T}\right\}, \\ A_2 &:= \left\{s : s_1 > \frac{r}{2},\ 0 < s_3 < \frac{1}{T} < s_2 < \delta\right\}, \\ A_3 &:= \left\{s : s_1 > \frac{r}{2},\ \frac{1}{T} < s_3 < s_2 < \delta\right\}, \\ A_4 &:= \left\{s : s_1 > \frac{r}{2},\ 0 < s_3 < \frac{1}{T} < \delta < s_2\right\}, \\ A_5 &:= \left\{s : s_1 > \frac{r}{2},\ \frac{1}{T} < s_3 < \delta < s_2\right\}, \\ A_6 &:= \left\{s : s_1 > \frac{r}{2},\ \frac{1}{T} < \delta < s_3 < s_2\right\}, \\ B_1 &:= \left\{s : 0 < s_1 - s_2 - s_3 < \frac{1}{T}\right\}, \\ B_2 &:= \left\{s : \frac{1}{T} < s_1 - s_2 - s_3 < \delta\right\}, \\ B_3 &:= \left\{s : \delta < s_1 - s_2 - s_3 < \frac{(s_1 - s_2)}{2}\right\}, \\ B_4 &:= \left\{s : \frac{(s_1 - s_2)}{2} \vee \delta < s_1 - s_2 - s_3 < s_1 - s_2 - \delta\right\}, \\ B_5 &:= \left\{s : \left(s_1 - s_2 - \delta\right) \vee \delta < s_1 - s_2 - s_3 < s_1 - s_2\right\}, \\ C_1 &:= \left\{s : 0 < s_1 - s_2 < \delta\right\}, \\ C_2 &:= \left\{s : \delta < s_1 - s_2 < \frac{s_1}{2}\right\}, \\ C_3 &:= \left\{s : \frac{s_1}{2} < s_1 - s_2 < s_1 - \delta\right\}, \\ C_4 &:= \left\{s : s_1 - \delta < s_1 - s_2 < s_1\right\} \end{aligned} \quad (36)$$

for a given small $\delta > 0$ and large T. Then we have to estimate the integral in (35) for $A_i, i = 1, \ldots, 6$. First of all observe that $A_i \cap B_j = \emptyset$ for $i = 1, 2, 3$ and $j = 1, 2$. On the set A_1 we have $s_2, s_3 < 1/T$ and so $s_1 - s_2 - s_3 > s_1/2$. Hence

$$\left|K_T^\theta(s)\right| \le CT^{2-\alpha} s_1^{-1} (s_1 - s_2 - s_3)^{-\alpha} \le CT^{2-\alpha} s_1^{-1-\alpha}. \quad (37)$$

Since f is locally bounded at x_j, we get by (37) that

$$\begin{aligned}
&\int_{A_1 \cap (B_3 \cup B_4 \cup B_5)} \prod_{j=1}^{3} \left|f(x_j - s_j) - f(x_j)\right| \left|K_T^\theta(s)\right| ds \\
&\quad \le CT^{2-\alpha} \int_{r/2}^{\infty} \int_0^{1/T} \int_0^{1/T} s_1^{-1-\alpha} \prod_{j=1}^{3} \left|f(x_j - s_j)\right| ds \\
&\quad \le CT^{-\alpha} \sum_{i=0}^{\infty} (i \vee 1)^{-1-\alpha} \int_i^{i+1} \left|f(x_1 - s_1)\right| ds_1 \\
&\quad \le CT^{-\alpha} \|f\|_{W(L_1, \ell_\infty)} \longrightarrow 0,
\end{aligned} \quad (38)$$

as $T \to \infty$. On A_2 we have

$$\begin{aligned}
\left|K_T^\theta(s)\right| &\le CT^{1-\alpha} s_1^{-1} s_2^{-1} (s_1 - s_2 - s_3)^{-\alpha} \\
&\le CT^{1-\alpha} s_1^{-1-\alpha} s_2^{-1}
\end{aligned} \quad (39)$$

and so

$$\begin{aligned}
&\int_{A_2 \cap (B_3 \cup B_4 \cup B_5)} \prod_{j=1}^{3} \left|f(x_j - s_j) - f(x_j)\right| \left|K_T^\theta(s)\right| ds \\
&\quad \le CT^{1-\alpha} \int_{r/2}^{\infty} \int_{1/T}^{\delta} \int_0^{1/T} s_1^{-1-\alpha} s_2^{-1} \prod_{j=1}^{3} \left|f(x_j - s_j)\right| ds \\
&\quad \le CT^{-\alpha} \ln T \sum_{i=0}^{\infty} (i \vee 1)^{-1-\alpha} \int_i^{i+1} \left|f(x_1 - s_1)\right| ds_1 \\
&\quad \le CT^{-\alpha} \ln T \|f\|_{W(L_1, \ell_\infty)} \longrightarrow 0.
\end{aligned} \quad (40)$$

Similarly,

$$\begin{aligned}
&\int_{A_3 \cap (B_3 \cup B_4 \cup B_5)} \prod_{j=1}^{3} \left|f(x_j - s_j) - f(x_j)\right| \left|K_T^\theta(s)\right| ds \\
&\quad \le CT^{-\alpha} \int_{r/2}^{\infty} \int_{1/T}^{\delta} \int_{1/T}^{\delta} s_1^{-1-\alpha} s_2^{-1} s_3^{-1} \prod_{j=1}^{3} \left|f(x_j - s_j)\right| ds \\
&\quad \le CT^{-\alpha} \ln^2 T \|f\|_{W(L_1, \ell_\infty)} \longrightarrow 0.
\end{aligned} \quad (41)$$

On the set $A_4 \cap B_1$ we have $s_1 - s_2 < 2/T < \delta$ if T is large enough and so $s_2 > s_1/2$. Then

$$\left|K_T^\theta(s)\right| \le CTs_1^{-1} s_2^{-1} \le CTs_1^{-2}, \quad (42)$$

$$\begin{aligned}
&\int_{A_4 \cap B_1} \prod_{j=1}^{3} \left|f(x_j - s_j) - f(x_j)\right| \left|K_T^\theta(s)\right| ds \\
&\quad \le CT \int_{r/2}^{\infty} \int_{s_1 - \delta}^{s_1} \int_0^{1/T} s_1^{-2} \prod_{j=1}^{3} \left|f(x_j - s_j)\right| ds \\
&\quad \le C \sum_{i=0}^{N_0 - 1} (i \vee 1)^{-2} \\
&\quad \cdot \int_i^{i+1} \int_{s_1 - \delta}^{s_1} \left|f(x_1 - s_1)\right| \left|f(x_2 - s_2)\right| ds_1\, ds_2 \\
&\quad + C \sum_{i=N_0}^{\infty} i^{-2} \\
&\quad \cdot \int_i^{i+1} \int_{s_1 - \delta}^{s_1} \left|f(x_1 - s_1)\right| \left|f(x_2 - s_2)\right| ds_1\, ds_2 \\
&\quad \le C \left\|f \times f 1_{\{(s_1, s_2): 0 < s_1 < N_0, s_1 - \delta < s_2 \le s_1\}}\right\|_{W(L_1, \ell_\infty)} \\
&\quad + CN_0^{-1} \|f \times f\|_{W(L_1, \ell_\infty)}.
\end{aligned} \quad (43)$$

The second term is less than ϵ if N_0 is large enough and the first term is less than ϵ if δ is small enough. The set $A_4 \cap B_2$ can be handled in the same way.

On $A_4 \cap (B_3 \cup B_4 \cup B_5)$ we have $s_1 - s_2 > \delta$. Thus $(B_3 \cup B_4 \cup B_5) \cap C_1 = \emptyset$ and $s_1 - s_2 - s_3 > s_1 - s_2 - 1/T > (s_1 - s_2)/2$. Then

$$\begin{aligned}
\left|K_T^\theta(s)\right| &\le CT^{1-\alpha} s_1^{-1} s_2^{-1} (s_1 - s_2 - s_3)^{-\alpha} \\
&\le CT^{1-\alpha} s_1^{-1} s_2^{-1} (s_1 - s_2)^{-\alpha}.
\end{aligned} \quad (44)$$

On C_2 we have

$$\begin{aligned}
\left|K_T^\theta(s)\right| &\le CT^{1-\alpha} s_1^{-2} (s_1 - s_2)^{-\alpha} \\
&\le CT^{1-\alpha} s_1^{-1-\eta} (s_1 - s_2)^{-1-\alpha+\eta},
\end{aligned} \quad (45)$$

where η is chosen such that $0 < \eta < 1 \wedge \alpha$. Hence

$$\begin{aligned}
&\int_{A_4 \cap (B_3 \cup B_4 \cup B_5) \cap C_2} \prod_{j=1}^{3} \left|f(x_j - s_j) - f(x_j)\right| \left|K_T^\theta(s)\right| ds \\
&\quad \le CT^{1-\alpha} \int_{r/2}^{\infty} \int_{s_1/2}^{s_1 - \delta} \int_0^{1/T} s_1^{-1-\eta} (s_1 - s_2)^{-1-\alpha+\eta} \\
&\quad \cdot \prod_{j=1}^{3} \left|f(x_j - s_j)\right| ds \le CT^{-\alpha} \sum_{i=0}^{\infty} (i \vee 1)^{-1-\eta} \\
&\quad \cdot \sum_{0 \le j < (i+1)/2} (j \vee 1)^{-1-\alpha+\eta} \cdot \int_i^{i+1} \int_{s_1 - j - 1}^{s_1 - j} \left|f(x_1 - s_1)\right| \\
&\quad \cdot \left|f(x_2 - s_2)\right| ds_1\, ds_2 \le CT^{-\alpha} \|f\|_{W(L_1, \ell_\infty)}^2 \\
&\quad \longrightarrow 0.
\end{aligned} \quad (46)$$

On C_3, (44) implies that

$$\begin{aligned}
&\left|K_T^\theta(s)\right| \le CT^{1-\alpha}s_1^{-1-\alpha}s_2^{-1} \le CT^{1-\alpha}s_1^{-1-\alpha+\eta}s_2^{-1-\eta},\\
&\int_{A_4\cap(B_3\cup B_4\cup B_5)\cap C_3}\prod_{j=1}^{3}\left|f\left(x_j-s_j\right)-f\left(x_j\right)\right|\left|K_T^\theta(s)\right|ds\\
&\quad\le CT^{1-\alpha}\int_{r/2}^{\infty}\int_{\delta}^{s_1/2}\int_0^{1/T}s_1^{-1-\alpha+\eta}s_2^{-1-\eta}\prod_{j=1}^{3}\left|f\left(x_j-s_j\right)\right|ds\\
&\quad\le CT^{-\alpha}\sum_{i=0}^{\infty}(i\vee 1)^{-1-\alpha+\eta}\sum_{0\le j<(i+1)/2}(j\vee 1)^{-1-\eta}\\
&\quad\cdot\int_i^{i+1}\int_j^{j+1}\left|f(x_1-s_1)\right|\left|f(x_2-s_2)\right|ds_1\,ds_2\\
&\quad\le CT^{-\alpha}\left\|f\right\|_{W(L_1,\ell_\infty)}^2\longrightarrow 0.
\end{aligned}\tag{47}$$

Observe that $s_2>\delta$ contradicts C_4.

Consider the set $A_5\cap B_1$. If $s_1-s_2>2/T$, then $s_3>s_1-s_2-1/T>(s_1-s_2)/2$ and if $s_1-s_2<2/T$, then $s_3>1/T>(s_1-s_2)/2$. Observe that $s_1-s_2<1/T+\delta<2\delta$. Hence

$$\left|K_T^\theta(s)\right|\le Cs_1^{-1}s_2^{-1}s_3^{-1}\le Cs_1^{-2}\left(s_1-s_2\right)^{-1/2}s_3^{-1/2}.\tag{48}$$

Since $s_3<\delta$, we can integrate in s_3 to obtain

$$\begin{aligned}
&\int_{A_5\cap B_1}\prod_{j=1}^{3}\left|f\left(x_j-s_j\right)-f\left(x_j\right)\right|\left|K_T^\theta(s)\right|ds\\
&\quad\le C\int_{r/2}^{\infty}\int_{s_1-2\delta}^{s_1}\int_{s_1-s_2-1/T}^{s_1-s_2}s_1^{-2}\left(s_1-s_2\right)^{-1/2}\\
&\quad\cdot s_3^{-1/2}\prod_{j=1}^{3}\left|f\left(x_j-s_j\right)\right|ds\\
&\quad\le C\int_{r/2}^{\infty}\int_{s_1-2\delta}^{s_1}s_1^{-2}\left|f(x_1-s_1)\right|\left|f(x_2-s_2)\right|ds_1\,ds_2\\
&\quad\le C\left\|f\times f1_{\{(s_1,s_2):0<s_1<N_0,s_1-2\delta<s_2\le s_1\}}\right\|_{W(L_1,\ell_\infty)}\\
&\quad+CN_0^{-1}\left\|f\times f\right\|_{W(L_1,\ell_\infty)}
\end{aligned}\tag{49}$$

as in (43). Similarly, $s_1-s_2<2\delta$ holds as well on $A_5\cap B_2$. If $s_1-s_2-s_3>s_3$, then

$$\begin{aligned}
\left|K_T^\theta(s)\right|&\le CT^{-\alpha}s_1^{-1}s_2^{-1}s_3^{-1}\left(s_1-s_2-s_3\right)^{-\alpha}\\
&\le CT^{-\alpha}s_1^{-2}s_3^{-1-\alpha}
\end{aligned}\tag{50}$$

and so

$$\begin{aligned}
&\int_{A_5\cap B_2}\prod_{j=1}^{3}\left|f\left(x_j-s_j\right)-f\left(x_j\right)\right|\left|K_T^\theta(s)\right|ds\\
&\quad\le CT^{-\alpha}\int_{r/2}^{\infty}\int_{s_1-2\delta}^{s_1}\int_{1/T}^{\delta}s_1^{-2}s_3^{-1-\alpha}\prod_{j=1}^{3}\left|f\left(x_j-s_j\right)\right|ds\\
&\quad\le C\int_{r/2}^{\infty}\int_{s_1-2\delta}^{s_1}s_1^{-2}\left|f(x_1-s_1)\right|\left|f(x_2-s_2)\right|ds_1\,ds_2\\
&\quad<\epsilon
\end{aligned}\tag{51}$$

as before. If $s_3>s_1-s_2-s_3$, then

$$\begin{aligned}
&\left|K_T^\theta(s)\right|\le CT^{-\alpha}s_1^{-2}\left(s_1-s_2-s_3\right)^{-1-\alpha},\\
&\int_{A_5\cap B_2}\prod_{j=1}^{3}\left|f\left(x_j-s_j\right)-f\left(x_j\right)\right|\left|K_T^\theta(s)\right|ds\\
&\quad\le CT^{-\alpha}\int_{r/2}^{\infty}\int_{s_1-2\delta}^{s_1}\int_{s_1-s_2-\delta}^{s_1-s_2-1/T}s_1^{-2}\left(s_1-s_2-s_3\right)^{-1-\alpha}\\
&\quad\cdot\prod_{j=1}^{3}\left|f\left(x_j-s_j\right)\right|ds\\
&\quad\le C\int_{r/2}^{\infty}\int_{s_1-2\delta}^{s_1}s_1^{-2}\left|f(x_1-s_1)\right|\left|f(x_2-s_2)\right|ds_1\,ds_2\\
&\quad<\epsilon.
\end{aligned}\tag{52}$$

If $s_1-s_2<2\delta$, then the integral on the set $A_5\cap(B_3\cup B_4\cup B_5)$ can be estimated in the same way. If $s_1-s_2>2\delta$, then $A_5\cap B_3=\emptyset$ because $s_3>(s_1-s_2)/2$ and $s_3<\delta$ on this set. Writing $1/T$ instead of δ in the sets B_i $(i=4,5)$, we obtain the definition of the sets B_i'. On B_4' we have by (50) that

$$\begin{aligned}
&\left|K_T^\theta(s)\right|\le CT^{-\alpha}s_1^{-1}s_2^{-1}s_3^{-1}\left(s_1-s_2\right)^{-\alpha},\\
&\int_{A_5\cap B_4'\cap C_2}\prod_{j=1}^{3}\left|f\left(x_j-s_j\right)-f\left(x_j\right)\right|\left|K_T^\theta(s)\right|ds\\
&\quad\le CT^{-\alpha}\int_{r/2}^{\infty}\int_{s_1/2}^{s_1-\delta}\int_{1/T}^{\delta}s_1^{-2}s_3^{-1}\left(s_1-s_2\right)^{-\alpha}\\
&\quad\cdot\prod_{j=1}^{3}\left|f\left(x_j-s_j\right)\right|ds\le CT^{-\alpha}\ln T\sum_{i=0}^{\infty}(i\vee 1)^{-1-\eta}\\
&\quad\cdot\sum_{0\le j<(i+1)/2}(j\vee 1)^{-1-\alpha+\eta}\int_i^{i+1}\int_{s_1-j-1}^{s_1-j}\left|f(x_1-s_1)\right|\\
&\quad\cdot\left|f(x_2-s_2)\right|ds_1\,ds_2\le CT^{-\alpha}\ln T\left\|f\right\|_{W(L_1,\ell_\infty)}^2\\
&\quad\longrightarrow 0,
\end{aligned}\tag{53}$$

$$\begin{aligned}
&\int_{A_5\cap B_4'\cap C_3}\prod_{j=1}^{3}\left|f\left(x_j-s_j\right)-f\left(x_j\right)\right|\left|K_T^\theta(s)\right|ds\\
&\quad\le CT^{-\alpha}\int_{r/2}^{\infty}\int_{\delta}^{s_1/2}\int_{1/T}^{\delta}s_1^{-1-\alpha}s_2^{-1}s_3^{-1}\prod_{j=1}^{3}\left|f\left(x_j-s_j\right)\right|ds\\
&\quad\le CT^{-\alpha}\ln T\sum_{i=0}^{\infty}(i\vee 1)^{-1-\alpha+\eta}\sum_{0\le j<(i+1)/2}(j\vee 1)^{-1-\eta}\\
&\quad\cdot\int_i^{i+1}\int_j^{j+1}\left|f(x_1-s_1)\right|\left|f(x_2-s_2)\right|ds_1\,ds_2\le CT^{-\alpha}\\
&\quad\cdot\ln T\left\|f\right\|_{W(L_1,\ell_\infty)}^2\longrightarrow 0.
\end{aligned}$$

Moreover, B_5' contradicts A_5, more exactly, $s_3>1/T$.

On A_6 $s_3>\delta$ and $s_1-s_2-s_3>0$, thus $s_1-s_2>\delta$. Similar to (48),

$$\left|K_T^\theta(s)\right|\le Cs_1^{-1}s_2^{-1}s_3^{-1}\le Cs_1^{-2}\left(s_1-s_2\right)^{-1}\tag{54}$$

on $A_6 \cap B_1 \cap C_2$. Then

$$\begin{aligned}
&\int_{A_6\cap B_1\cap C_2} \prod_{j=1}^{3} \left|f\left(x_j - s_j\right) - f\left(x_j\right)\right| \left|K_T^{\theta}(s)\right| ds \\
&\quad \le C \int_{r/2}^{\infty} \int_{s_1/2}^{s_1-\delta} \int_{s_1-s_2-1/T}^{s_1-s_2} s_1^{-2} \left(s_1 - s_2\right)^{-1} \\
&\quad \cdot \prod_{j=1}^{3} \left|f\left(x_j - s_j\right)\right| ds \le C \sum_{i=0}^{\infty} (i \vee 1)^{-3/2} \\
&\quad \cdot \sum_{0\le j<(i+1)/2} (j \vee 1)^{-3/2} \\
&\quad \cdot \int_{i}^{i+1} \int_{s_1-j-1}^{s_1-j} \int_{s_1-s_2-1/T}^{s_1-s_2} \left|f\left(x_1 - s_1\right)\right| \left|f\left(x_2 - s_2\right)\right| \\
&\quad \cdot \left|f\left(x_3 - s_3\right)\right| ds_1\, ds_2\, ds_3 \le C \left\| f \times f \right. \\
&\quad \times f 1_{\{s:0<s_1<N_0, s_1/2<s_2\le s_1-\delta, s_1-s_2-\delta<s_3<s_1-s_2\}} \Big\|_{W(L_1,\ell_\infty)} \\
&\quad + C N_0^{-1/2} \left\| f \times f \times f \right\|_{W(L_1,\ell_\infty)},
\end{aligned} \tag{55}$$

which is small enough if N_0 is large and δ is small enough. On $A_6 \cap B_1 \cap C_3$ we use the estimation

$$\left|K_T^{\theta}(s)\right| \le C s_1^{-1} s_2^{-1} s_3^{-1} \le C s_1^{-2} s_2^{-1} \tag{56}$$

to obtain

$$\begin{aligned}
&\int_{A_6\cap B_1\cap C_3} \prod_{j=1}^{3} \left|f\left(x_j - s_j\right) - f\left(x_j\right)\right| \left|K_T^{\theta}(s)\right| ds \\
&\quad \le C \int_{r/2}^{\infty} \int_{\delta}^{s_1/2} \int_{s_1-s_2-1/T}^{s_1-s_2} s_1^{-2} s_2^{-1} \prod_{j=1}^{3} \left|f\left(x_j - s_j\right)\right| ds \\
&\quad \le C \left\| f \times f \right. \\
&\quad \times f 1_{\{s:0<s_1<N_0, \delta<s_2\le s_1/2, s_1-s_2-\delta<s_3<s_1-s_2\}} \Big\|_{W(L_1,\ell_\infty)} \\
&\quad + C N_0^{-1/2} \left\| f \times f \times f \right\|_{W(L_1,\ell_\infty)} < \epsilon,
\end{aligned} \tag{57}$$

as just before.

On $A_6 \cap B_2$ $s_1 - s_2 > \delta$ again. If $s_1 - s_2 > 2\delta$, then $s_3 > s_1 - s_2 - \delta > (s_1 - s_2)/2$ and if $s_1 - s_2 < 2\delta$, then $s_3 > \delta > (s_1 - s_2)/2$. This case can be handled in the same way as the set $A_6 \cap B_1$.

On $A_6 \cap B_3$ we get

$$\begin{aligned}
\left|K_T^{\theta}(s)\right| &\le C T^{-\alpha} s_1^{-1} s_2^{-1} s_3^{-1} \left(s_1 - s_2 - s_3\right)^{-\alpha} \\
&\le C T^{-\alpha} s_1^{-1} s_2^{-1} \left(s_1 - s_2\right)^{-1} \left(s_1 - s_2 - s_3\right)^{-\alpha},
\end{aligned} \tag{58}$$

which implies that

$$\begin{aligned}
&\int_{A_6\cap B_3\cap C_2} \prod_{j=1}^{3} \left|f\left(x_j - s_j\right) - f\left(x_j\right)\right| \left|K_T^{\theta}(s)\right| ds \\
&\quad \le C T^{-\alpha} \int_{r/2}^{\infty} \int_{s_1/2}^{s_1-\delta} \int_{(s_1-s_2)/2}^{s_1-s_2-\delta} s_1^{-2} \left(s_1 - s_2\right)^{-1} \\
&\quad \cdot \left(s_1 - s_2 - s_3\right)^{-\alpha} \prod_{j=1}^{3} \left|f\left(x_j - s_j\right)\right| ds \\
&\quad \le C T^{-\alpha} \sum_{i=0}^{\infty} (i \vee 1)^{-1-\mu} \sum_{0\le j<(i+1)/2} (j \vee 1)^{-1-\mu} \\
&\quad \cdot \sum_{0\le k<(j+1)/2} (k \vee 1)^{-1-\alpha+2\mu} \\
&\quad \cdot \int_{i}^{i+1} \int_{s_1-j-1}^{s_1-j} \int_{s_1-s_2-k-1}^{s_1-s_2-k} \left|f\left(x_1 - s_1\right)\right| \left|f\left(x_2 - s_2\right)\right| \\
&\quad \cdot \left|f\left(x_3 - s_3\right)\right| ds_1\, ds_2\, ds_3 \\
&\quad \le C T^{-\alpha} \left\| f \right\|_{W(L_1,\ell_\infty)}^{3} \longrightarrow 0,
\end{aligned} \tag{59}$$

where $0 < \mu < 1/2 \wedge \alpha/2$. Similarly,

$$\begin{aligned}
&\int_{A_6\cap B_3\cap C_3} \prod_{j=1}^{3} \left|f\left(x_j - s_j\right) - f\left(x_j\right)\right| \left|K_T^{\theta}(s)\right| ds \\
&\quad \le C T^{-\alpha} \int_{r/2}^{\infty} \int_{\delta}^{s_1/2} \int_{(s_1-s_2)/2}^{s_1-s_2-\delta} s_1^{-2} s_2^{-1} \left(s_1 - s_2 - s_3\right)^{-\alpha} \\
&\quad \cdot \prod_{j=1}^{3} \left|f\left(x_j - s_j\right)\right| ds \le C T^{-\alpha} \sum_{i=0}^{\infty} (i \vee 1)^{-1-\mu} \\
&\quad \cdot \sum_{0\le j<(i+1)/2} (j \vee 1)^{-1-\mu} \sum_{0\le k<(j+1)/2} (k \vee 1)^{-1-\alpha+2\mu} \\
&\quad \cdot \int_{i}^{i+1} \int_{j}^{j+1} \int_{s_1-s_2-k-1}^{s_1-s_2-k} \left|f\left(x_1 - s_1\right)\right| \left|f\left(x_2 - s_2\right)\right| \\
&\quad \cdot \left|f\left(x_3 - s_3\right)\right| ds_1\, ds_2\, ds_3 \\
&\quad \le C T^{-\alpha} \left\| f \right\|_{W(L_1,\ell_\infty)}^{3} \longrightarrow 0.
\end{aligned} \tag{60}$$

Moreover, on $A_6 \cap B_4$

$$\left|K_T^{\theta}(s)\right| \le C T^{-\alpha} s_1^{-1} s_2^{-1} s_3^{-1} \left(s_1 - s_2\right)^{-\alpha},$$

$$\begin{aligned}
&\int_{A_6\cap B_4\cap C_2} \prod_{j=1}^{3} \left|f\left(x_j - s_j\right) - f\left(x_j\right)\right| \left|K_T^{\theta}(s)\right| ds \\
&\quad \le C T^{-\alpha} \int_{r/2}^{\infty} \int_{s_1/2}^{s_1-\delta} \int_{\delta}^{(s_1-s_2)/2} s_1^{-2} \left(s_1 - s_2\right)^{-\alpha} \\
&\quad \cdot s_3^{-1} \prod_{j=1}^{3} \left|f\left(x_j - s_j\right)\right| ds \le C T^{-\alpha} \sum_{i=0}^{\infty} (i \vee 1)^{-1-\mu}
\end{aligned}$$

$$\cdot \sum_{0\le j<(i+1)/2} (j\vee 1)^{-1-\alpha+2\mu} \sum_{0\le k<(j+1)/2} (k\vee 1)^{-1-\mu}$$
$$\cdot \int_i^{i+1}\int_{s_1-j-1}^{s_1-j}\int_k^{k+1} |f(x_1-s_1)|\,|f(x_2-s_2)|$$
$$\cdot |f(x_3-s_3)|\,ds_1\,ds_2\,ds_3 \le CT^{-\alpha}\|f\|^3_{W(L_1,\ell_\infty)}$$
$$\longrightarrow 0,$$

$$\int_{A_6\cap B_4\cap C_3}\prod_{j=1}^{3}\left|f(x_j-s_j)-f(x_j)\right|\left|K_T^\theta(s)\right|ds$$
$$\le CT^{-\alpha}\int_{r/2}^{\infty}\int_\delta^{s_1/2}\int_\delta^{(s_1-s_2)/2} s_1^{-1-\alpha}s_2^{-1}s_3^{-1}\prod_{j=1}^{3}\left|f(x_j-s_j)\right|ds$$
$$\le CT^{-\alpha}\sum_{i=0}^{\infty}(i\vee 1)^{-1-\alpha/2}\sum_{0\le j<(i+1)/2}(j\vee 1)^{-1-\alpha/2}$$
$$\cdot \sum_{0\le k<(j+1)/2}(k\vee 1)^{-1-\alpha/2}\int_i^{i+1}\int_j^{j+1}\int_k^{k+1}|f(x_1-s_1)|$$
$$\cdot |f(x_2-s_2)|\,|f(x_3-s_3)|\,ds_1\,ds_2\,ds_3$$
$$\le CT^{-\alpha}\|f\|^3_{W(L_1,\ell_\infty)} \longrightarrow 0. \tag{61}$$

Since $A_6\cap B_5=\emptyset$, the proof of the theorem is complete. □

Corollary 6. *Suppose that (14) and (25) hold for some even $d\in\mathbb{N}$ and $0<\alpha<\infty$. Let $f\in W(L_1,\ell_q)(\mathbb{R})$ for some $1\le q<\infty$. If $x\in\mathbb{R}$ is a Lebesgue point of f and f is locally bounded at x, then*

$$\lim_{T\to\infty}\frac{-1}{T}\int_0^\infty \theta'\left(\frac{t}{T}\right)\left|s_tf(x)-f(x)\right|^d dt=0. \tag{62}$$

If f is almost everywhere locally bounded, then the corollary holds almost everywhere. Unfortunately, it is not true that an integrable function is almost everywhere locally bounded (see [21]). The strong summability can be extended to all exponents in the usual way (see [21]).

Corollary 7. *Suppose that (14) holds for all $d\in\mathbb{N}$, θ is nonincreasing, and $r>0$. Under the same conditions as in Corollaries 4 or 6, respectively, one gets that*

$$\lim_{T\to\infty}\frac{-1}{T}\int_0^\infty \theta'\left(\frac{t}{T}\right)\left|s_tf(x)-f(x)\right|^r dt=0. \tag{63}$$

Note that under the same conditions we get for the Fejér summation that

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T \left|s_tf(x)-f(x)\right|^r dt=0. \tag{64}$$

5. Strong Summability of Fourier Series

In this section we formulate the above results for Fourier series. For an integrable function f the kth *Fourier coefficient* is defined by

$$\hat{f}(k)=\frac{1}{2\pi}\int_{\mathbb{T}} f(x)e^{-\imath kx}dx \quad (k\in\mathbb{Z}), \tag{65}$$

where $\mathbb{T}$ denotes the torus. For $f\in L_1(\mathbb{T})$ and $n\in\mathbb{N}$ the nth partial sum s_nf is introduced by

$$\begin{aligned} s_nf(x) &:= \sum_{k=-n}^{n}\hat{f}(k)e^{\imath kx} \\ &= \frac{1}{2\pi}\int_{\mathbb{T}} f(x-u)D_n(u)\,du, \end{aligned} \tag{66}$$

where

$$D_n(u) := \sum_{k=-n}^{n} e^{\imath k\cdot u} = 2\frac{\sin(n+1/2)x}{\sin(x/2)} \quad (n\in\mathbb{N}) \tag{67}$$

is the *Dirichlet kernel*. Let

$$\theta=\left(\theta(k,n),\ k\in\mathbb{N},\ n\in\mathbb{N}_+\right) \tag{68}$$

be a two-parameter sequence of real numbers satisfying

$$\begin{gathered} \theta(0,n)=1, \\ \lim_{n\to\infty}\theta(k,n)=1, \\ \lim_{k\to\infty}k^d\theta(k,n)=0, \\ \sum_{k=0}^{\infty}(k+1)^\beta\left|\Delta_1\theta(k,n)\right|\le Cn^\beta \quad \text{for } \beta=0,d, \end{gathered} \tag{69}$$

where

$$\Delta_1\theta(k,n) := \theta(k,n)-\theta(k+1,n). \tag{70}$$

The *θ-means* of $f\in L_1(\mathbb{T})$ are defined by

$$\begin{aligned} \sigma_n^\theta f(x) &:= \sum_{k=-\infty}^{\infty}\theta(|k|,n)\hat{f}(k)e^{\imath kx} \\ &= \frac{1}{2\pi}\int_{\mathbb{T}} f(x-u)K_n^\theta(u)\,du, \end{aligned} \tag{71}$$

where the *θ-kernel* is given by

$$K_n^\theta(x)=\sum_{j=0}^{\infty}\Delta_1\theta(j,n)D_j(x). \tag{72}$$

Hence

$$\sigma_n^\theta f(x)=\sum_{j=0}^{\infty}\Delta_1\theta(j,n)s_jf(x). \tag{73}$$

Basically, we will consider two types of examples. The first type of examples is the Cesàro summation, which is generated by a sequence θ.

Example 8 (Cesàro summation). For $k\in\mathbb{N}$, $\alpha\ne -1,-2,\ldots$, let

$$\begin{aligned} A_k^\alpha &:= \binom{k+\alpha}{k} = \frac{(\alpha+1)(\alpha+2)\cdots(\alpha+k)}{k!} \\ &= O(k^\alpha) \quad (k\in\mathbb{N}) \end{aligned} \tag{74}$$

(see Zygmund [26, p. 77]). Let

$$\theta(k,n) = \begin{cases} \dfrac{A^{\alpha}_{n-k-1}}{A^{\alpha}_{n-1}} & \text{if } |k| \le n-1 \\ 0 & \text{if } |k| \ge n \end{cases} \tag{75}$$

for some $0 < \alpha < \infty$. Since $\Delta_1\theta(k,n) = A^{\alpha-1}_{n-k-1}/A^{\alpha}_{n-1}$, the Cesàro operators can be given by

$$\sigma^{\theta}_n f(x) = \frac{1}{A^{\alpha}_{n-1}} \sum_{k=0}^{n-1} A^{\alpha-1}_{n-1-k} s_k f(x). \tag{76}$$

If $\alpha = 1$, we get back the Fejér means.

The other type is generated by a function $\theta : \mathbb{R}_+ \to \mathbb{R}$ (see all other examples in this paper). Let θ be continuous on $\mathbb{R}_+$ and

$$\theta(k,n) := \theta\left(\frac{k}{n}\right) \quad (k \in \mathbb{N},\ n \in \mathbb{N}_+). \tag{77}$$

Suppose that

$$\begin{gathered} \theta(0) = 1, \\ \lim_{t\to\infty} t^d \theta(t) = 0, \\ \sum_{k=0}^{\infty} (k+1)^{\beta} \left|\Delta_1\theta\left(\frac{k}{n}\right)\right| \le Cn^{\beta} \quad \text{for } \beta = 0, d. \end{gathered} \tag{78}$$

Note that for the Fejér means we get the usual definition

$$\sigma^{\theta}_n f(x) = \frac{1}{n}\sum_{j=0}^{n-1} s_j f(x). \tag{79}$$

All examples of this paper satisfy (26) and (27).

Theorem 9. *Suppose that (26), (27), and (69) are satisfied for some $d \in \mathbb{N}$ and $0 < \alpha < \infty$. Let $f \in L_p(\mathbb{T})$ for some $1 < p < \infty$. If x_j is a p-Lebesgue point of f for all $j = 1, \dots, d$, then*

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} \Delta_1\theta(k,n) \prod_{j=1}^{d} \left(s_k f(x_j) - f(x_j)\right) = 0. \tag{80}$$

Moreover, if d is even, then

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} \Delta_1\theta(k,n) \left|s_k f(x_1) - f(x_1)\right|^d = 0. \tag{81}$$

Theorem 10. *Suppose that (26), (27), and (69) are satisfied for some $d \in \mathbb{N}$ and $0 < \alpha < \infty$. If $f \in L_1(\mathbb{T})$, x_j is a Lebesgue point of f, and f is locally bounded at x_j for all $j = 1, \dots, d$, then (80) and (81) hold.*

Corollary 11. *Suppose that $r > 0$, $\Delta_1\theta(k,n) \le 0$, and (69) is satisfied for all d. Under the same conditions as in Theorems 9 or 10, respectively, one gets that*

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} \Delta_1\theta(k,n) \left|s_k f(x) - f(x)\right|^r = 0. \tag{82}$$

Note that under the same conditions we get for the Fejér summation that

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \left|s_k f(x) - f(x)\right|^r = 0. \tag{83}$$

Finally we note that the Weierstrass, Abel, Picard, Bessel, Fejér, de La Vallée-Poussin, Rogosinski, and Riesz summations can be considered as special cases of θ-summation (see Weisz [21]).

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

The author would like to thank the referee for reading the paper carefully and for calling his attention to some papers of the references. This research was supported by the Hungarian Scientific Research Funds (OTKA) no. K115804.

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