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Research Article

Multiplicative Isometries on Classes $M^p(X)$ of Holomorphic Functions

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In (Iida and Kasuga 2013), the authors described multiplicative (but not necessarily linear) isometries of $M^p(X)$ onto $M^p(X)$ in the case of positive integer $p \in \mathbb{N}$, where $M^p(X)$ $(p \geq 1)$ is included in the Smirnov class $N_*(X)$. In this paper, we will generalize the result to arbitrary (not necessarily positive integer) value of the exponents $0 < p < \infty$.

1. Introduction

Let n be a positive integer. The space of n-complex variables $z = (z_1, \ldots, z_n)$ is denoted by $\mathbb{C}^n$. The unit polydisk $\{z \in \mathbb{C}^n : |z_j| < 1, 1 \leq j \leq n\}$ is denoted by U^n and the distinguished boundary $\mathbb{T}^n$ is $\{z \in \mathbb{C}^n : |z_j| = 1, 1 \leq j \leq n\}$. The unit ball $\{z \in \mathbb{C}^n : \sum_{j=1}^{n} |z_j|^2 < 1\}$ is denoted by B_n and S_n is its boundary. In this paper X denotes the unit polydisk or the unit ball for $n \geq 1$ and ∂X denotes $\mathbb{T}^n$ for $X = U^n$ or S_n for $X = B_n$. The normalized (in the sense that $\sigma(\partial X) = 1$) Lebesgue measure on ∂X is denoted by $d\sigma$.

The Hardy space on X is denoted by $H^q(X)$ $(0 < q \leq \infty)$ and $\|\cdot\|_q$ denotes the norm on $H^q(X)$ $(1 \leq q \leq \infty)$.

The Nevanlinna class $N(X)$ on X is defined as the set of all holomorphic functions f on X such that

$$\sup_{0 \leq r < 1} \int_{\partial X} \log\left(1 + \left|f(rz)\right|\right) d\sigma(z) < \infty \tag{1}$$

holds. It is known that $f \in N(X)$ has a finite nontangential limit, also denoted by f, almost everywhere on ∂X.

The Smirnov class $N_*(X)$ is defined as the set of all $f \in N(X)$ which satisfy the equality

$$\sup_{0 \leq r < 1} \int_{\partial X} \log\left(1 + \left|f(rz)\right|\right) d\sigma(z) = \int_{\partial X} \log\left(1 + \left|f(z)\right|\right) d\sigma(z). \tag{2}$$

Define a metric

$$d_{N_*(X)}(f, g) = \int_{\partial X} \log\left(1 + \left|f(z) - g(z)\right|\right) d\sigma(z) \tag{3}$$

for $f, g \in N_*(X)$. With the metric $d_{N_*(X)}(\cdot, \cdot)$ the Smirnov class $N_*(X)$ is an F-algebra. Recall that an F-algebra is a topological algebra in which the topology arises from a complete metric. Complex-linear isometries on the Smirnov class are characterized by Stephenson in [1].

The Privalov class $N^p(X)$, $1 < p < \infty$, is defined as the set of all holomorphic functions f on X such that

$$\sup_{0 \leq r < 1} \int_{\partial X} \left(\log\left(1 + \left|f(rz)\right|\right)\right)^p d\sigma(z) < \infty \tag{4}$$

holds. It is well-known that $N^p(X)$ is a subalgebra of $N_*(X)$; hence, every $f \in N^p(X)$ has a finite nontangential limit almost everywhere on ∂X. Under the metric defined by

$$d_{N^p(X)}(f,g) = \left(\int_{\partial X} (\log(1+|f(z)-g(z)|))^p \, d\sigma(z)\right)^{1/p} \quad (5)$$

for $f, g \in N^p(X)$, $N^p(X)$ becomes an F-algebra (cf. [2]). Complex-linear isometries on $N^p(X)$ are investigated by Iida and Mochizuki [3] for one-dimensional case and by Subbotin [2, 4] for a general case.

Now we define the class $M^p(X)$. For $0 < p < \infty$, the class $M^p(X)$ is defined as the set of all holomorphic functions f on X such that

$$\int_{\partial X} \left(\log\left(1+\sup_{0\le r<1}|f(rz)|\right)\right)^p d\sigma(z) < \infty. \quad (6)$$

The class $M^p(X)$ with $p = 1$ in the case where $n = 1$ was introduced by Kim in [5]. As for $p > 0$ and $n > 1$, the class was considered in [6, 7]. For $f, g \in M^p(X)$, define a metric

$$d_{M^p(X)}(f,g) = \left\{\int_{\partial X} \left(\log\left(1+\sup_{0\le r<1}|f(rz)-g(rz)|\right)\right)^p d\sigma(z)\right\}^{\alpha_p/p}, \quad (7)$$

where $\alpha_p = \min(1, p)$. With this metric $M^p(X)$ is also an F-algebra (see [2]). Complex-linear surjective isometries on $M^p(X)$ are investigated by Subbotin [2, 4, 8].

It is well-known that the following inclusion relations hold:

$$H^q(X) \subsetneq N^p(X) \subsetneq M^1(X) \subsetneq N_*(X) \quad (0 < q \le \infty,\ p > 1). \quad (8)$$

Moreover, it is known that $N(X) \subsetneq M^p(X)$ $(0 < p < 1)$ [8]. As shown in [6], for any $p > 1$ the class $M^p(X)$ coincides with the class $N^p(X)$ and the metrics $d_{M^p(X)}$ and $d_{N^p(X)}$ are equivalent. Therefore, the topologies induced by these metrics are identical on the set $M^p(X) = N^p(X)$. But we note that in [4, Theorems 1 and 4] it is implied that the sets of linear isometries on $M^p(X)$ and $N^p(X)$ are different. It is known that $H^\infty(X)$ is a dense subalgebra of $M^p(X)$. The convergence in the metric is stronger than uniform convergence on compact subsets of X.

In [9], the authors described multiplicative (but not necessarily linear) isometries of $M^p(X)$ onto $M^p(X)$ in the case of positive integer $p \in \mathbb{N}$. In this paper, we will generalize the result to arbitrary (not necessarily positive integer) value of the exponents $0 < p < \infty$.

2. The Results

Proposition 1. *Let n be a positive integer and let X be either B_n or U^n. Let $0 < p < \infty$ and suppose that $A : M^p(X) \to M^p(X)$ is a surjective isometry. If A is 2-homogeneous in the sense that $A(2f) = 2A(f)$ holds for every $f \in M^p(X)$, then either*

$$A(f) = \alpha f \circ \Phi \quad \text{for every } f \in M^p(X) \quad (9)$$

or

$$A(f) = \alpha \overline{f \circ \overline{\Phi}} \quad \text{for every } f \in M^p(X), \quad (10)$$

where α is a complex number with the unit modulus and for $X = B_n$, Φ is a unitary transformation; for $X = U^n$, $\Phi(z_1, \ldots, z_n) = (\lambda_1 z_{i_1}, \ldots, \lambda_n z_{i_n})$, where $|\lambda_j| = 1$, $1 \le j \le n$ and $(i_1, \ldots, i_n)$ is some permutation of the integers from 1 through n.

To prove Proposition 1, we need the following lemmas.

Lemma 2 (see [4]). *Let $f \in H^p(X)$, $p \ge 1$. Then the norm*

$$\|f\|_{H^p_m} := \left\{\int_{\partial X} \sup_{0\le r<1} |f(rz)|^p \, d\sigma(z)\right\}^{1/p} \quad (11)$$

is equivalent to the standard norm in $H^p(X)$.

Lemma 3 (see [4]). *Let $0 < p < \infty$. Then*

$$\lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon^{p+1}} \left\{(\varepsilon t)^p - (\log(1+\varepsilon t))^p\right\} = \frac{p}{2} t^{p+1}, \quad t \ge 0. \quad (12)$$

We recall that the function $\varphi(x)$ on the interval $(0, +\infty)$ is said to be *completely monotone* if it is infinitely differentiable on $(0, +\infty)$ and

$$(-1)^n \varphi^{(n)}(x) \ge 0, \quad x > 0,\ n \in \mathbb{Z}_+. \quad (13)$$

Lemma 4 (see [8]). *The functions $((\log(1+x))/x)^p$ are completely monotone for all $p > 0$.*

Lemma 5 (see [8]). *If a completely monotone function $\varphi(x)$ on $(0, +\infty)$ can be continued to an infinitely differentiable function on $[0, +\infty)$, then the inequality*

$$\frac{(-1)^n}{x^n}\left\{\varphi(x) - \varphi(0) - \varphi'(0)x - \cdots - \frac{\varphi^{(n-1)}(0)}{(n-1)!}x^{n-1}\right\} \ge 0, \quad x > 0, \quad (14)$$

holds for any $n \in \mathbb{N}$ and $\varphi^{(n)}(0) \ne 0$ if only φ is not constant.

Lemma 6. *Let C be a cone of measurable functions on a measurable space with a measure (G, μ), and let A be a mapping from C to the set of measurable functions. Suppose that A is homogeneous with positive coefficients and*

$$\int_G (\log(1+|Af(x)|))^p \mu(dx) = \int_G (\log(1+|f(x)|))^p \mu(dx), \quad f \in C, \quad (15)$$

for some $p > 0$. *Then*

$$\int_G |Af(x)|^q \mu(dx) = \int_G |f(x)|^q \mu(dx), \quad f \in C, \quad (16)$$

for all $q = p + l$, *where* $l \in \mathbb{Z}_+$.

Proof. We follow [2, Lemma 2]. A is homogeneous with positive coefficients, so we have, using (15),

$$\begin{aligned} &\int_G \frac{(\log(1 + t|Af(x)|))^p}{t^p} \mu(dx) \\ &\quad = \int_G \frac{(\log(1 + t|f(x)|))^p}{t^p} \mu(dx), \quad f \in C, \end{aligned} \quad (17)$$

for any $t > 0$. Letting $t \to 0^+$, we obtain

$$\int_G |Af(x)|^p \mu(dx) = \int_G |f(x)|^p \mu(dx), \quad f \in C. \quad (18)$$

Next we argue by induction. Let $k \in \mathbb{N}$ and suppose that (16) holds for $q = p + 1, p + 2, \ldots, p + k - 1$. Then, for any $t > 0$ and any function $f \in C$, we have

$$\begin{aligned} &\int_G \frac{1}{t^{p+k}} \Bigg\{ (\log(1 + t|Af(x)|))^p \\ &\quad - \sum_{l=0}^{k-1} c_l (t|Af(x)|)^{p+l} \Bigg\} \mu(dx) \\ &\quad = \int_G \frac{1}{t^{p+k}} \Bigg\{ (\log(1 + t|f(x)|))^p \\ &\quad - \sum_{l=0}^{k-1} c_l (t|f(x)|)^{p+l} \Bigg\} \mu(dx), \end{aligned} \quad (19)$$

where c_l are the Taylor coefficients of the function $((\log(1 + x))/x)^p$ at zero.

Assume first that $f \in L^{p+k}(G,\mu)$. It is easy to see that the integrand on the right-hand side of (19) converges to $c_k|f(x)|^{p+k}$ as $t \to 0^+$; this integrand is of fixed sign by Lemmas 4 and 5 and is dominated by the function $C_k|f(x)|^{p+k}$ with some constant C_k. By the Lebesgue theorem on dominated convergence, the right-hand side of (19) converges to the integral of $c_k|f(x)|^{p+k}$. Therefore, the left-hand side of (19) has a finite limit and, by the Fatou theorem, the function $c_k|Af(x)|^{p+k}$ is integrable. Since $c_k \neq 0$ for any k by Lemmas 4 and 5, we deduce that $Af \in L^{p+k}(G,\mu)$, and repeating the above arguments, we see that the left-hand side of (19) converges to the integral of $c_k|Af(x)|^{p+k}$ as $t \to 0^+$. Therefore, passing to the limit in (19) as $t \to 0^+$ and dividing the result by $c_k \neq 0$, we obtain (16) for $q = p + k$. The case of $Af \in L^{p+k}(G,\mu)$ can be considered in a similar way. For $f, Af \notin L^{p+k}(G,\mu)$, relation (16) with $q = p + k$ is trivial. □

Proof of Proposition 1. Suppose first that $p \geq 1$. Let $f, g \in H^p(X)$. We easily confirm that, by utilizing Lemma 2 and the celebrated theorem of Mazur and Ulam [10], $A|_{H^p(X)}$ is a real-linear isometry in a way similar to [9, Proposition 1].

Next suppose that $0 < p < 1$. We define the class $H^p_m(X)$ $(0 < p < 1)$ as the set of all holomorphic functions f on X such that

$$d_{H^p_m}(f, 0) := \int_{\partial X} \sup_{0 \leq r < 1} |f(rz)|^p \, d\sigma(z) < \infty \quad (20)$$

and define a metric $d_{H^p_m}(f, g) = d_{H^p_m}(f - g, 0)$ for $f, g \in H^p_m(X)$. If $A : M^p(X) \to M^p(X)$ is a surjective isometry and A is 2-homogeneous in the sense that $A(2f) = 2A(f)$ holds for every $f \in M^p(X)$, we confirm that $A : H^p_m(X) \to H^p_m(X)$ is also a surjective isometry.

For $0 < p < 1$, let $f, g \in H^p(X)$. We have $A(f)/2^m = A(f/2^m)$ $(m \in \mathbb{N})$ since $A(2f) = 2A(f)$. Then the following equality holds:

$$\begin{aligned} &\int_{\partial X} \sup_{0 \leq r < 1} \left| \frac{f(rz)}{2^m} \right|^p d\sigma(z) \\ &\quad - \int_{\partial X} \left(\log\left(1 + \sup_{0 \leq r < 1} \left| \frac{f(rz)}{2^m} \right| \right) \right)^p d\sigma(z) \\ &\quad = \int_{\partial X} \sup_{0 \leq r < 1} \left| \frac{(Af)(rz)}{2^m} \right|^p d\sigma(z) \\ &\quad - \int_{\partial X} \left(\log\left(1 + \sup_{0 \leq r < 1} \left| \frac{(Af)(rz)}{2^m} \right| \right) \right)^p d\sigma(z). \end{aligned} \quad (21)$$

Therefore, it follows that

$$\begin{aligned} &\int_{\partial X} \frac{1}{(1/2^m)^{p+1}} \Bigg\{ \left(\frac{1}{2^m} \sup_{0 \leq r < 1} |f(rz)| \right)^p \\ &\quad - \left(\log\left(1 + \frac{1}{2^m} \sup_{0 \leq r < 1} |f(rz)| \right) \right)^p \Bigg\} d\sigma(z) \\ &\quad = \int_{\partial X} \frac{1}{(1/2^m)^{p+1}} \Bigg\{ \left(\frac{1}{2^m} \sup_{0 \leq r < 1} |(Af)(rz)| \right)^p \\ &\quad - \left(\log\left(1 + \frac{1}{2^m} \sup_{0 \leq r < 1} |(Af)(rz)| \right) \right)^p \Bigg\} d\sigma(z). \end{aligned} \quad (22)$$

Using the elementary inequalities $\log(1 + xy) \leq \log(1 + x) + \log(1 + y)$ $(x \geq 0, y \geq 0)$ and $(a + b)^p \leq 2^p(a^p + b^p)$ $(a \geq 0, b \geq 0, p > 0)$, we confirm that the integrand on the left-hand side of (22) is dominated by an $L^1(\partial X)$-function. The integrand on the right-hand side of (22) is also dominated by an $L^1(\partial X)$-function in the same way. Applying the Lebesgue theorem on dominated convergence and Lemma 3 on both sides of (22), we have the equality

$$\begin{aligned} &\int_{\partial X} \frac{p}{2} \left(\sup_{0 \leq r < 1} |f(rz)| \right)^{p+1} d\sigma(z) \\ &\quad = \int_{\partial X} \frac{p}{2} \left(\sup_{0 \leq r < 1} |(Af)(rz)| \right)^{p+1} d\sigma(z). \end{aligned} \quad (23)$$

Hence, we obtain

$$\int_{\partial X} \sup_{0\le r<1} |f(rz)|^{p+1}\, d\sigma(z) = \int_{\partial X} \sup_{0\le r<1} |(Af)(rz)|^{p+1}\, d\sigma(z). \tag{24}$$

The equivalence of the norms $\|\cdot\|_{p+1}$ and $\|\cdot\|_{H_m^{p+1}}$ guarantees that $A : H^{p+1}(X) \to H^{p+1}(X)$ is a surjective isometry. By using Mazur-Ulam theorem again, $A|_{H^{p+1}(X)}$ is a real-linear isometry since $H^{p+1}(X)$ is a normed vector space and $A(0) = 0$.

We consider an arbitrary function $f \in M^p(X)$ $(0 < p < \infty)$ and the cone $C_f := \{\lambda Mf(\zeta) \mid \lambda > 0\}$ generated by f. Here $Mf(\zeta) = \sup_{0\le r<1}|f(r\zeta)|$ is the radial maximal function for f. Moreover, consider the following mapping on this cone:

$$\widetilde{A} : \lambda Mf(\zeta) \longmapsto \lambda M(Af)(\zeta) = M(A[\lambda f])(\zeta), \quad \zeta \in \partial X,\ \lambda \ge 0. \tag{25}$$

Since A is isometric with respect to the metric d_{M^p}, it follows that the assumptions of Lemma 6 hold on the cone C_f.

$A : M^p(X) \to M^p(X)$ is a surjective isometry, so the equation

$$\int_{\partial X} \left(\log\left(1 + \sup_{0\le r<1} |f(rz)|\right)\right)^p d\sigma(z) = \int_{\partial X} \left(\log\left(1 + \sup_{0\le r<1} |(Af)(rz)|\right)\right)^p d\sigma(z) \tag{26}$$

guarantees the equalities

$$\int_{\partial X} \left(\sup_{0\le r<1} |f(rz)|\right)^{p+l} d\sigma(z) = \int_{\partial X} \left(\sup_{0\le r<1} |(Af)(rz)|\right)^{p+l} d\sigma(z) \tag{27}$$

for all $l \in \mathbb{Z}_+$. Therefore, A is isometric in the norm H_m^{p+l} $(p > 0, l = 0, 1, 2, \ldots)$.

Since $d\sigma$ is a finite measure, we verify that

$$\lim_{l\to\infty} \|f\|_{H_m^{p+l}} = \|f\|_{H_m^\infty} \tag{28}$$

holds for every $f \in H^\infty(X)$, and it is clear that $\|f\|_{H_m^\infty} = \|f\|_\infty$. Moreover, $\|f\|_p = \|A(f)\|_p$ for every $f \in H^\infty(X)$ and $\lim_{p\to\infty}\|A(f)\|_p = \|A(f)\|_\infty$, so we have $A(f) \in H^\infty(X)$ and $\|f\|_\infty = \|A(f)\|_\infty$ for every $f \in H^\infty(X)$. Similarly we see that $f \in H^\infty(X)$ if $A(f)$ belongs to $H^\infty(X)$. Therefore, $A|_{H^\infty(X)}$ is a surjective isometry with respect to $\|\cdot\|_\infty$ from $H^\infty(X)$ onto itself. We may suppose that $H^\infty(X)$ is a uniform algebra on the maximal ideal space and the maximal ideal space is connected by the Šilov idempotent theorem; hence, we see that $A|_{H^\infty(X)}$ is complex-linear or conjugate linear by [11, Theorem]. As for the rest of this proof, we follow the proof of [9, Proposition 1]. □

Finally we consider multiplicative isometries from $M^p(X)$ $(0 < p < \infty)$ onto itself. Recall that $A : M^p(X) \to M^p(X)$ is *multiplicative* if $A(fg) = A(f)A(g)$ for every $f, g \in M^p(X)$.

The following theorem is proved by the same method as [9, Theorem 2]; therefore, we do not prove it here.

Theorem 7. *Let $0 < p < \infty$ and A be a multiplicative (not necessarily linear) isometry from $M^p(X)$ onto itself. Then there exists a holomorphic automorphism Φ on X such that either of the following holds:*

$$A(f) = f \circ \Phi \quad \text{for every } f \in M^p(X) \tag{29}$$

or

$$A(f) = \overline{f \circ \overline{\Phi}} \quad \text{for every } f \in M^p(X), \tag{30}$$

where Φ is a unitary transformation for $X = B_n$; $\Phi(z_1, \ldots, z_n) = (\lambda_1 z_{i_1}, \ldots, \lambda_n z_{i_n})$ for $X = U^n$, where $|\lambda_j| = 1$ for every $1 \le j \le n$ and $(i_1, \ldots, i_n)$ is some permutation of the integers from 1 through n.

Remark 8. We note that surjective multiplicative isometries of the class $M^p(X)$ $(0 < p < \infty)$ have the same form as surjective multiplicative isometries of $M^p(X)$ $(p \in \mathbb{N})$ [9, Theorem 2], the Smirnov class [12, Theorem 2.2], and the Privalov class [13, Corollary 3.4].

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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References

[1] K. Stephenson, "Isometries of the Nevanlinna class," *Indiana University Mathematics Journal*, vol. 26, no. 2, pp. 307–324, 1977.

[2] A. V. Subbotin, "Isometries of Privalov spaces of holomorphic functions of several variables," *Journal of Mathematical Sciences*, vol. 135, no. 1, pp. 2794–2802, 2006.

[3] Y. Iida and N. Mochizuki, "Isometries of some F-algebras of holomorphic functions," *Archiv der Mathematik*, vol. 71, no. 4, pp. 297–300, 1998.

[4] A. V. Subbotin, "Linear isometry groups of Privalov's spaces of holomorphic functions of several variables," *Doklady Mathematics*, vol. 60, no. 1, pp. 77–79, 1999.

[5] H. O. Kim, "On an F-algebra of holomorphic functions," *Canadian Journal of Mathematics*, vol. 40, no. 3, pp. 718–741, 1988.

[6] B. R. Choe and H. O. Kim, "On the boundary behavior of functions holomorphic on the ball," *Complex Variables, Theory and Application*, vol. 20, no. 1–4, pp. 53–61, 1992.

[7] H. O. Kim and Y. Y. Park, "Maximal functions of plurisubharmonic functions," *Tsukuba Journal of Mathematics*, vol. 16, no. 1, pp. 11–18, 1992.

[8] A. V. Subbotin, "Groups of linear isometries of spaces M^q of holomorphic functions of several complex variables," *Mathematical Notes*, vol. 83, no. 3, pp. 437–440, 2008.

[9] Y. Iida and K. Kasuga, "Multiplicative isometries on some F-algebras of holomorphic functions," *Journal of Function Spaces and Applications*, vol. 2013, Article ID 681637, 3 pages, 2013.

[10] S. Mazur and S. Ulam, "Sur les transformationes isométriques d'espaces vectoriels normés," *Comptes Rendus de l'Académie des Sciences Paris*, vol. 194, pp. 946–948, 1932.

[11] A. J. Ellis, "Real characterizations of function algebras amongst function spaces," *Bulletin of the London Mathematical Society*, vol. 22, no. 4, pp. 381–385, 1990.

[12] O. Hatori and Y. Iida, "Multiplicative isometries on the Smirnov class," *Central European Journal of Mathematics*, vol. 9, no. 5, pp. 1051–1056, 2011.

[13] O. Hatori, Y. Iida, S. Stević, and S.-I. Ueki, "Multiplicative isometries on F-algebras of holomorphic functions ," *Abstract and Applied Analysis*, vol. 2012, Article ID 125987, 16 pages, 2012.