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Research Article

A Generalization for Theorems of Datko and Barbashin Type

Pham Viet Hai

MAS-SPMS, Nanyang Technological University, 21 Nanyang Link, Singapore 637371

Correspondence should be addressed to Pham Viet Hai; phamviethai86@gmail.com

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The goal of the paper is to give some characterizations for the uniform exponential stability of evolution families by unifying the discrete-time versions of the Barbashin-type theorem and the Datko-type theorem.

1. Introduction

In operator theory, a bounded linear operator family $\{\mathcal{U}(t,s)\}_{t\geq s\geq 0}$ $(\mathcal{U})$ is called an evolution family if

(i) $\mathcal{U}(t,t) = I$, the identity;

(ii) $\mathcal{U}(t,s) = \mathcal{U}(t,r)\mathcal{U}(r,s)$, the evolution property;

(iii) for each element x, the orbit $\mathcal{U}(\cdot,\cdot)x$ is continuous.

This notion occupies a particularly important role in representing solutions of the nonautonomous linear differential equation

$$u' = A(t)u \tag{1}$$

on Banach spaces when the operators $A(t)$, $t \geq 0$, are linear and unbounded. It is difficult to prove the existence of evolution families when studying on infinite dimensional Banach spaces. In fact, the conditions are obtained in several special cases of (1). We will not continue, in this present paper, the existence problem. Instead, we will assume that the evolution family exists and then study its stability.

We say that $\mathcal{U}$ admits a uniform exponential growth if there are two positive numbers M and ω such that, for every $t \geq s \geq 0$, $\|\mathcal{U}(t,s)\| \leq Me^{\omega(t-s)}$. When $\omega < 0$, $\mathcal{U}$ is called uniformly exponentially stable (u.e.s.). During the past decade, an increasing attention is devoted to the stability of evolution families. For example, in 1970, Datko showed that the evolution family with the uniform exponential growth is u.e.s. if, for each element x,

$$\sup_{s\geq 0}\int_s^{\infty}\|\mathcal{U}(\tau,s)x\|\,d\tau < \infty. \tag{2}$$

In his result, the integral is taken according to the first variable of the evolution family. A similar characterization is named by Barbashin in 1967 when integrating with respect to the second variable:

$$\sup_{t\geq 0}\int_0^{t}\|\mathcal{U}(t,\tau)\|\,d\tau < \infty. \tag{3}$$

Two results become the starting point for the works [1–3], where the discrete versions were established:

(i) (2) $\Leftrightarrow \sup_s \sum_{j=0}^{\infty}\|\mathcal{U}(j+s,s)x\| < \infty$.

(ii) (3) $\Leftrightarrow \sup_s \sum_{j=0}^{n}\|\mathcal{U}(n+s,j+s)\| < \infty$.

Particularly, we can find in [4] the initial studies on unifying the discrete-time versions of the Datko-type theorem and the Barbashin-type theorem. Let us restate the following: the uniform exponential stability is equivalent to the condition

$$\sup_{n,m}\sum_{j=0}^{n}\left\|\mathcal{U}\left(a_j+b_n+m,b_j+m\right)\right\| < \infty, \tag{4}$$

where the nondecreasing sequences (a_n) and (b_n) belong to class $\mathcal{S}$; that is, $(c_n)\in\mathcal{S}$ if $\sup_n(c_{n+1}-c_n) < \infty$. Following this idea, [5] provides another characterization:

$$\sup_{n,m}\sum_{j=0}^{n}\left\|\mathcal{U}\left(a_j+m,m\right)\right\|\left\|\mathcal{U}\left(b_n+m,b_j+m\right)\right\| < \infty. \tag{5}$$

It is worth mentioning here that conditions (4) and (5) become the Datko-type theorem when $b_n = 0$. When $a_n = 0$,

we get the Barbashin-type theorem. Naturally, we raise a question to consider the following assumption:

$$\sup_{n,m}\sum_{j=0}^{n}\left\|\mathcal{U}\left(a_jb_n+m,a_0b_j+m\right)\right\|<\infty. \tag{6}$$

The goal of the paper is to show that condition (6) is equivalent to the uniform exponential stability of the evolution family. The obtained result is an extension of classical theorems due to Barbashin and Datko.

2. Preliminaries

We start the paper with some notations. As usual, we denote by $\mathbb{R}_+$ and $\mathbb{N}$ the set of positive numbers and positive integers, respectively. We write $\mathbb{N}_p$ for the set of integers j with $j\geq p$. Let $(\mathbb{X},\|\cdot\|)$ be a Banach space. The norm on the space of bounded linear operators on $\mathbb{X}$ is also denoted by $\|\cdot\|$. Let $\mathcal{S}_\alpha$ denote the set of all nondecreasing positive sequences (b_n) with $\sup_{n\in\mathbb{N}}(b_{n+1}-b_n)\leq\alpha$. Let us denote by $\mathcal{F}$ the set of functionals F defined on the set of positive sequences with the following conditions:

(i) $0\leq F(s_1)\leq F(s_2)$ if $s_1\leq s_2$.

(ii) there is $c>0$ so that $F(\alpha\mathcal{X}_{\{n\}})\geq c\alpha$ for every $n\in\mathbb{N}$ and every $\alpha\geq 0$.

(iii) $\lim_{n\to\infty}F(\alpha\mathcal{X}_{\{0,\dots,n\}})=\infty$ for every $\alpha>0$.

Notation $\mathcal{X}_A$ means the characteristic function of a set A. The following lemma is derived from [6, 7].

Lemma 1. *If $F\in\mathcal{F}$ and $l>0$ then $\lim_{n\to\infty}R_F^l(n)=\infty$, where $R_F^l(n):=\inf_{\alpha\in(0,l]}(F(\alpha\mathcal{X}_{\{0,\dots,n\}})/\alpha^2)$.*

By $\mathbb{U}$, we denote the set of all functions $g(\cdot,\cdot):T:=\{(t,s):t\geq s\geq 0\}\to\mathbb{R}_+$ with the following properties:

(i) $g(t,r)\leq g(t,s)g(s,r)$ for all $t\geq s\geq r\geq 0$.

(ii) for each $\alpha>0$, there exists $M_\alpha>0$ such that $\sup_{t\in[0,\alpha],s\geq 0}g(t+s,s)\leq M_\alpha<\infty$.

For simplicity, we will use the symbol "sup" instead of $\sup_{n,m\in\mathbb{N}}$. In the whole paper, we always assume that the evolution family admits the uniform exponential growth with the constants M and ω. The following lemmas play an important part in the proof.

Lemma 2. *Let $g\in\mathbb{U}$. Assume that there exist $l,\delta>0$ and $(c_n)\in\mathcal{S}_\delta$ satisfying*

$$\begin{gathered}\lim_{n\to\infty}c_n=\infty,\\ \sup g\left(c_n+m,m\right)\leq l<\infty.\end{gathered} \tag{7}$$

Then g is bounded on T.

Proof. In the first step, we demonstrate that $\sup_{t\in\mathbb{R}_+,m\in\mathbb{N}}g(t+m,m)<\infty$. Indeed, for each $t\geq 0$, we have two cases. The first case is $t\leq c_0$. In this case, $g(t+m,m)\leq M_{c_0}$. The second case is $t\geq c_0$. By $\lim_{n\to\infty}c_n=\infty$ we can fix $n\in\mathbb{N}$ such that $t\in[c_n,c_{n+1}]$. Thus, $g(t+m,m)\leq g(t+m,c_n+m)g(c_n+m,m)\leq M_\delta l$. Hence, $\sup_{t\in\mathbb{R}_+,m\in\mathbb{N}}g(t+m,m)\leq\max\{M_{c_0},M_\delta l\}$. Next, we evaluate that $\sup_{t,s\in\mathbb{R}_+}g(t+s,s)<\infty$. For each $(t,s)\in\mathbb{R}_+^2$, there are also two cases.

If $t\leq 1$ then $g(t+s,s)\leq M_1$.

If $t\geq 1$ then $t+s\geq 1+[s]>s$. We estimate

$$\begin{aligned}g(t+s,s)&\leq g(t+s,1+[s])\,g(1+[s],s)\\&\leq M_1g(t+s,1+[s])\\&\leq M_1\max\left\{M_{c_0},M_\delta l\right\}.\end{aligned} \tag{8}$$

This implies the desired result. □

Lemma 3. *Let $g\in\mathbb{U}$. Assume that there are two constants $p\in\mathbb{N}_1$ and $q\in\mathbb{N}$ and $c<1$ such that $\sup_{m\in\mathbb{N}_q}g(p+m,m)\leq c$. Then there exist $K,\nu>0$ such that*

$$g(t+s,s)\leq Ke^{-\nu t},\quad\forall(t,s)\in\mathbb{R}_+^2. \tag{9}$$

Proof.

Step 1. By induction, we now prove that

$$g(kp+m,m)\leq c^k,\quad\forall m\in\mathbb{N}_q,\ k\in\mathbb{N}. \tag{10}$$

For $k=0$ and $k=1$, (10) holds. Assume that (10) is true when $k=n$. Consider $k=n+1$. We have that

$$\begin{aligned}&g((k+1)p+m,m)\\&\quad\leq g((k+1)p+m,kp+m)\,g(kp+m,m)\leq c\cdot c^k\\&\quad=c^{k+1}.\end{aligned} \tag{11}$$

Hence, (10) is true.

Step 2. We prove that there exists $K_1,\nu_1>0$ such that

$$g(t+m,m)\leq K_1e^{-\nu_1t},\quad\forall t\geq 0,\ m\in\mathbb{N}_q. \tag{12}$$

Indeed, for each $t\geq 0$, there exist $k\in\mathbb{N}$ and $r\in[0,p)$ such that $t=kp+r$. We have that

$$\begin{aligned}g(t+m,m)&\leq g(t+m,kp+m)\,g(kp+m,m)\\&\leq M_pc^k=M_pe^{k\ln c}\leq K_1e^{-\nu_1t},\end{aligned} \tag{13}$$

where $K_1:=M_p/c$ and $\nu_1:=-\ln c/p$.

Step 3. We prove that there exist $K_2,\nu_2>0$ such that

$$g(t+m,m)\leq K_2e^{-\nu_2t},\quad\forall t\geq 0,\ m\in\mathbb{N}. \tag{14}$$

If $q=0$ then, from the second step, we obtain the desired result. Now we consider the case as $q\in\mathbb{N}_1$. For each $(m,t)\in\mathbb{N}\times\mathbb{R}_+$, there are two situations as follows. The first situation is $t<q$. With this situation, we estimate

$$g(t+m,m)\leq M_q=M_qe^{\nu_1t}e^{-\nu_1t}\leq M_qe^{\nu_1q}e^{-\nu_1t}. \tag{15}$$

The second situation is $t \geq q$. We estimate

$$\begin{aligned} g(t+m,m) &\leq g(t+m,q+m)\, g(q+m,m) \\ &\leq M_q K_1 e^{-\nu_1(t-q)} = K_1 M_q e^{\nu_1 q} e^{-\nu_1 t}. \end{aligned} \tag{16}$$

We rewrite

$$g(t+m,m) \leq K_1 M_q e^{\nu_1 q} e^{-\nu_1 t}. \tag{17}$$

From (15) and (17), we can choose $\nu_2 := \nu_1$ and $K_2 := \max\{K_1 M_q e^{\nu_1 q}, M_q e^{\nu_1 q}\}$.

Step 4. For each $t, s \geq 0$, there are two situations. The first situation is $t \leq 1$. Then $g(t+s,s) \leq M_1$. The second situation is $t \geq 1$. Then $t+s \geq 1+[s] > s$. We have that

$$\begin{aligned} g(t+s,s) &\leq g(t+s,1+[s])\, g(1+[s],s) \\ &\leq K_2 e^{-\nu_2(t+s-1-[s])} M_1 \leq K_2 e^{-\nu_2 t} M_1 e^{\nu_2}. \end{aligned} \tag{18}$$

This implies the desired result. □

As a consequence, we have the following.

Lemma 4. *If there exist $l, \delta > 0$, and $(c_n) \in \mathcal{S}_\delta$ satisfying the hypotheses*

$$\begin{gathered} \lim_{n\to\infty} c_n = \infty, \\ \sup \left\| \mathcal{U}(c_n+m,m) \right\| \leq l < \infty, \end{gathered} \tag{19}$$

then $\mathcal{U}$ is uniformly bounded.

Lemma 5. *If there are two constants $p \in \mathbb{N}_1$ and $q \in \mathbb{N}$, and $c < 1$ such that $\|\mathcal{U}(p+m,m)\| \leq c$ for all $m \in \mathbb{N}_q$ then $\mathcal{U}$ is uniformly exponentially stable.*

3. Main Results

Given an evolution family $\mathcal{U}$, we define the mapping $\varphi(\cdot,\cdot,\cdot) : \mathbb{N}^3 \to \mathbb{R}_+$ by

$$\varphi(m,n,j) := \begin{cases} \left\| \mathcal{U}\left(c_j + a_n b_j + m, a_j b_0 + m\right) \right\|, & j \in \{0,\ldots,n\}, \\ 0, & j \notin \{0,\ldots,n\}. \end{cases} \tag{20}$$

The first characterization is given by the following.

Theorem 6. *$\mathcal{U}$ is u.e.s. if and only if there exist $\alpha, \beta, \delta, \mathcal{K}, L_1, L_2, L_3 > 0$, $(a_n) \in \mathcal{S}_\alpha$, $(b_n) \in \mathcal{S}_\beta$, $(c_n) \in \mathcal{S}_\delta$, and $F \in \mathcal{F}$ such that*

(1)

$$\begin{aligned} &\sup \sup_{j\leq n} \left\| \mathcal{U}\left(c_n + a_n b_n + m, c_j + a_n b_j + m\right) \right\| \leq L_1 \\ &\quad < \infty, \\ &\sup \left\| \mathcal{U}(a_n b_0 + m, m) \right\| \leq L_2 < \infty, \\ &\sup \left\| \mathcal{U}(c_n + a_n b_n + m, m) \right\| \leq L_3 < \infty, \end{aligned} \tag{21}$$

(2) $\sup F(\varphi(m,n,\cdot)) < \mathcal{K}$.

Proof. Let us prove the necessity. Suppose that $\mathcal{U}$ is uniformly exponentially stable; that is, $\|\mathcal{U}(t+s,s)\| \leq K e^{-\nu t}$. We have that

$$\sum_{j=0}^{n} \left\| \mathcal{U}(nj+m,m) \right\| \leq \sum_{j=0}^{n} K e^{-\nu n j} \leq \sum_{j=0}^{\infty} K e^{-\nu j}. \tag{22}$$

Thus we only take $a_j = b_j = j$, $\alpha = \beta = 1$, $\mathcal{K} = \sum_{j=0}^{\infty} K e^{-\nu j}$, and $L_j = K$; $F(s) = \sum_{n=0}^{\infty} s(n)$.

Now let us prove the sufficiency. Fix $(m,n) \in \mathbb{N}^2$. Let $j \in \{0,\ldots,n\}$. Denote $l := L_3/L_1L_2$. From the evolution property

$$\begin{aligned} &\mathcal{U}(c_n + a_n b_n + m, m) \\ &\quad = \mathcal{U}\left(c_n + a_n b_n + m, c_j + a_n b_j + m\right) \\ &\quad \cdot \mathcal{U}\left(c_j + a_n b_j + m, a_j b_0 + m\right) \mathcal{U}\left(a_j b_0 + m, m\right), \end{aligned} \tag{23}$$

we estimate

$$\begin{aligned} &\left\| \mathcal{U}(c_n + a_n b_n + m, m) \right\| \\ &\quad \leq L_1 L_2 \left\| \mathcal{U}\left(c_j + a_n b_j + m, a_j b_0 + m\right) \right\|. \end{aligned} \tag{24}$$

It follows that

$$\varphi(m,n,\cdot) \geq \left(\frac{\left\| \mathcal{U}(c_n + a_n b_n + m, m) \right\|}{L_1 L_2} \right) \mathcal{X}_{\{0,\ldots,n\}}. \tag{25}$$

Acting F on two both sides, we have that

$$\begin{aligned} \mathcal{K} &\geq F(\varphi(m,n,\cdot)) \\ &\geq F\left(\left(\frac{\left\| \mathcal{U}(c_n + a_n b_n + m, m) \right\|}{L_1 L_2} \right) \mathcal{X}_{\{0,\ldots,n\}} \right) \\ &\geq \left(\frac{\left\| \mathcal{U}(c_n + a_n b_n + m, m) \right\|}{L_1 L_2} \right)^2 R_F^l(n). \end{aligned} \tag{26}$$

Hence,

$$\sqrt{R_F^l(n)} \left\| \mathcal{U}(c_n + a_n b_n + m, m) \right\| \leq \sqrt{\mathcal{K}} L_1 L_2. \tag{27}$$

By Lemma 1, we can fix n_1 such that $\sqrt{R_F^l(n_1)} \geq 2Me^{\omega}\sqrt{\mathcal{K}} L_1 L_2$. Now we derive that $2Me^{\omega}\|\mathcal{U}(c_{n_1} + a_{n_1} b_{n_1} + m, m)\| \leq 1$. From the evolution property

$$\begin{aligned} &\mathcal{U}\left(\left[c_{n_1} + a_{n_1} b_{n_1}\right] + 1 + m, m\right) \\ &\quad = \mathcal{U}\left(\left[c_{n_1} + a_{n_1} b_{n_1}\right] + 1 + m, c_{n_1} + a_{n_1} b_{n_1} + m\right) \\ &\quad \cdot \mathcal{U}\left(c_{n_1} + a_{n_1} b_{n_1} + m, m\right), \end{aligned} \tag{28}$$

we estimate

$$\begin{aligned} &\left\| \mathcal{U}\left(\left[c_{n_1} + a_{n_1} b_{n_1}\right] + 1 + m, m\right) \right\| \\ &\quad \leq \left\| \mathcal{U}\left(\left[c_{n_1} + a_{n_1} b_{n_1}\right] + 1 + m, c_{n_1} + a_{n_1} b_{n_1} + m\right) \right\| \\ &\quad \cdot \left\| \mathcal{U}\left(c_{n_1} + a_{n_1} b_{n_1} + m, m\right) \right\| \\ &\quad \leq M e^{\omega} \left\| \mathcal{U}\left(c_{n_1} + a_{n_1} b_{n_1} + m, m\right) \right\| \leq \frac{1}{2}. \end{aligned} \tag{29}$$

Taking into account that $[c_{n_1} + a_{n_1} b_{n_1}] + 1$ does not depend on m. Using Lemma 5, $\mathcal{U}$ is uniformly exponentially stable. The proof completes. □

Our main result is as follows.

Theorem 7. *$\mathcal{U}$ is u.e.s. if and only if there exist $\alpha, \beta, \delta, \mathcal{K} > 0$, $(a_n) \in \mathcal{S}_\alpha$, $(b_n) \in \mathcal{S}_\beta$, $(c_n) \in \mathcal{S}_\delta$, and $F \in \mathcal{F}$ such that*

$$\sup F(\varphi(m,n,\cdot)) < \mathcal{K}. \tag{30}$$

Proof. The necessity is clear. Let us prove the sufficiency. Fix $(m,n) \in \mathbb{N}^2$. Let $z \in \{0,\ldots,n\}$. It is obvious that

$$\varphi(m,n,\cdot) \geq \|\mathcal{U}(c_z + a_n b_z + m, a_z b_0 + m)\| \mathcal{X}_{\{z\}}. \tag{31}$$

Acting F on two both sides of the inequality above, we obtain

$$\begin{aligned} \mathcal{K} &\geq F(\varphi(m,n,\cdot)) \\ &\geq F(\|\mathcal{U}(c_z + a_n b_z + m, a_z b_0 + m)\| \mathcal{X}_{\{z\}}) \\ &\geq c\|\mathcal{U}(c_z + a_n b_z + m, a_z b_0 + m)\|. \end{aligned} \tag{32}$$

Therefore,

$$\|\mathcal{U}(c_z + a_n b_z + m, a_z b_0 + m)\| \leq \frac{\mathcal{K}}{c}. \tag{33}$$

On the other hand, from the evolution property

$$\begin{aligned} &\mathcal{U}(c_z + a_n b_z + m, m) \\ &\quad = \mathcal{U}(c_z + a_n b_z + m, a_z b_0 + m)\,\mathcal{U}(a_z b_0 + m, m), \end{aligned} \tag{34}$$

we estimate

$$\begin{aligned} &\|\mathcal{U}(c_z + a_n b_z + m, m)\| \\ &\quad \leq \|\mathcal{U}(c_z + a_n b_z + m, a_z b_0 + m)\| \|\mathcal{U}(a_z b_0 + m, m)\| \\ &\quad \leq M e^{\omega a_z b_0} \|\mathcal{U}(c_z + a_n b_z + m, a_z b_0 + m)\| \\ &\quad \leq \frac{\mathcal{K} M e^{\omega a_z b_0}}{c}. \end{aligned} \tag{35}$$

Now we rewrite the last inequality

$$\|\mathcal{U}(c_z + a_n b_z + m, m)\| \leq \frac{\mathcal{K} M e^{\omega a_z b_0}}{c}. \tag{36}$$

We consider three cases as follows.

Case 1 ($\sup a_n = \infty$). In (36), let $z = 0$. We get

$$\|\mathcal{U}(c_0 + a_n b_0 + m, m)\| \leq \frac{\mathcal{K} M e^{\omega a_0 b_0}}{c}. \tag{37}$$

Using Lemma 5 for the case of the sequence $u_n := c_0 + a_n b_0$, there exists $L > 0$ such that

$$\sup_{t,s\geq 0} \|\mathcal{U}(t+s,s)\| \leq L < \infty. \tag{38}$$

Applying Theorem 6, $\mathcal{U}$ is uniformly exponentially stable.

Case 2 ($\mathcal{A} := \sup a_n < \infty$ and $\sup a_n b_n = \infty$). With these conditions, it follows from (36) that

$$\|\mathcal{U}(c_z + a_n b_z + m, m)\| \leq \frac{\mathcal{K} M e^{\omega a_z b_0}}{c} \leq \frac{\mathcal{K} M e^{\omega \mathcal{A} b_0}}{c}. \tag{39}$$

Let $n \to \infty$. Using the continuity of the mapping $(t,s) \to U(t,s)x$, we obtain

$$\|\mathcal{U}(c_z + \mathcal{A} b_z + m, m)\| \leq \frac{\mathcal{K} M e^{\omega \mathcal{A} b_0}}{c}. \tag{40}$$

Hence, by Lemma 5 for the case of the sequence $u_n := c_n + \mathcal{A} b_n$, there exists $L > 0$ such that

$$\sup_{t,s\geq 0} \|\mathcal{U}(t+s,s)\| \leq L < \infty. \tag{41}$$

Using Theorem 6, $\mathcal{U}$ is uniformly exponentially stable.

Case 3 ($\mathcal{D} := \sup a_n b_n < \infty$, $\sup c_n = \infty$). In (36), let $z = n$. We obtain

$$\|\mathcal{U}(c_n + a_n b_n + m, m)\| \leq \frac{\mathcal{K} M e^{\omega a_n b_0}}{c} \leq \frac{\mathcal{K} M e^{\omega \mathcal{D}}}{c}. \tag{42}$$

Let $n \to \infty$. Using the continuity of the mapping $(t,s) \to U(t,s)x$, we obtain

$$\|U(c_n + \mathcal{D} + m, m)\| \leq \frac{\mathcal{K} M e^{\omega \mathcal{D}}}{c}. \tag{43}$$

Using Lemma 5 for the case of the sequence $u_n := c_n + \mathcal{D}$, there exists $L > 0$ such that

$$\sup_{t,s\geq 0} \|U(t+s,s)\| \leq L < \infty. \tag{44}$$

Using Theorem 6, $\mathcal{U}$ is uniformly exponentially stable.

Case 4 ($\mathcal{D} := \sup a_n b_n < \infty$, $\mathcal{C} := \sup c_n < \infty$). In this case, we see that

$$\begin{aligned} &\|\mathcal{U}(c_n + a_n b_n + m, c_j + a_n b_j + m)\| \\ &\quad \leq M e^{\omega(c_n - c_j + a_n b_n - a_n b_j)} \leq M e^{\omega(\mathcal{C}+\mathcal{D})}, \\ &\|\mathcal{U}(a_n b_0 + m, m)\| \leq M e^{\omega a_n b_0} \leq M e^{\omega a_n b_n} \leq M e^{\omega \mathcal{D}}, \\ &\|U(c_n + a_n b_n + m, m)\| \leq M e^{\omega(c_n + a_n b_n)} \leq M e^{\omega(\mathcal{C}+\mathcal{D})}. \end{aligned} \tag{45}$$

Now we again apply Theorem 6 to obtain the desired result. □

From Theorem 6, we get the following.

Corollary 8. *$\mathcal{U}$ is u.e.s. if and only if there exist $\alpha, \beta > 0$, $(a_j) \in \mathcal{S}_\alpha$, and $(b_j) \in \mathcal{S}_\beta$ such that*

$$\sup \sum_{j=0}^{n} \|\mathcal{U}(a_n b_j + m, a_j b_0 + m)\| < \infty. \tag{46}$$

Proof. Using Theorem 7 for $c_j = 0$ and $F(s) = \sum_{j=0}^{\infty} s(j)$, the proof completes. □

Remark 9. Note that if we choose in Corollary 8 $b_j = 1$ ($a_j = 1$) then we obtain Barbashin's theorem (Datko's theorem).

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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