Hindawi Publishing Corporation
Journal of Function Spaces
Volume 2015, Article ID 478437, 7 pages
http://dx.doi.org/10.1155/2015/478437

Research Article

Generalized Projections on Closed Nonconvex Sets in Uniformly Convex and Uniformly Smooth Banach Spaces

Messaoud Bounkhel

Department of Mathematics, College of Science, King Saud University, P.O. Box 2455, Riyadh 11451, Saudi Arabia

Correspondence should be addressed to Messaoud Bounkhel; bounkhel@ksu.edu.sa

Received 6 April 2015; Accepted 30 May 2015

Academic Editor: Calogero Vetro

The present paper is devoted to the study of the generalized projection $\pi_K : X^* \to K$, where X is a uniformly convex and uniformly smooth Banach space and K is a nonempty closed (not necessarily convex) set in X. Our main result is the density of the points $x^* \in X^*$ having unique generalized projection over nonempty close sets in X. Some minimisation principles are also established. An application to variational problems with nonconvex sets is presented.

1. Introduction

In 1994, Alber [1] (see also [2]) introduced and studied an appropriate extension of the projection operator over closed convex sets from Hilbert spaces to uniformly convex and uniformly smooth Banach spaces. It is called *generalized projection* operator. He proved various properties and extended many existing results from Hilbert spaces to uniformly convex and uniformly smooth Banach spaces.

In 2005, Li [3] extended and studied this concept from uniformly convex and uniformly smooth Banach spaces to reflexive Banach spaces. This concept has been used successfully in many applications such as variational inequalities, minimization principles, and differential inclusions (see [1, 2, 4–7] and the references therein). The main result in [1–3] is the existence property of the operator π_K for closed convex sets in reflexive Banach spaces (resp., in uniformly convex and uniformly smooth Banach spaces) in [3] (resp., in [1]). Our main aim is to study the existence of π_K for nonempty closed sets not necessarily convex. An application of our main result to variational problems with nonconvex sets is presented at the end of the paper.

2. Preliminaries

Let X be a Banach space with topological dual space X^*. We denote by $\mathbf{B}$ and $\mathbf{B}_*$ the closed unit ball in X and X^*, respectively. We recall some definitions and results on uniformly convex and uniformly smooth Banach spaces (see, e.g., [8, 9]). The moduli of convexity and smoothness of X are defined, respectively, by

$$\delta_X(\epsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : \|x\| = \|y\| = 1,\ \|x-y\| = \epsilon\right\},\quad 0 \le \epsilon \le 2, \tag{1}$$

$$\rho_X(t) = \sup\left\{\frac{1}{2}\left(\|x+y\| + \|x-y\|\right) - 1 : \|x\| = 1,\ \|y\| = t\right\},\quad t > 0.$$

The space X is said to be uniformly convex whenever $\delta_X(\epsilon) > 0$ for all $0 < \epsilon \le 2$ and is said to be uniformly smooth whenever $\lim_{t \downarrow 0} \rho_X(t) = 0$. Let $p, q > 1$ be real numbers. The space X is said to be p-uniformly convex (resp., q-uniformly smooth) if there is a constant $c > 0$ such that

$$\delta_X(\epsilon) \ge c\epsilon^p \quad (\text{resp., } \rho_X(t) \le ct^q). \tag{2}$$

Obviously from the definition of p-uniform convexity and q-uniform smoothness the constants p and q satisfy $q \in (1, 2]$ and $p \ge 2$. It is known (see, e.g., [8, 9]) that uniformly convex

Banach spaces are reflexive strictly convex and that uniformly smooth Banach spaces are reflexive. If X is a p-uniformly convex Banach space, then X^* is a p'-uniformly smooth Banach space, where $p' = p/(p-1)$ is the conjugate number of p. If X is a q-uniformly smooth Banach space, then X^* is a q'-uniformly convex Banach space, where $q' = q/(q-1)$.

The normalized duality mapping $J : X \rightrightarrows X^*$ is defined by

$$J(x) = \left\{ j(x) \in X^* : \langle j(x), x \rangle = \|x\|^2 = \|j(x)\|^2 \right\}. \quad (3)$$

Many properties of the normalized duality mapping J have been studied. For the details, one may see Takahashi's book [10] or Vainberg's book [11]. We list some properties of J:

- (J_1) For any $x \in X$, $J(x)$ is nonempty.
- (J_2) For any $x \in X$ and any real number α, $J(\alpha x) = \alpha J(x)$.
- (J_3) If X is reflexive, then J is a mapping of X onto X^*.
- (J_4) If X^* is strictly convex (i.e., the unit sphere in X^* is strictly convex; i.e., the inequality $\|x^* + y^*\| < 2$ holds for all $x^*, y^* \in X^*$ such that $\|x^*\| = \|y^*\| = 1$, $x^* \neq y^*$), then J is a single valued mapping.
- (J_5) J is a continuous operator in smooth Banach spaces.
- (J_6) If X is strictly convex, then J is one-to-one.
- (J_7) If X is a reflexive strictly convex space with strictly convex dual space X^* and if $J^* : X^* \rightrightarrows X$ is a normalized duality mapping in X^*, then $J^{-1} = J^*$.
- (J_8) J is the identity operator in Hilbert spaces.

It is known (see [8, 9]) that a reflexive Banach space X is smooth if and only if X^* is strictly convex. Hence by (J_3) and (J_4), if X is a reflexive smooth Banach space, then J is a single valued mapping from X onto X^*. And, by (J_7), if X is reflexive smooth strictly convex Banach space, then $J^{-1} = J^*$. Let $V : X^* \times X \to \mathbf{R}$ be defined by

$$V(x^*, x) = \|x^*\|^2 - 2\langle x^*, x \rangle + \|x\|^2. \quad (4)$$

First, we mention that, in Hilbert spaces ($X^* = X$), the functional V has the form $V(x^*, x) = \|x^* - x\|^2$, $\forall x, x^* \in X$.

We list now some important properties of V needed in our proofs, when X is a reflexive smooth Banach space:

(i) $V(x^*, x) \geq 0$.

(ii) $(\|x^*\| - \|x\|)^2 \leq V(x^*, x) \leq (\|x^*\| + \|x\|)^2$.

(iii) $V(J(x), x) = 0$.

(iv) $V(x^*, x)$ is continuous and V is convex with respect to x when x^* is fixed and convex with respect to x^* when x is fixed.

(v) $V(x^*, x)$ is differentiable with respect to x when x^* is fixed.

(vi) $\operatorname{grad}_x V(x^*, x) = 2(J(x) - x^*)$. This property is true whenever the space X is smooth which is the case for uniformly convex spaces.

(vii) $V(x^*, x) = 0$ if and only if $x^* = J(x)$.

Let $f : X \to \mathbf{R} \cup \{+\infty\}$ be a function and $x \in X$ where f is finite. We recall from [4] that the V-proximal subdifferential $\partial^\pi f(x)$ (called in [4] the analytical proximal subdifferential) is the set of all $x^* \in X^*$ for which there exists $\sigma > 0$ such that

$$\left\langle x^*, x' - x \right\rangle \leq f(x') - f(x) + \sigma V(J(x), x'), \quad (5)$$

for all x' around x. Recall also [4] that the V-proximal normal cone (called in [4] the proximal normal cone) of a nonempty closed subset S in X at $x \in S$ is defined by $N^\pi(S; x) = \partial^\pi \psi_S(x)$, where ψ_S is the indicator function of S. It has been proved in [4] that $N^\pi(S; x)$ coincides with the normal cone in the sense of convex analysis $N(S; x)$ given by $N(S; x) = \{x^* \in X^* : \langle x^*, y - x \rangle \leq 0, \ \forall x \in S\}$.

Based on the functional V, a set $\pi_S(x^*)$ of generalized projections of $x^* \in X^*$ onto S is defined as follows (see [1]).

Definition 1. Let S be a nonempty subset of X and $x^* \in X^*$. If there exists a point $\overline{x} \in S$ satisfying

$$V(x^*, \overline{x}) = \inf_{x \in S} V(x^*, x), \quad (6)$$

then $\overline{x}$ is called a generalized projection of x^* onto S. The set of all such points is denoted by $\pi_S(x^*)$.

The following lemma is needed in our proofs and for its proof we refer to [1].

Lemma 2. *If E is a uniformly convex Banach space, then the inequality*

$$V(J(x), y) \geq 8C^2 \delta_E \left(\frac{\|x - y\|}{4C} \right) \quad (7)$$

holds for all x and y in E, where $C = \sqrt{(\|x\|^2 + \|y\|^2)/2}$.

We end this section with the following important result proved in [4]. It proves the density of the set $\operatorname{dom}(\partial^\pi f)$ in $\operatorname{dom} f$, that is, the set of points x in $\operatorname{dom} f$ at which $\partial^\pi f(x) \neq \emptyset$ is dense in $\operatorname{dom} f$.

Theorem 3. *Let $p \geq 2$, let $q \in (0, 2]$, let X be a p-uniformly convex and q-uniformly smooth Banach space, and let $f : X \to \mathbf{R} \cup \{+\infty\}$ be a lower semicontinuous function. Let $x_0 \in \operatorname{dom} f$, and let $\epsilon > 0$ be given. Then there exists a point $y \in x_0 + \epsilon \mathbf{B}$ satisfying $\partial^\pi f(y) \neq \emptyset$ and $f(x_0) - \epsilon \leq f(y) \leq f(x_0)$. Consequently, $\operatorname{dom}(\partial^\pi f)$ is dense in $\operatorname{dom} f$.*

3. Minimization Principles in Banach Spaces

Given a lower semicontinuous function $f : X \to \mathbf{R} \cup \{+\infty\}$ that is bounded below and $\alpha > 0$, we define two functions $f_\alpha : X \to \mathbf{R}$ and $f_\alpha^* : X^* \to \mathbf{R}$ by

$$f_\alpha(x) := \inf_{y \in X} \{ f(y) + \alpha V(J(y), x) \}, \quad (8)$$

$$f_\alpha^*(x^*) := \inf_{y \in X} \{ f(y) + \alpha V(x^*, y) \}. \quad (9)$$

These functions f_α and f_α^* coincide, in Hilbert spaces, with the inf-convolution of the function f and the function $x \mapsto \alpha\|x\|^2$, which is due to the relation $V(J(y), x) = V(J(x), y) = \|y - x\|^2$ in Hilbert spaces. In [4], the authors studied the function f_α and they derived some minimization principles in p-uniformly convex and q-uniformly smooth Banach spaces. In this section, we establish similar results for the function f_α^* that will be used to prove our main theorem in this paper. We start with the following theorem proved in [4].

Theorem 4. *Let $p' \geq 2$, let $q' \in (1, 2]$, let E be a p'-uniformly convex and q'-uniformly smooth Banach space, and let $f : E \to \mathbf{R} \cup \{+\infty\}$ be a proper lower semicontinuous function. Suppose that f is bounded below by some constant c. Then f_α is bounded below by c and is Lipschitz on each bounded subset of E. Furthermore, suppose that $x \in E$ is such that $\partial^\pi f_\alpha(x)$ is nonempty. Then there exists a point $\overline{y} \in E$ satisfying the following:*

(i) *If $\{y_i\}$ is a minimizing sequence in E for the infimum in (8), then $\lim_{i \to +\infty} y_i = \overline{y}$.*

(ii) *The infimum in (8) is attained uniquely at $\overline{y}$.*

Let X be a p-uniformly convex and q-uniformly smooth Banach space with $p \geq 2$ and $q \in (0, 2]$. Then X is reflexive; that is, $X^{**} = X$, and J is one-to-one from X to X^* with $J^{-1} = J^*$. Thus, observe that the function f_α^* can be rewritten as follows:

$$\begin{aligned} f_\alpha^*(x^*) &= \inf_{y \in X} \{f(y) + \alpha V(x^*, y)\} \\ &= \inf_{y^* \in X^*} \{f(J^*(y^*)) + \alpha V(x^*, J^*(y^*))\} \\ &= \inf_{y^* \in X^*} \{(f \circ J^*)(y^*) + \alpha V_*(J^*(y^*), x^*)\} \\ &= \inf_{y^* \in X^*} \{F(y^*) + \alpha V_*(J^*(y^*), x^*)\} \\ &= F_\alpha(x^*), \end{aligned} \tag{10}$$

where $F : X^* \to \mathbf{R}$ is defined by $F := f \circ J^*$ and $V_* : X^{**} \times X^* \to \mathbf{R}$ is defined by

$$\begin{aligned} V_*(J_*(y^*), x^*) = \|J_*(y^*)\|^2 - 2\langle J_*(y^*); x^* \rangle \\ + \|x^*\|^2. \end{aligned} \tag{11}$$

Using this observation together with Theorem 4 with $E = X^*$ and F playing the role of f we can prove the following theorem.

Theorem 5. *Let $p \geq 2$, let $q \in (1, 2]$, let X be a p-uniformly convex and q-uniformly smooth Banach space, and let $f : X \to \mathbf{R} \cup \{+\infty\}$ be a proper lower semicontinuous function which is bounded below by some constant c. Then f_α^* is bounded below by c and is Lipschitz on each bounded subset of X^*. Furthermore, for any $x^* \in X^*$ with $\partial^\pi f_\alpha^*(x^*) \neq \emptyset$ there exists a point $\overline{y} \in X$ satisfying the following:*

$(i)^*$ *If $\{y_i\}$ is a minimizing sequence in X for the infimum in (9), then $\lim_{i \to +\infty} y_i = \overline{y}$.*

$(ii)^*$ *The infimum in (9) is attained uniquely at $\overline{y} \in X$; that is,*

$$f_\alpha^*(x^*) = f(\overline{y}) + \alpha V(x^*, \overline{y}). \tag{12}$$

Proof. Let $E = X^*$ and $F = f \circ J^*$. Clearly F is a proper l.s.c. function on E and is bounded below by the same constant c. Then by Theorem 4 the function $F_\alpha = f_\alpha^*$ is bounded below by c and is Lipschitz on each bounded subset in X^*. Furthermore, for any $x^* \in X^*$ with $\partial^\pi f_\alpha^*(x^*) \neq \emptyset$, we have $\partial^\pi F_\alpha(x^*) \neq \emptyset$ and so by Theorem 4 there exists a point $z^* \in X^*$ such that

(i) if $\{y_n^*\}$ is a minimizing sequence in X^* for the infimum in

$$F_\alpha(x^*) = \inf_{y^* \in X^*} \{F(y^*) + \alpha V_*(J^*(y^*), x^*)\}, \tag{13}$$

then $\lim_{i \to +\infty} y_n^* = z^*$;

(ii) the infimum in (13) is attained uniquely at $z^* \in X^*$; that is,

$$F_\alpha(x^*) = F(z^*) + \alpha V_*(J^*(z^*), x^*). \tag{14}$$

The proof will be complete by taking $\overline{y} := J^*(z^*)$ and by using the fact that J^* is continuous in smooth Banach spaces. □

By taking different forms of the function f_α^*, we can obtain various types of minimization principles. We state here the two following types. The first one is Stegall's minimization principle and the second one is a variant of the smooth Borwein-Preiss variational principle in Banach spaces (see [4] for different variants in Banach spaces and see [12] for those principles in Hilbert spaces).

Theorem 6. *Let $q \in (1, 2]$ and $p \geq 2$, let X be a p-uniformly convex and q-uniformly smooth Banach space, and let $f : X \to \mathbf{R}$ be a lower semicontinuous function. Suppose that f is bounded below on the bounded closed set $S \subset X$, with $S \cap \operatorname{dom} f \neq \emptyset$. Then, for any $\epsilon > 0$, there exists $x^* \in X^*$ with $\|x^*\| < \epsilon$ such that the function $y \mapsto f(y) + \langle x^*, y \rangle$ attains a unique minimum over S.*

Proof. Define on X^* the function

$$\begin{aligned} &g(x^*) \\ &:= \inf_{y \in X} \left\{ f(y) + \psi_S(y) - \frac{1}{2}\|y\|^2 + \frac{1}{2} V(x^*, y) \right\}, \end{aligned} \tag{15}$$

which is of the form h_α^* with $h = f + \psi_S - (1/2)\|\cdot\|^2$ and $\alpha = 1/2$. Furthermore, expression (15) for g can be rewritten as

$$g(x^*) = \inf_{y \in S} \{f(y) - \langle x^*, y \rangle\} + \frac{1}{2}\|x^*\|^2. \tag{16}$$

Let $\epsilon > 0$ and let $z \in S \cap \operatorname{dom} f$. Then for any $x^* \in X^*$ we have

$$g(x^*) \leq f(z) - \langle x^*, z \rangle + \frac{1}{2}\|x^*\|^2 < \infty; \tag{17}$$

that is, $\operatorname{dom} g = X^*$. Now, by the density theorem of the V-proximal subdifferential in Theorem 3, there exists $y^* \in \operatorname{dom} \partial^\pi g$; that is, $\partial^\pi g(y^*) \neq \emptyset$ with $\|y^*\| < \epsilon$ and

$$\inf_{y\in S} f(y) - \epsilon = g(0) - \epsilon \leq g(y^*) \leq g(0) = \inf_{y\in S} f(y). \quad (18)$$

Using now Theorem 4(ii), we deduce that the infimum in (15) and (16) is attained at a unique point $\overline{y} \in S$; that is,

$$g(x^*) = f(\overline{y}) - \langle x^*, \overline{y}\rangle + \frac{1}{2}\|x^*\|^2. \quad (19)$$

Therefore, by taking $x^* = -y^*$, we obtain $\|x^*\| < \epsilon$ and the function $y \mapsto f(y) + \langle x^*, y\rangle$ attains a unique minimum over S at $\overline{y} \in S$ and so the proof is complete. □

The following theorem is a different variant of the smooth Borwein-Preiss variational principle in which the perturbation is given in terms of the functional V.

Theorem 7. *Let $p \geq 2$, let $q \in (1, 2]$, let X be a p-uniformly convex and q-uniformly smooth Banach space, and let $f : X \to \mathbf{R}$ be a lower semicontinuous function bounded below, and $\epsilon > 0$. Suppose that $\overline{x}$ is a point satisfying $f(\overline{x}) < \inf_{x\in X} f(x) + \epsilon$. Then for any $\lambda > 0$ there exist points $\overline{y} \in X$ and $z^* \in X^*$ such that*

(i) $\|z^* - J(\overline{x})\| < \lambda$, $V(z^*, \overline{y}) < \lambda$, $f(\overline{y}) \leq f(\overline{x})$,

(ii) $f + (\epsilon/\lambda)V(z^*; \cdot)$ *has a unique minimum at* $\overline{y}$.

Proof. Let $\epsilon > 0$ be as in the statement of Theorem 7 and let $\lambda > 0$. Put $\alpha = \epsilon/\lambda$ and consider the function

$$f_\alpha^*(x^*) := \inf_{y\in X}\left\{f(y) + \frac{\epsilon}{\lambda}V(x^*, y)\right\}. \quad (20)$$

Since f_α^* is l.s.c. on X^* and by the density result in Theorem 3, there exists $z^* \in J(\overline{x}) + \lambda\mathbf{B}$ satisfying $f_\alpha^*(J(\overline{x})) - \epsilon \leq f_\alpha^*(z^*) \leq f_\alpha^*(J(\overline{x})) \leq f(\overline{x})$ with $\partial^\pi f_\alpha^*(z^*) \neq \emptyset$. By Theorem 5 there is a unique point $\overline{y} \in X$ satisfying

$$f_\alpha^*(z^*) = f(\overline{y}) + \frac{\epsilon}{\lambda}V(z^*; \overline{y}) \leq f(\overline{x}) < \inf_{x\in X} f(x) + \epsilon \quad (21)$$

and so

$$\frac{\epsilon}{\lambda}V(z^*; \overline{y}) \leq \inf_{x\in X} f(x) - f(\overline{y}) + \epsilon < \epsilon \quad (22)$$

and so

$$V(z^*; \overline{y}) < \lambda. \quad (23)$$

Thus,

$$f(\overline{y}) + \frac{\epsilon}{\lambda}V(z^*; \overline{y}) = f_\alpha^*(z^*) \leq f(x) + \frac{\epsilon}{\lambda}V(z^*; x), \quad \forall x \in X, \quad (24)$$

and so the function $x \mapsto f(x) + (\epsilon/\lambda)V(z^*; x)$ has a unique minimum at $x = \overline{y}$. □

4. Generalized Projections on Closed Nonconvex Sets

Let us start with the following example showing that $\pi_K(\varphi)$ may be empty for nonconvex closed sets in uniformly convex and uniformly smooth Banach spaces.

Example 8. Let $X = \ell_p$ $(p \geq 1)$, let $\theta = (0, \ldots, 0, \ldots) \in l_p$, and let $S := \{e_1, e_2, \ldots, e_n, \ldots\}$ with $e_j = (0, \ldots, 0, (j+1)/j, 0, \ldots)$. Then S is a closed nonconvex subset in X with $\pi_S(\theta) = \emptyset$.

Proof. Clearly S is closed and not convex. Let x be any element in $\{e_1, e_2, \ldots, e_n, \ldots\}$; that is, $x = e_n$ for some $n \geq 1$, $\|x\| = \|e_n\| = 1 + (1/n) > 1$. Then for any $x \in S$ we have $\|x\| > 1$ and so $V(\theta, x) = \|x\|^2 > 1$, $\forall x \in S$; that is,

$$1 \leq \inf_{x\in S} V(\theta, x) \leq \liminf_{n\to\infty} V(\theta, e_n) = \liminf_{n\to\infty}\left(1 + \frac{1}{n}\right) = 1 \quad (25)$$

and so

$$\inf_{x\in S} V(\theta, x) = 1. \quad (26)$$

This ensures that $\pi_S(\theta) = \emptyset$. □

From the previous example, we see that even in uniformly convex and uniformly smooth Banach spaces the generalised projection $\pi_K(x^*)$ may be empty for closed nonconvex sets and so there is no hope of getting the conclusion of Theorem 2.1 in [3] saying that $\pi_K(x^*) \neq \emptyset$, $\forall x^* \in X^*$, whenever the set S is closed convex in reflexive Banach spaces. However, we are going to prove that, for closed nonconvex sets, the set of points $x^* \in X^*$ for which $\pi_S(x^*) \neq \emptyset$ is dense in X^*. We are going to prove our main result in the following theorem. It is an analogue result to Lau's theorem for metric projections in reflexive Banach spaces [13].

Theorem 9. *Let $p \geq 2$ and $q \in (1, 2]$, let X be a p-uniformly convex and q-uniformly smooth Banach space, and let S be any closed nonempty set of X. Then there is a dense set of points in X^* admitting unique generalised projection on S; that is, for any $x^* \in X^*$, there exists $x_n^* \to x^*$ with $\pi_S(x_n^*) \neq \emptyset$, $\forall n$.*

Proof. Observe that

$$d_S^V(x^*) = \inf_{y\in X}\{\psi_S(y) + V(x^*, y)\}, \quad (27)$$

which means that d_S^V has the form f_α^* with $\alpha = 1$ and $f = \psi_S$. Since f is proper l.s.c. and is bounded below, we can apply Theorem 5 to get for any $x^* \in X^*$ with $\partial^\pi d_S^V(x^*) \neq \emptyset$ the existence of some $\overline{y} \in X$ satisfying

$$d_S^V(x^*) = \psi_S(\overline{y}) + V(x^*, \overline{y}); \quad (28)$$

that is, $\overline{y} \in S$ and $d_S^V(x^*) = V(x^*, \overline{y})$, which means that $\overline{y} \in \pi_S(x^*)$. Using now the density result in Theorem 3, to get the density of the set $\operatorname{dom} \partial^\pi d_S^V(\cdot)$ in X^*, that is, for any $x^* \in X^*$, there exists $x_n^* \to x^*$ with $\partial^\pi d_S^V(x_n^*) \neq \emptyset$. Therefore, by what precedes, there exists $y_n \in \pi_S(x_n^*)$ $\forall n$; that is, $\pi_S(x_n^*) \neq \emptyset$, $\forall n$. This proves the conclusion of the theorem. □

5. Applications to Nonconvex Variational Problems

Let $p \geq 2$ and $q \in (1,2]$ and let X be a p-uniformly convex and q-uniformly smooth Banach space. Let $F : X \rightrightarrows X^*$ be a set-valued mapping and let $S \subset X$ be a nonempty closed set not necessarily convex. Our aim is to use the main result in the previous section to study the following nonconvex variational problem:

$$\text{Find } \overline{x} \in S \text{ such that } N^{\pi}(S;\overline{x}) \cap [-F(\overline{x})] \neq \emptyset. \tag{29}$$

First we show that in the convex case (29) coincides with the usual variational inequality

$$\begin{aligned} &\text{Find } \overline{x} \in S,\ y^* \in F(\overline{x}) \\ &\text{such that } \langle y^*, y - \overline{x} \rangle \geq 0, \\ &\qquad \forall y \in S. \end{aligned} \tag{30}$$

Proposition 10. *Whenever S is a closed convex set, one has (29) $\Leftrightarrow$ (30).*

Proof. The proof follows from the fact that $N^{\pi}(S;x)$ coincides with the convex normal cone which can be characterised as $N(S;x) = \{x^* \in X^* : \langle x^*, s - x\rangle \leq 0,\ \forall s \in S\}$. □

We suggest the following algorithm to solve the proposed problem (29) under some natural and appropriate assumptions on S and F.

Algorithm 11. Let $\delta_n \downarrow 0$ with δ_0 being too small:

(i) Select $x_0 \in S$, $y_0^* \in F(x_0)$, and $\rho > 0$.

(ii) For $n \geq 0$,

(a) compute $z_{n+1} := J^*(J(x_n) - \rho y_n^*)$;

(b) choose $u_{n+1} \in J^*(J(z_{n+1}) + \delta_n \mathbf{B}_*)$ with $\pi_S(J(u_{n+1})) \neq \emptyset$;

(c) compute $x_{n+1} := \pi_S(J(u_{n+1}))$ and $y_{n+1}^* \in F(x_{n+1})$.

Since S is not necessarily convex, the generalised projection π_S does not exist necessarily for any $x^* \in X^* \setminus J(S)$. However, our previous algorithm is well defined as we will prove in the following proposition.

Proposition 12. *Assume that X is uniformly convex and uniformly smooth Banach space. The above algorithm is well defined.*

Proof. Let $n \geq 0$ and let $x_n \in S$ with $y_n^* \in F(x_n)$ be given. The point z_{n+1} is well defined since J and J^* are well defined and one-to-one because the space X is assumed to be uniformly convex and uniformly smooth. Now, since the generalised projection of $J(z_{n+1})$ is not ensured we use our main result in Theorem 9 to choose some point $J(u_{n+1}) \in X^*$ too close to $J(z_{n+1})$ so that $\|J(z_{n+1}) - J(u_{n+1})\| \leq \delta_n$ and $\pi_S(J(u_{n+1})) \neq \emptyset$. Then by the same theorem we have the uniqueness of the generalised projection so we can take $x_{n+1} := \pi_S(J(u_{n+1}))$ and then we are done. □

After proving the well definedness of the algorithm without any additional assumptions on S and F we add some natural assumptions on the data to prove the convergence of the sequence $\{x_n\}_n$ to a solution of (29).

In our analysis we need the following assumptions on S and F:

Assumptions $\mathscr{A}$

(1) The solution set of (29) is nonempty.

(2) The set S is ball compact; that is, any bounded set in S is relatively compact.

(3) F is bounded on S by some constant $L > 0$.

(4) F is β-Lipschitz on S; that is,

$$\begin{aligned} &\left\|y_1^* - y_2^*\right\| \leq \beta \left\|x_1 - x_2\right\|, \\ &\qquad \forall x_i \in S,\ \forall y_i \in F(x_i),\ i = 1,2. \end{aligned} \tag{31}$$

(5) F is α-J-strongly monotone on S; that is,

$$\begin{aligned} &\langle J^*(y_1^* - y_2^*); J(x_1) - J(x_2)\rangle \\ &\quad \geq \alpha \left\|J(x_1) - J(x_2)\right\|^2, \\ &\qquad \forall x_i \in S,\ \forall y_i^* \in F(x_i),\ i = 1,2. \end{aligned} \tag{32}$$

(6) There exist some constants $\mu > 0$ and $\xi > 0$ such that

$$\begin{aligned} &\left\|\pi_S(u_1^*) - \pi_S(u_2^*)\right\| \leq \xi \left\|u_1^* - u_2^*\right\|, \\ &\qquad \forall u_1^*, u_2^* \in J(S) + \mu \mathbf{B}_*. \end{aligned} \tag{33}$$

(7) The constants μ, δ_0, α, c, ξ, and β satisfy

$$\begin{aligned} &\xi c > \beta(\xi - 1), \\ &0 < \delta_0 < \mu. \end{aligned} \tag{34}$$

Theorem 13. *Assume that X is 2-uniformly smooth. Let $\{x_n\}_n$ be a sequence generated by Algorithm 11. Assume that Assumptions $\mathscr{A}$ hold and that the parameter ρ satisfies the inequalities*

$$\frac{\alpha}{\beta^2} - \overline{\epsilon} < \rho < \min\left\{\frac{\mu - \delta_0}{L}, \frac{\alpha}{\beta^2} + \overline{\epsilon}\right\}, \tag{35}$$

where $\overline{\epsilon} := \sqrt{\alpha^2 - \beta^2(1 - c/\xi^2)}/\beta^2$. Then there exists a subsequence of $\{x_n\}_n$ converging to a solution $\overline{x}$ of (29).

Proof. Let $\overline{x}$ be a solution of (29); that is, there exists $\overline{y}^* \in F(\overline{x})$ such that $-\overline{y}^* \in N^{\pi}(S;\overline{x})$. Hence by definition of the V-proximal normal cone there exists $r > 0$ such that $\overline{x} \in \pi_S(J(\overline{x}) - r\overline{y}^*)$. Without loss of generality we may assume that ρ is too small so that $\rho \in (0,r]$. First we claim that $\overline{x} \in \pi_S(J(\overline{x}) - \rho\overline{y}^*)$; that is, $V(J(\overline{x}) - \rho\overline{y}^*, \overline{x}) = \inf_{s\in S} V(J(\overline{x}) - \rho\overline{y}^*, s)$. Let $\lambda := \rho/r \in (0,1]$. Then for any $s \in S$ we have

$$\begin{aligned} &2\langle J(\overline{x}) - \rho\overline{y}^* - J\overline{x}; s - \overline{x}\rangle \\ &\quad = 2\langle \lambda(J(\overline{x}) - r\overline{y}^*) + (1-\lambda)J(\overline{x}) - J\overline{x}; s - \overline{x}\rangle \\ &\quad = 2\lambda\langle (J(\overline{x}) - r\overline{y}^*) - J(\overline{x}); s - \overline{x}\rangle. \end{aligned} \tag{36}$$

We distinguish two cases.

Case 1. If $\langle (J(\overline{x}) - r\overline{y}^*) - J(\overline{x}); s - \overline{x}\rangle \le 0$, then obviously we have

$$2\left\langle J(\overline{x}) - \rho\overline{y}^* - J\overline{x}; s - \overline{x}\right\rangle \le 0 \le V(J(\overline{x}), s). \tag{37}$$

Case 2. If $\langle (J(\overline{x}) - r\overline{y}^*) - J(\overline{x}); s - \overline{x}\rangle \ge 0$, then since $0 < \lambda \le 1$ we have

$$\begin{aligned} &2\lambda\left\langle \left(J(\overline{x}) - r\overline{y}^*\right) - J(\overline{x}); s - \overline{x}\right\rangle \\ &\quad \le 2\left\langle \left(J(\overline{x}) - r\overline{y}^*\right) - J(\overline{x}); s - \overline{x}\right\rangle \end{aligned} \tag{38}$$

and so we obtain

$$\begin{aligned} &2\left\langle J(\overline{x}) - \rho\overline{y}^* - J\overline{x}; s - \overline{x}\right\rangle \\ &\quad \le 2\left\langle \left(J(\overline{x}) - r\overline{y}^*\right) - J(\overline{x}); s - \overline{x}\right\rangle \\ &\quad \le \left\|J(\overline{x}) - r\overline{y}^*\right\|^2 - 2\left\langle \left(J(\overline{x}) - r\overline{y}^*\right); \overline{x}\right\rangle + \|\overline{x}\|^2 \\ &\qquad + 2\left\langle \left(J(\overline{x}) - r\overline{y}^*\right); s\right\rangle - \left\|J(\overline{x}) - r\overline{y}^*\right\|^2 - \|s\|^2 \\ &\qquad + \|s\|^2 - 2\left\langle J(\overline{x}); s - \overline{x}\right\rangle - \|\overline{x}\|^2 \\ &\quad \le V\left(J(\overline{x}) - r\overline{y}^*, \overline{x}\right) - V\left(J(\overline{x}) - r\overline{y}^*, s\right) \\ &\qquad + V(J(\overline{x}), s) \\ &\quad \le \inf_{z\in S} V\left(J(\overline{x}) - r\overline{y}^*, z\right) - V\left(J(\overline{x}) - r\overline{y}^*, s\right) \\ &\qquad + V(J(\overline{x}), s) \le V(J(\overline{x}), s). \end{aligned} \tag{39}$$

Therefore, in both cases, we have

$$2\left\langle J(\overline{x}) - \rho\overline{y}^* - J\overline{x}; s - \overline{x}\right\rangle \le V(J(\overline{x}), s). \tag{40}$$

Hence

$$2\left\langle J(\overline{x}) - \rho\overline{y}^* - J\overline{x}; s - \overline{x}\right\rangle - V(J(\overline{x}), s) \le 0. \tag{41}$$

On the other hand, simple decomposition yields

$$\begin{aligned} &2\left\langle J(\overline{x}) - \rho\overline{y}^* - J\overline{x}; s - \overline{x}\right\rangle - V(J(\overline{x}), s) \\ &\quad = V\left(J(\overline{x}) - \rho\overline{y}^*, \overline{x}\right) - V\left(J(\overline{x}) - \rho\overline{y}^*, s\right). \end{aligned} \tag{42}$$

Consequently, we have

$$\begin{aligned} &V\left(J(\overline{x}) - \rho\overline{y}^*, \overline{x}\right) - V\left(J(\overline{x}) - \rho\overline{y}^*, s\right) \le 0, \\ &\qquad\qquad \text{for any } s \in S, \end{aligned} \tag{43}$$

which means that $\overline{x} \in \pi_S(J(\overline{x}) - \rho\overline{y}^*)$. Set $\overline{z} := J^*(J(\overline{x}) - \rho\overline{y}^*)$. Since X is 2-uniformly smooth we have the 2-uniform convexity of the dual space X^* and so $\delta_{X^*}(\tau) \ge 2c\tau^2$ for some constant c depending only on the space X^*. On the other hand, by Lemma 2, we have

$$\begin{aligned} &V_*\left(J^*x^*, y^*\right) \ge 8C^2\delta_{X^*}\left(\frac{\left\|x^* - y^*\right\|}{4C}\right), \\ &\qquad\qquad \forall x^*, y^* \in X^*, \end{aligned} \tag{44}$$

where $C = \sqrt{(\|x^*\|^2 + \|y^*\|^2)/2}$. Hence

$$\begin{aligned} V_*\left(J^*x^*, y^*\right) &\ge 8C^2\delta_{X^*}\left(\frac{\left\|x^* - y^*\right\|}{4C}\right) \\ &\ge c\left\|x^* - y^*\right\|^2, \quad \forall x^*, y^* \in X^*. \end{aligned} \tag{45}$$

Therefore, we obtain

$$\begin{aligned} &c\left\|J(z_{n+1}) - J(\overline{z})\right\|^2 \\ &\quad = c\left\|J(x_n) - J(\overline{x}) - \rho\left(y_n^* - \overline{y}^*\right)\right\|^2 \\ &\quad \le V_*\left(\rho J^*\left(y_n^* - \overline{y}^*\right); J(x_n) - J(\overline{x})\right) \\ &\quad \le \rho^2\left\|y_n^* - \overline{y}^*\right\|^2 \\ &\qquad - 2\rho\left\langle J^*\left(y_n^* - \overline{y}^*\right); J(x_n) - J(\overline{x})\right\rangle \\ &\qquad + \left\|J(x_n) - J(\overline{x})\right\|^2. \end{aligned} \tag{46}$$

Using now the β-Lipschitz property and the α-J-strong monotony of F, we write

$$\begin{aligned} \left\|J(z_{n+1}) - J(\overline{z})\right\|^2 &\le c^{-1}\left[\left\|J(x_n) - J(\overline{x})\right\|^2\right. \\ &\left. - 2\rho\alpha\left\|J(x_n) - J(\overline{x})\right\|^2 + \rho^2\beta^2\left\|J(x_n) - J(\overline{x})\right\|^2\right] \end{aligned} \tag{47}$$

and so

$$\begin{aligned} &\left\|J(z_{n+1}) - J(\overline{z})\right\| \\ &\quad \le \sqrt{c^{-1}\left(1 - 2\rho\alpha + \rho^2\beta^2\right)}\left\|J(x_n) - J(\overline{x})\right\|. \end{aligned} \tag{48}$$

Since x_n and $\overline{x}$ belong to S (by construction) we have $J(x_n)$, $J(\overline{x}) \in J(S)$ and so by our assumptions on the constants L, δ_0, and μ and the choice of ρ we obtain

$$\begin{aligned} d_{J(S)}(J(u_n)) &\le d_{J(S)}(J(z_n)) + \left\|J(u_n) - J(z_n)\right\| \\ &\le \rho\left\|y_n^*\right\| + \delta_n < \rho L + \delta_0 < \mu, \\ d_{J(S)}(J(\overline{z})) &= d_{J(S)}\left(J(\overline{x}) - \rho\overline{y}^*\right) \le \rho\left\|\overline{y}^*\right\| < \rho L \\ &< \mu \end{aligned} \tag{49}$$

which ensures that $J(u_n)$ and $J(\overline{z})$ belong to $J(S) + \mu\mathbf{B}_*$. This yields with the Lipschitz assumption of the generalised projection on $J(S) + \mu\mathbf{B}_*$ that

$$\begin{aligned} &\left\|J(x_n) - J(\overline{x})\right\| \\ &\quad = \left\|J\left(\pi_S(J(u_n))\right) - J\left(\pi_S(J(\overline{z}))\right)\right\| \\ &\quad \le \xi\left\|J(u_n) - J(\overline{z})\right\| \\ &\quad \le \xi\left[\left\|J(z_n) - J(\overline{z})\right\| + \left\|J(u_n) - J(z_n)\right\|\right] \\ &\quad \le \xi\left\|J(z_n) - J(\overline{z})\right\| + \xi\delta_{n-1}. \end{aligned} \tag{50}$$

And consequently inequality (48) becomes

$$\begin{aligned}\left\|J\left(z_{n+1}\right)-J\left(\overline{z}\right)\right\| &\leq \xi\sqrt{c^{-1}\left(1-2\rho\alpha+\rho^2\beta^2\right)}\left\|J\left(z_n\right)-J\left(\overline{z}\right)\right\| \\ &\quad+\xi\delta_{n-1}\sqrt{c^{-1}\left(1-2\rho\alpha+\rho^2\beta^2\right)}.\end{aligned} \tag{51}$$

Set $\zeta := \xi\sqrt{c^{-1}(1-2\rho\alpha+\rho^2\beta^2)}$, $\overline{\delta}_n := \xi\delta_{n-1}\sqrt{c^{-1}(1-2\rho\alpha+\rho^2\beta^2)}$, and $\Phi_n := \|J(z_n)-J(\overline{z})\|$. Then for any $n \geq 1$ we have

$$\Phi_{n+1} \leq \zeta\Phi_n + \overline{\delta}_n. \tag{52}$$

By mathematical induction we get

$$\begin{aligned}\Phi_{n+1} &\leq \sum_{k=0}^{n-1}\zeta^k\overline{\delta}_{n-k}+\zeta^n\Phi_1 \leq \sum_{k=0}^{n-1}\zeta^k+\zeta^n\Phi_1 \\ &\leq \frac{1-\zeta^n}{1-\zeta}+\zeta^n\Phi_1.\end{aligned} \tag{53}$$

Our Assumptions $\mathscr{A}$ and the choice of ρ ensure that $0 < \zeta < 1$ and hence the sequence Φ_n is bounded and so the sequences $\{z_n\}_n$ and $\{x_n\}_n$ are bounded and since the set S is ball compact then the sequence $\{x_n\}_n$ is compact and hence there exists a subsequence $\{x_{n_k}\}_k$ converging to some limit $\widetilde{x} \in S$. By Lipschitz property of F we can check easily that the sequence $\{y_{n_k}\}_k$ is convergent to some limit $\widetilde{y}^*$ belonging to $F(\widetilde{x})$ and so $\lim_k J(u_{n_k+1}) = \lim_k J(z_{n_k+1}) = J(\widetilde{x}) - \rho\widetilde{y}^*$. Set $\widetilde{z} := J^*(J(\widetilde{x}) - \rho\widetilde{y}^*)$. To complete the proof we have to prove that $\widetilde{x}$ is a solution of (29). By Algorithm 11 the subsequence $\{x_{n_k}\}_k$ satisfies $x_{n_k+1} = \pi_S(J(u_{n_k+1}))$ and so by the Lipschitz property of the generalised projection on the set $J(S)+\mu\mathbf{B}_*$ we can write

$$\begin{aligned}\left\|\pi_S\left(J\left(\widetilde{z}\right)\right)-\widetilde{x}\right\| &\leq \left\|\pi_S\left(J\left(\widetilde{z}\right)\right)-\pi_S\left(J\left(u_{n_k+1}\right)\right)\right\| \\ &\quad+\left\|x_{n_k+1}-\widetilde{x}\right\| \\ &\leq \xi\left\|J\left(u_{n_k+1}\right)-J\left(\widetilde{z}\right)\right\|+\left\|x_{n_{k+1}}-\widetilde{x}\right\| \\ &\longrightarrow 0\end{aligned} \tag{54}$$

and so $\widetilde{x} = \pi_S(J(\widetilde{z})) = \pi_S(J(\widetilde{x}) - \rho\widetilde{y}^*)$ which ensures by definition of the V-proximal normal cone that

$$\begin{aligned}0 &= \rho^{-1}\left(\left[J\left(\widetilde{x}\right)-\rho\widetilde{y}^*\right]-J\left(\widetilde{x}\right)\right)+\widetilde{y}^* \\ &\in N^{\pi}\left(S;\widetilde{x}\right)+F\left(\widetilde{x}\right).\end{aligned} \tag{55}$$

Thus

$$N^{\pi}\left(S;\widetilde{x}\right)\cap\left[-F\left(\widetilde{x}\right)\right] \neq \emptyset; \tag{56}$$

that is, $\widetilde{x}$ is a solution of (29). Thus the proof is complete. □

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

The author would like to thank both referees for the complete reading of the first version of this work and for their suggestions allowing him to improve the presentation of the paper. The author extends his appreciations to the Deanship of Scientific Research at King Saud University for funding the work through the Research Group Project no. RGP-024.

References

[1] Y. I. Alber, "Generalized projection operators in Banach spaces: properties and applications," *Functional Differential Equations*, vol. 1, no. 1, pp. 1–21, 1994.

[2] Y. I. Alber, "Metric and generalized projection operators in Banach spaces: properties and appklications," in *Theory and Applications of Nonlinear Operators of Accretive and Monotone Type*, Lecture Notes in Pure and Applied Mathematics, pp. 15–50, Dekker, New York, NY, USA, 1996.

[3] J. Li, "The generalized projection operator on reflexive Banach spaces and its applications," *Journal of Mathematical Analysis and Applications*, vol. 306, no. 1, pp. 55–71, 2005.

[4] M. Bounkhel and R. Al-Yusof, "Proximal analysis in reflexive smooth Banach spaces," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 73, no. 7, pp. 1921–1939, 2010.

[5] M. Bounkhel and R. Al-Yusof, "First and second order convex sweeping processes in reflexive smooth Banach spaces," *Set-Valued and Variational Analysis*, vol. 18, no. 2, pp. 151–182, 2010.

[6] J. Li, "On the existence of solutions of variational inequalities in Banach spaces," *Journal of Mathematical Analysis and Applications*, vol. 295, no. 1, pp. 115–126, 2004.

[7] Y. Alber and J. L. Li, "The connection between the metric and generalized projection operators in Banach spaces," *Acta Mathematica Sinica, English Series*, vol. 23, no. 6, pp. 1109–1120, 2007.

[8] R. Deville, G. Godefroy, and V. Zizler, *Smoothness and Renormings in Banach Spaces*, vol. 64 of *Pitman Monographs and Surveys in Pure and Applied Mathematics*, Longman Scientific & Technical, Harlow, UK, 1993.

[9] J. Diestel, *Geometry of Banach Spaces—Selected Topics*, Lecture Notes in Mathematics, Vol. 485, Springer, Berlin, Germany, 1975.

[10] W. Takahashi, *Nonlinear Functional Analysis*, Yokohama Publishers, 2000.

[11] M. M. Vainberg, *Variational Methods and Method of Monotone Operators in the Theory of Nonlinear Equations*, John Wiley & Sons, New York, NY, USA, 1973.

[12] F. H. Clarke, Y. S. Ledyaev, R. J. Stern, and P. R. Wolenski, *Nonsmooth Analysis and Control Theory*, vol. 178 of *Graduate Texts in Mathematics*, Springer, Berlin, Germany, 1998.

[13] K. S. Lau, "Almost Chebyshev subsets in reflexive Banach spaces," *Indiana University Mathematics Journal*, vol. 27, no. 5, pp. 791–795, 1978.