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Research Article

Properties of Functions in the Wiener Class $BV_p[a,b]$ for $0<p<1$

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We will investigate properties of functions in the Wiener class $BV_p[a,b]$ with $0 < p < 1$. We prove that any function in $BV_p[a,b]$ $(0 < p < 1)$ can be expressed as the difference of two increasing functions in $BV_p[a,b]$. We also obtain the explicit form of functions in $BV_p[a,b]$ and show that their derivatives are equal to zero a.e. on $[a,b]$.

1. Introduction

Let $0 < p < \infty$. We say that a real valued function f on $[a,b]$ is of bounded p-variation and is denoted by $f \in BV_p[a,b]$, if

$$V_p f = \sup_T \left(\sum_{k=1}^{n} \left| f(x_k) - f(x_{k-1}) \right|^p \right)^{1/p} < \infty, \qquad (1)$$

where the supremum is taken over all partitions $T : a = x_0 < x_1 < \cdots < x_n = b$. When $p = 1$, we get the well-known Jordan bounded variation $BV[a,b]$; and when $1 < p < \infty$, we get Wiener's definition of bounded p-variation. There are many other generalizations of BV, such as bounded Φ-variation in the sense of Young (see [1]) and Waterman's Λ-bounded variation (see [2]). The class BV_p and generalizations of BV have been studied mainly because of their applicability to the theory of Fourier series and some good approximative properties (see, e.g., [1–7]).

However, it should be mentioned that results of most papers deal mostly with the case $p \ge 1$. This is because that in this case $BV_p[a,b]$ is a Banach space with the norm $\|f\|_{BV_p} = |f(a)| + V_p f$ (see, e.g., [3]). In the case $0 < p < 1$, $BV_p[a,b]$ is no longer a Banach space and has not been studied as far as we know. Nevertheless, functions in $BV_p[a,b]$ $(0 < p < 1)$ have many interesting properties; for example, their derivatives are equal to zero a.e. on $[a,b]$.

In this paper, we will investigate properties of functions in the class $BV_p[a,b]$ with $0 < p < 1$. We will show that $BV_p[a,b]$ is a Frechet space with the quasinorm

$$q(f) = |f(a)|^p + (V_p f)^p. \qquad (2)$$

We will get the Jordan type decomposition theorem which says that any function in $BV_p[a,b]$ $(0 < p < 1)$ can be expressed as the difference of two increasing functions in $BV_p[a,b]$. We also get the representation theorem which gives the explicit form of functions in $BV_p[a,b]$ $(0 < p < 1)$.

2. Statement of Main Results

Clearly, for any fixed $p \in (0,1)$, the Wiener class $BV_p[a,b]$ is a linear space. We define the functional q on $BV_p[a,b]$ by

$$q(f) = |f(a)|^p + (V_p f)^p = |f(a)|^p + \sup_T \sum_{k=1}^{n} \left| f(x_k) - f(x_{k-1}) \right|^p, \qquad (3)$$
$$f \in BV_p[a,b].$$

From the inequality $(a+b)^p \le a^p + b^p$ $(a, b \ge 0,\ 0 < p < 1)$, we get that $q(f+g) \le q(f) + q(g)$. It then follows that q is a quasinorm on $BV_p[a,b]$.

Our first result claims that $BV_p[a,b]$ $(0 < p < 1)$ equipped with the quasinorm q is a Frechet space.

Theorem 1. *The Wiener class $BV_p[a,b]$ $(0 < p < 1)$ equipped with the quasinorm q is a Frechet space.*

From the inequality

$$\left(\sum_{i=1}^{\infty} a_i^{p_2}\right)^{1/p_2} \le \left(\sum_{i=1}^{\infty} a_i^{p_1}\right)^{1/p_1}, \quad a_i \ge 0,\ 0 < p_1 \le p_2 < \infty, \tag{4}$$

we get that, for any $f \in BV_{p_1}[a,b]$,

$$V_{p_2} f \le V_{p_1} f, \tag{5}$$

which means that $BV_{p_1}[a,b] \subseteq BV_{p_2}[a,b]$. Specially, for $0 < p < 1$, $BV_p[a,b] \subseteq BV_1[a,b] \equiv BV[a,b]$. This implies that $BV_p[a,b]$ functions are bounded, and the discontinuities of a $BV_p[a,b]$ function are simple and, therefore, at most denumerable (see [8, Theorem 13.7 and Lemma 13.2]). By the Jordan decomposition theorem, we know that every function f in $BV[a,b]$ can be expressed as the difference of two increasing functions g and h defined on $[a,b]$ (see [8, Corollary 13.6]). If $f \in BV_p[a,b] \subseteq BV[a,b]$, we can require that the above increasing functions g and h are still in $BV_p[a,b]$. This is our next theorem.

Theorem 2 (Jordan type decomposition theorem). *Any function in $BV_p[a,b]$ $(0 < p < 1)$ can be expressed as the difference of two increasing functions in $BV_p[a,b]$.*

Let $t \in [a,b]$, $d > 0$, and $0 \le d' \le d$. We set

$$h_{t,d,d'}(x) = \begin{cases} 0, & x < t, \\ d', & x = t, \\ d, & x > t. \end{cases} \tag{6}$$

Then $h_{t,d,d'}(x)$ is increasing on $[a,b]$ with only one discontinuity point t. Also, $(h_{t,d,d'}(x))' = 0$ for $x \ne t$.

Let f be an increasing function in $BV_p[a,b]$ $(0 < p < 1)$. Denote by $A \equiv A(f)$ the set of points of discontinuity of f. Then A is at most countable (see [8, Theorem 2.17]). Since f is increasing, we get that, for any $t \in A$, the right and left limits $f(t+0)$ and $f(t-0)$ of the function f at t exist, $f(t+0) - f(t-0) > 0$, and $0 \le f(t) - f(t-0) \le f(t+0) - f(t-0)$. For $t \in A$, we define

$$\widetilde{h_t}(x) \equiv \widetilde{h_{t,f}}(x) = h_{t, f(t+0)-f(t-0), f(t)-f(t-0)}(x). \tag{7}$$

Our next theorem characterizes the form of an increasing function in $BV_p[a,b]$. Any increasing function f in $BV_p[a,b]$ must be as follows:

$$f(x) = \sum_{n=1}^{N} h_{t_n,d_n,d_n'}(x) + c, \tag{8}$$

where $N \le \infty$, $t_n \in [a,b]$, $d_n > 0$, $d_n' \in [0,d_n]$, and $\sum_{n=1}^{N} d_n^p < \infty$.

Theorem 3. *(1) If $f(x) = c + \sum_{n=1}^{N} h_{t_n,d_n,d_n'}(x)$, where $N \le \infty$, $t_n \in [a,b]$, $d_n > 0$, and $d_n' \in [0,d_n]$, then $f \in BV_p[a,b]$ $(0 < p < 1)$ if and only if $\sum_{n=1}^{N} d_n^p < \infty$. In this case,*

$$\left(\sum_{n=1}^{N} d_n^p\right)^{1/p} \le V_p(f) \le \left(2\sum_{n=1}^{N} d_n^p\right)^{1/p}. \tag{9}$$

(2) Let f be an increasing function in $BV_p[a,b]$ $(0 < p < 1)$. Then $f(x) = \sum_{t\in A} \widetilde{h_t}(x) + c$, where c is a constant, A is the set of points of discontinuity of f, and $\widetilde{h_t}(x)$ is defined by (7).

Finally, for an increasing function f in $BV_p[a,b]$ $(0 < p < 1)$, by Theorem 3 we have $f(x) = \sum_{t\in A} \widetilde{h_t}(x) + c$, where A is the set of points of discontinuity of f and at most countable. Since $(\widetilde{h_t}(x))' = 0$, a.e. $x \in [a,b]$, by the Fubini term by term differentiation theorem (see [9, Proposition 4.6]), we get $f'(x) = 0$, a.e. $x \in [a,b]$. By Theorem 2, any function f in $BV_p[a,b]$ can be expressed as the difference of two increasing functions $g(x)$ and $r(x)$ in $BV_p[a,b]$. Applying Theorem 3, we get the representation theorem of functions in $BV_p[a,b]$ $(0 < p < 1)$ as follows.

Corollary 4. *Let $f \in BV_p[a,b]$ $(0 < p < 1)$. Then f can be expressed in the following form:*

$$f(x) = g(x) - r(x) = \sum_{t\in A_1} \widetilde{h_{t,g}}(x) - \sum_{t\in A_2} \widetilde{h_{t,r}}(x) + c, \tag{10}$$

where c is a constant, $g(x)$, $r(x)$ are increasing functions in $BV_p[a,b]$, $\widetilde{h_{t,g}}(x)$ and $\widetilde{h_{t,r}}$ are defined by (7), $A_1, A_2 \subseteq A$, and A_1, A_2, A are the sets of points of discontinuity of g, r, and f, respectively. Furthermore, $f'(x) = 0$, a.e. $x \in [a,b]$.

3. Proofs of Theorems 1–3

Proof of Theorem 1. It suffices to prove that $BV_p[a,b]$ is complete. Let $\{f_n\}$ be a Cauchy sequence in $BV_p[a,b]$; that is, $q(f_n - f_m) = |f_n(a) - f_m(a)|^p + (V_p(f_n - f_m))^p \to 0$ as $n, m \to \infty$. For any $\xi \in [a,b]$, using the partition $T : a \le \xi \le b$ and the definition of $V_p f$, we get that $\{f_n(\xi)\}$ is a Cauchy sequence in $\mathbb{R}$ and converges to a number denoted by $f(\xi)$. For any $\varepsilon > 0$, there exists an integer N such that $q(f_n - f_m) \le \varepsilon$ for $m, n > N$. Let $T : a = x_0 < x_1 < \cdots < x_k = b$ be an arbitrary partition of $[a,b]$. Then

$$\begin{aligned} &\left|f_m(a) - f_n(a)\right|^p \\ &\quad + \sum_{i=1}^{k} \left|(f_m - f_n)(x_i) - (f_m - f_n)(x_{i-1})\right|^p \\ &\quad \le q(f_n - f_m) \le \varepsilon. \end{aligned} \tag{11}$$

Letting $m \to \infty$, we get that

$$\left|f(a) - f_n(a)\right|^p + \sum_{i=1}^{k} \left|(f - f_n)(x_i) - (f - f_n)(x_{i-1})\right|^p \le \varepsilon. \tag{12}$$

Taking the supremum over all partitions T, we have $q(f - f_n) \le \varepsilon$ for $n > N$. This means that $f = (f - f_n) + f_n \in BV_p[a,b]$, and $q(f - f_n) \to 0$ as $n \to \infty$. Hence, $BV_p[a,b]$ $(0 < p < 1)$ is complete. Theorem 1 is proved. □

Proof of Theorem 2. Suppose that $f \in BV_p[a,b]$ $(0 < p < 1)$. Since $f \in BV_p[a,b] \subset BV[a,b]$, by the Jordan decomposition theorem (see [8, Corollary 13.6]), we have $f(x) = g(x) - r(x)$, where $g(x)$, $r(x)$ are increasing functions on $[a,b]$. Indeed, we can choose $g(x)$ to be $V_a^x(f)$, the total variation function of f defined by

$$V_a^x(f) = \sup_T \left\{ \sum_{i=1}^n \left| f(x_i) - f(x_{i-1}) \right| \right\}, \tag{13}$$

where the supremum is taken over all partitions $T : a = x_0 < x_1 < \cdots < x_n = x$ of $[a,x]$, $r(x) = V_a^x(f) - f(x)$. It suffices to show that $g(x) = V_a^x(f) \in BV_p[a,b]$. For any fixed partition $T : a = x_0 < x_1 < \cdots < x_n = b$, we note that

$$\begin{aligned} \left| g(x_i) - g(x_{i-1}) \right|^p &= \left| V_{x_{i-1}}^{x_i} f \right|^p \\ &= \sup_{T_i} \left(\sum_{j=1}^{m_i} \left| f(\xi_{i,j}) - f(\xi_{i,j-1}) \right| \right)^p \\ &\le \sup_{T_i} \sum_{j=1}^{m_i} \left| f(\xi_{i,j}) - f(\xi_{i,j-1}) \right|^p, \end{aligned} \tag{14}$$

where the supremum is taken over all partitions $T_i : x_{i-1} = \xi_{i,1} < \xi_{i,2} < \cdots < \xi_{i,m_i} = x_i$ of $[x_{i-1}, x_i]$. It follows that

$$\begin{aligned} \sum_{i=1}^n \left| g(x_i) - g(x_{i-1}) \right|^p &\le \sum_{i=1}^n \sup_{T_i} \sum_{j=1}^{m_i} \left| f(\xi_{i,j}) - f(\xi_{i,j-1}) \right|^p \\ &= \sup_{T_i,\ 1 \le i \le n} \sum_{i=1}^n \sum_{j=1}^{m_i} \left| f(\xi_{i,j}) - f(\xi_{i,j-1}) \right|^p \\ &\le \left(V_p f \right)^p, \end{aligned} \tag{15}$$

which implies $g \in BV_p[a,b]$. This completes the proof of Theorem 2. □

To prove Theorem 3, we introduce the next lemma.

Lemma 5. *If $f \in BV_p[a,b] \bigcap C[a,b]$ $(0 < p < 1)$, then f is a constant function.*

Proof. It suffices to show that, for any $d \in [a,b]$, $f(d) = f(a)$. Assume that there exists $d \in (a,b]$ such that $f(d) \ne f(a)$. Without loss of generality, we assume that $f(a) < f(d)$. Since $f \in C[a,b]$, there exist $n - 1$ points $\xi_1, \xi_2, \ldots, \xi_{n-1}$ such that $a = \xi_0 < \xi_1 < \cdots < \xi_{n-1} < \xi_n = d$ and $f(\xi_i) = f(a) + ((f(d) - f(a))/n)i$. Hence,

$$\begin{aligned} \left(V_p f \right)^p &\ge \sum_{i=1}^n \left| f(\xi_i) - f(\xi_{i-1}) \right|^p \\ &= n^{1-p} \left| f(d) - f(a) \right|^p \longrightarrow \infty, \end{aligned} \tag{16}$$

as $n \to \infty$, which implies that $f \notin BV_p[a,b]$. This leads to a contradiction. Lemma 5 is proved. □

Proof of Theorem 3. (1) Without loss of generality, we may assume that $N = \infty$. Let $T : a = y_0 < y_1 < \cdots < y_m = b$ be a partition of $[a,b]$. For j, $1 \le j \le m$, we note that

$$\begin{aligned} &\left| f(y_j) - f(y_{j-1}) \right|^p \\ &\quad = \left| \sum_{n=1}^\infty \left(h_{t_n,d_n,d_n'}(y_j) - h_{t_n,d_n,d_n'}(y_{j-1}) \right) \right|^p \\ &\quad = \left| \sum_{\substack{n \\ y_{j-1} < t_n < y_j}} d_n + \sum_{\substack{n \\ t_n = y_{j-1}}} \left(d_n - d_n' \right) + \sum_{\substack{n \\ t_n = y_j}} d_n' \right|^p \\ &\quad \le \sum_{\substack{n \\ y_{j-1} \le t_n \le y_j}} d_n^p, \end{aligned} \tag{17}$$

where an empty sum denotes 0. It follows that

$$\sum_{j=1}^m \left| f(y_j) - f(y_{j-1}) \right|^p \le \sum_{j=1}^m \left(\sum_{\substack{n \\ y_{j-1} \le t_n \le y_j}} d_n^p \right) \le 2 \sum_{n=1}^\infty d_n^p. \tag{18}$$

Taking the supremum over all partitions of $[a,b]$, we obtain that

$$\left(V_p f \right)^p \le 2 \sum_{n=1}^\infty d_n^p. \tag{19}$$

On the other hand, for any fixed m, by renumbering $\{t_n\}_{n=1}^m$ if necessary, we may assume that $a \le t_1 < t_2 < \cdots < t_m \le b$. We set $y_i = ((t_i + t_{i+1})/2)$ $(1 \le i \le m - 1)$. Then $T : a = y_0 < y_1 < y_2 < \cdots < y_{m-1} < y_m = b$ is a partition of $[a,b]$. It follows that

$$\begin{aligned} \left(V_p f \right)^p &\ge \sum_{j=1}^m \left| f(y_j) - f(y_{j-1}) \right|^p \ge \sum_{j=1}^m \left(\sum_{\substack{n \\ y_{j-1} < t_n < y_j}} d_n \right)^p \\ &\ge \sum_{j=1}^m d_j^p. \end{aligned} \tag{20}$$

Letting $m \to \infty$, we get

$$V_p f \ge \left(\sum_{n=1}^\infty d_n^p \right)^{1/p}. \tag{21}$$

Combining (19) with (21), we get (9). Hence, $f \in BV_p[a,b]$ $(0 < p < 1)$ if and only if $\sum_{n=1}^\infty d_n^p < \infty$.

(2) Let f be an increasing function in $BV_p[a,b]$ $(0 < p < 1)$ and A the set of points of discontinuity of f on $[a,b]$. We set $h_f(x) = \sum_{t \in A} \widetilde{h}_t(x)$, where $\widetilde{h}_t(x)$ is defined by (7). Similar to the proof of (21), we have

$$\sum_{t \in A} \left(f(t+0) - f(t-0) \right)^p \le \left(V_p f \right)^p < \infty. \tag{22}$$

Applying the above proved result, we obtain that $h_f(x) \in BV_p[a,b]$. We set $g(x) = f(x) - h_f(x)$; then $g \in BV_p[a,b]$. We will show that $g(x)$ is continuous on $[a,b]$.

Indeed, for $x \in [a,b]$, we have

$$\begin{aligned} \sum_{t\in A} \widetilde{h_t}(x) &\le \sum_{t\in A} \left(f(t+0) - f(t-0)\right) \\ &\le \left(\sum_{t\in A} \left(f(t+0) - f(t-0)\right)^p\right)^{1/p} \\ &\le V_p f < \infty. \end{aligned} \tag{23}$$

By Weierstrass M-test (see [10, Theorem 7.10]), we get that the series $\sum_{t\in A} \widetilde{h_t}(x)$ converges uniformly on $[a,b]$. For $x_0 \in [a,b]\setminus A$, $\widetilde{h_t}(x)$ $(t \in A)$ is continuous at x_0, so $h_f(x) = \sum_{t\in A} \widetilde{h_t}(x)$ is also continuous at x_0. It follows that $g(x)$ is continuous at x_0 for $x_0 \in [a,b]\setminus A$.

For $x_0 \in A$, we set $u(x) = \sum_{t\in A\setminus\{x_0\}} \widetilde{h_t}(x)$. Then $u(x)$ is continuous at x_0 and $h_f(x) = u(x) + \widetilde{h_{x_0}}(x)$. Hence,

$$\begin{gathered} h_f(x_0+0) = u(x_0) + \left(f(x_0+0) - f(x_0-0)\right), \\ h_f(x_0-0) = u(x_0), \\ h_f(x_0) = u(x_0) + \left(f(x_0) - f(x_0-0)\right). \end{gathered} \tag{24}$$

Thus,

$$g(x_0+0) = g(x_0) = g(x_0-0) = f(x_0-0) - u(x_0), \tag{25}$$

from which we can deduce that g is continuous at x_0. Hence, $g(x) \in C[a,b]$.

Since $g(x) \in C[a,b] \cap BV_p[a,b]$, it follows from Lemma 5 that $g(x)$ is a constant c. Thus $f(x) = h_f(x) + c = \sum_{t\in A} \widetilde{h_t}(x) + c$. The proof of Theorem 3 is complete. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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