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Research Article

Three-Point Boundary Value Problems for Conformable Fractional Differential Equations

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We study a fractional differential equation using a recent novel concept of fractional derivative with initial and three-point boundary conditions. We first obtain Green's function for the linear problem and then we study the nonlinear differential equation.

1. Introduction

In this paper, we study a class of differential equations supplemented with three-point boundary conditions. Precisely, we consider the following problem:

$$\mathrm{D}^{\alpha}(\mathrm{D}+\lambda)x(t) = f(t, x(t)), \quad t \in [0,1], \tag{1}$$

$$x(0) = 0, \qquad x'(0) = 0, \qquad x(1) = \beta x(\eta), \tag{2}$$

where D^{α} is the conformable fractional derivative of order $\alpha \in (1,2]$, D is the ordinary derivative, $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a known continuous function, λ and β are real numbers, $\lambda > 0$, and $\eta \in (0,1)$.

Fractional calculus and fractional differential equations are relevant areas of research. There are several concepts of fractional derivatives, some classical, such as Riemann-Liouville or Caputo definitions, and some novel, such as conformable fractional derivative [1], β-derivative [2], or a new definition [3, 4]. The relation between these definitions and their potential applications needs further study.

The conformable fractional derivative aims at extending the usual derivative satisfying some natural properties (see [1]) and gives a new solution for some fractional differential equations. In this paper we present a boundary value problem involving this fractional derivative.

Sequential fractional differential equations have been considered for other types of fractional derivatives, see, for example, [5, 6].

The paper is organized as follows. In Section 2 we recall some concepts relative to the conformable fractional calculus. In Section 3 we solve the corresponding linear problem and obtain Green's function. In Section 4 we study the nonlinear problem and finally, in Section 5, we present an example to illustrate the applicability of our results.

2. Preliminaries

We recall some definitions and results concerning conformable fractional derivative.

Definition 1 (see [1]). Given a function $x : [0,+\infty) \to \mathbb{R}$, the conformable fractional derivative of order $\alpha \in (0,1]$ of x at t is defined by

$$\mathrm{D}^{\alpha}x(t) = \lim_{\varepsilon \to 0} \frac{x\left(t + \varepsilon t^{1-\alpha}\right) - x(t)}{\varepsilon}, \tag{3}$$

for all $t > 0$. If f is α-differentiable in some interval $(0,a)$ with $a > 0$, then we define

$$\mathrm{D}^{\alpha}x(0) = \lim_{t \to 0^+} \mathrm{D}^{\alpha}x(t), \tag{4}$$

whenever the limit of the right hand side exists.

We remark that if x is differentiable, then

$$\mathrm{D}^{\alpha} x(t) = t^{1-\alpha} x'(t). \tag{5}$$

Reciprocally, if $\mathrm{D}^{\alpha} x(t)$ exists, then for $t \neq 0$ we have

$$\begin{aligned} x'(t) &= \lim_{\delta \to 0} \frac{x(t+\delta) - x(t)}{\delta} \\ &= \lim_{\varepsilon \to 0} \frac{x\left(t + \varepsilon t^{1-\alpha} - x(t)\right)}{\varepsilon t^{1-\alpha}} = t^{\alpha-1} \mathrm{D}^{\alpha} x(t). \end{aligned} \tag{6}$$

Hence, $\mathrm{D}^{\alpha} x(t) = t^{1-\alpha} x'(t)$. Of course, for $t = 0$ this is not valid and it would be useful to deal with equations and solutions with singularities.

Definition 2. Let $\alpha \in (n, n+1]$ and let x be an n-differentiable function at $t > 0$; the fractional conformable derivative of order α at $t > 0$ is given by

$$\mathrm{D}^{\alpha} x(t) = \lim_{\varepsilon \to 0} \frac{x^{(\lceil \alpha \rceil - 1)}\left(t + \varepsilon t^{\lceil \alpha \rceil - \alpha}\right) - x^{(\lceil \alpha \rceil - 1)}(t)}{\varepsilon}, \tag{7}$$

where $\lceil \alpha \rceil$ is the smallest integer greater than or equal to α. For $t = 0$ we proceed in a similar way as in Definition 1.

Definition 3. Given $\alpha \in (0, 1]$, the fractional integral of order α at $t \geq 0$ is given by

$$\mathrm{D}^{-\alpha} x(t) \equiv \mathrm{I}^{\alpha} x(t) = \mathrm{I}^{1}\left(t^{\alpha-1} x\right)(t) = \int_0^t \frac{x(s)}{s^{1-\alpha}} ds. \tag{8}$$

Definition 4. Given $\alpha \in (n-1, n]$, the fractional integral of order α is given by

$$\mathrm{D}^{-\alpha} x(t) \equiv \mathrm{I}^{\alpha} x(t) = \mathrm{I}^{n}\left(t^{\alpha - \lceil \alpha \rceil} x\right)(t), \tag{9}$$

where I^n denotes the operator I^1 (usual integration) of order n.

Remark 5. Some authors (see [7, 8]) have argued that conformable fractional derivative is not a truly fractional operator. This question seems today to still be open and perhaps it is a philosophical issue. However, in any case, the study of boundary value problems involving this new derivative has, in our opinion, a point of interest and deserves to be researched in more detail.

In [1, Theorem 3.1], authors have proved that for $\alpha \in (0, 1]$ and x a given continuous function, $\mathrm{D}^{\alpha} \mathrm{I}^{\alpha} x(t) = x(t)$ for $t \geq 0$. In this paper, we consider $\alpha \in (1, 2]$, so we need the following results.

Lemma 6. *Given $\alpha \in (1, 2]$ and x a continuous function defined in the domain of I^{α}, one has that $D^{\alpha} I^{\alpha} x(t) = x(t)$ for $t \geq 0$.*

Proof. Since x is continuous, then $\mathrm{I}^{\alpha} x(t)$ is twice differentiable. In view of [1, Remark 2.1] we have

$$\begin{aligned} \mathrm{D}^{\alpha}\left(\mathrm{I}^{\alpha} x\right)(t) &= t^{2-\alpha} \frac{d^2}{dt^2} \int_0^t \int_0^{t_1} x(s) s^{\alpha-2} ds\, dt_1 \\ &= t^{2-\alpha} \frac{d}{dt} \int_0^t x(s) s^{\alpha-2} ds \\ &= t^{2-\alpha} x(t) t^{\alpha-2} = x(t). \end{aligned} \tag{10}$$

Thus, statement of Lemma 6 has been proved. □

Lemma 7. *Given $\alpha \in (1, 2]$ and $x : [0, +\infty) \to \mathbb{R}$ an α-differentiable function, one has that $D^{\alpha} x(t) = 0$ if and only if $x(t) = c_1 t + c_2$, where $c_1, c_2 \in \mathbb{R}$.*

Proof. This fact follows easily in view of the mean value theorem for conformable fractional differentiable functions (see [1, Theorem 2.4]). □

3. Linear Boundary Problem

In order to study boundary value problem (1)-(2), we consider now the linear equation

$$\mathrm{D}^{\alpha}(\mathrm{D} + \lambda) x(t) = \sigma(t), \quad t \in [0, 1], \tag{11}$$

where $1 < \alpha \leq 2$ and $\sigma \in \mathscr{C}[0, 1]$.

Lemma 8. *Consider*

$$\beta \neq \frac{\lambda + e^{-\lambda} - 1}{\lambda\eta + e^{-\lambda\eta} - 1}. \tag{12}$$

Then, the unique solution of (11) subject to the boundary conditions (2) is given by

$$\begin{aligned} x(t) = &\int_0^t e^{-\lambda(t-s)} \left(\int_0^s \sigma(u) u^{\alpha-2} (s-u) du\right) ds \\ &+ A(t) \left[\beta \int_0^{\eta} e^{-\lambda(\eta-s)} \left(\int_0^s \sigma(u) u^{\alpha-2} (s-u) du\right) ds \right. \\ &\left. - \int_0^1 e^{-\lambda(1-s)} \left(\int_0^s \sigma(u) u^{\alpha-2} (s-u) du\right) ds\right], \end{aligned} \tag{13}$$

where

$$\begin{aligned} A(t) &= \frac{1}{\Delta}\left(\lambda t + e^{-\lambda t} - 1\right), \\ \Delta &= \lambda + e^{-\lambda} - 1 - \beta\left(\lambda\eta + e^{-\lambda\eta} - 1\right) \neq 0. \end{aligned} \tag{14}$$

Proof. Integrating (11) we obtain

$$(\mathrm{D} + \lambda) x(t) = \mathrm{I}^{\alpha} \sigma(t) + c_1 t + c_2. \tag{15}$$

Thus, in view of Lemmas 6 and 7, every solution of (15) is a solution for (11).

Let $y(t) = e^{\lambda t}x(t)$. Equation (15) can be rewritten as

$$\mathrm{D}y(t) = \left(\mathrm{I}^{\alpha}\sigma(t) + c_1 t + c_2\right)e^{\lambda t}. \tag{16}$$

Integrating from 0 to t, we obtain

$$\begin{aligned} x(t) = {} & \frac{c_1}{\lambda^2}\left(\lambda t - 1 + e^{-\lambda t}\right) + \frac{c_2}{\lambda}\left(1 - e^{-\lambda t}\right) + c_3 \\ & + \int_0^t e^{-\lambda(t-s)}\left(\int_0^s \sigma(u)\, u^{\alpha-2}(s-u)\, du\right) ds. \end{aligned} \tag{17}$$

Imposing boundary conditions (2), we conclude that $c_3 = 0$, $c_2 = 0$, and

$$\begin{aligned} c_1 = \frac{\lambda^2}{\Delta}\Bigg[& \beta\int_0^{\eta} e^{-\lambda(\eta-s)}\left(\int_0^s \sigma(u)\, u^{\alpha-2}(s-u)\, du\right) ds \\ & - \int_0^1 e^{-\lambda(1-s)}\left(\int_0^s \sigma(u)\, u^{\alpha-2}(s-u)\, du\right) ds\Bigg]. \end{aligned} \tag{18}$$

Substituting these values of c_1, c_2, and c_3 in (17), we finally obtain (13) and that expression gives the unique solution. □

We now obtain Green's function corresponding to the fractional differential equations (11) of order $\alpha + 1$ with $1 < \alpha \le 2$ subject to boundary conditions (2).

By changing the order of integration, we note that

$$\begin{aligned} & \int_0^t e^{-\lambda(t-s)}\left(\int_0^s \sigma(u)\, u^{\alpha-2}(s-u)\, du\right) ds \\ & \quad = \int_0^t \left(\int_u^t e^{\lambda(s-t)}(s-u)\, ds\right)\sigma(u)\, u^{\alpha-2} du. \end{aligned} \tag{19}$$

Hence, solution (17) with $c_2 = c_3 = 0$ takes the form

$$x(t) = \frac{c_1}{\lambda^2}\left(\lambda t - 1 + e^{-\lambda t}\right) + \int_0^t k(t,s)\,\sigma(s)\, s^{\alpha-2} ds, \tag{20}$$

where

$$k(t,s) = \int_s^t e^{\lambda(u-t)}(u-s)\, du = \frac{e^{\lambda(s-t)} - \lambda s - 1 + \lambda t}{\lambda^2}. \tag{21}$$

Now, using the boundary condition $x(1) = \beta x(\eta)$, we get

$$c_1 = \frac{\lambda^2}{\Delta}\left(\beta\int_0^{\eta} k(\eta,s)\,\sigma(s)\, s^{\alpha-s} ds - \int_0^1 k(1,s)\,\sigma(s)\, s^{\alpha-2} ds\right). \tag{22}$$

Therefore, we finally conclude the following:

$$\begin{aligned} x(t) = {} & \frac{A(t)}{\Delta}\left(\beta\int_0^{\eta} k(\eta,s)\,\sigma(s)\, s^{\alpha-s} ds \right. \\ & \left. - \int_0^1 k(1,s)\,\sigma(s)\, s^{\alpha-2} ds\right) \\ & + \int_0^t k(t,s)\,\sigma(s)\, s^{\alpha-2} ds; \end{aligned} \tag{23}$$

so, we deduce the following result.

Theorem 9. *The unique solution of (11) subject to boundary conditions (2) is given by*

$$x(t) = \int_0^1 G(t,s)\,\sigma(s)\, s^{\alpha-2} ds, \tag{24}$$

where

$$G(t,s) = \begin{cases} -k(1,s)\,\psi(t), \\ \qquad \text{if } 0 \le \max\{\eta, t\} < s \le 1; \\ -k(1,s)\,\psi(t) + k(t,s), \\ \qquad \text{if } 0 \le \eta < s < t \le 1; \\ \left(\beta k(\eta,s) - k(1,s)\right)\psi(t), \\ \qquad \text{if } 0 \le t < s \le \eta \le 1; \\ \left(\beta k(\eta,s) - k(1,s)\right)\psi(t) + k(t,s), \\ \qquad \text{if } 0 \le s < \min\{\eta, t\} \le 1, \end{cases} \tag{25}$$

with

$$\psi(t) = \frac{A(t)}{\Delta}, \quad t \in [0,1]. \tag{26}$$

Remark 10. In other words, corresponding Green's function for the homogeneous problem (11) satisfying the boundary conditions (2) is given by (25).

Remark 11. Note that $G(t,s)$ is independent of α, but the solution depends, of course, on α.

4. Nonlinear Problem

Let $\mathscr{C} = \mathscr{C}[0,1]$ be the Banach space of all continuous functions defined in $[0,1]$ endowed with the usual supremum norm defined by $\|x\| = \sup\{|x(t)|,\ t \in [0,1]\}$.

For the sake of convenience, we set

$$B = \frac{1 + A_1\left[|\beta|\,\eta^{\alpha}\left(1 - e^{-\lambda\eta}\right) + 1 - e^{-\lambda}\right]}{\lambda\alpha(\alpha-1)} > 0, \tag{27}$$

with

$$A_1 = \sup_{t\in[0,1]} |A(t)|, \tag{28}$$

where $A(t)$ is given by (14).

In view of Lemma 8, we transform boundary value problem (1)-(2) into

$$x = \mathscr{T}x, \quad x \in \mathscr{C}, \tag{29}$$

where $\mathscr{T} : \mathscr{C} \to \mathscr{C}$ is defined by

$$\begin{aligned} & (\mathscr{T}x)(t) \\ & = \int_0^t e^{-\lambda(t-s)}\left(\int_0^s f(u, x(u))\, u^{\alpha-2}(s-u)\, du\right) ds \end{aligned}$$

$$+ A(t)\left[\beta\int_0^{\eta} e^{-\lambda(\eta-s)}\left(\int_0^s f(u,x(u))\,u^{\alpha-2}(s-u)\,du\right)ds - \int_0^1 e^{-\lambda(1-s)}\left(\int_0^s f(u,x(u))\,u^{\alpha-2}(s-u)\,du\right)ds\right]. \tag{30}$$

Observe that problem (1)-(2) has solutions if the operator (30) has fixed points.

Theorem 12. *Let $f:[0,1]\times\mathbb{R}\to\mathbb{R}$ be a jointly continuous function satisfying the condition*

$$|f(t,v)-f(t,w)|\le L|v-w| \quad \forall t\in[0,1],\ v,w\in\mathbb{R}, \tag{31}$$

where $L>0$ is the Lipschitz constant. Then, boundary value problem (1)-(2) has a unique solution if $B<1/L$, where B is given by (27).

Proof. First, for $\mathcal{T}$ defined by (30), we show that $\mathcal{T}(\mathcal{B}_r)\subset\mathcal{B}_r$, where $\mathcal{B}_r$ is the closed ball of radius $r>0$ in $\mathcal{C}$; that is, $\mathcal{B}_r=\{x\in\mathcal{C}:\|x\|\le r\}$. Now we consider $M>\sup\{|f(t,0)|:t\in[0,1]\}$ and we choose

$$r>\frac{MB}{1-LB}. \tag{32}$$

For $x\in\mathcal{B}_r$, we have

$$\begin{aligned}
&\|\mathcal{T}x\| \\
&= \sup_{t\in[0,1]}\left|\int_0^t e^{-\lambda(t-s)}\left(\int_0^s f(u,x(u))\,u^{\alpha-2}(s-u)\,du\right)ds + A(t)\left[\beta\int_0^{\eta} e^{-\lambda(\eta-s)}\cdot\left(\int_0^s f(u,x(u))\,u^{\alpha-2}(s-u)\,du\right)ds - \int_0^1 e^{-\lambda(1-s)}\cdot\left(\int_0^s f(u,x(u))\,u^{\alpha-2}(s-u)\,du\right)ds\right]\right| \\
&\le \sup_{t\in[0,1]}\int_0^t e^{-\lambda(t-s)}\cdot\left(\int_0^s\left(|f(u,x(u))-f(u,0)|+|f(u,0)|\right)\cdot u^{\alpha-2}(s-u)\,du\right)ds \\
&\quad + \sup_{t\in[0,1]}|A(t)|\left[|\beta|\int_0^{\eta} e^{-\lambda(\eta-s)}\cdot\left(\int_0^s\left(|f(u,x(u))-f(u,0)|+|f(u,0)|\right)u^{\alpha-2}(s-u)\,du\right)ds + \int_0^1 e^{-\lambda(1-s)}\cdot\left(\int_0^s\left(|f(u,x(u))-f(u,0)|+|f(u,0)|\right)u^{\alpha-2}(s-u)\,du\right)ds\right] \\
&\le \sup_{t\in[0,1]}\int_0^t e^{-\lambda(t-s)}\cdot\left(\int_0^s\left(L|x(u)|+|f(u,0)|\right)u^{\alpha-2}(s-u)\,du\right)ds \\
&\quad + A_1\left[|\beta|\int_0^{\eta} e^{-\lambda(\eta-s)}\cdot\left(\int_0^s\left(L|x(u)|+|f(u,0)|\right)u^{\alpha-2}(s-u)\,du\right)ds + \int_0^1 e^{-\lambda(1-s)}\left(\int_0^s\left(L|x(u)|+|f(u,0)|\right)\cdot u^{\alpha-2}(s-u)\,du\right)ds\right] \\
&\le (Lr+M)\left\{\sup_{t\in[0,1]}\int_0^t e^{-\lambda(t-s)}\left(\int_0^s u^{\alpha-2}(s-u)\,du\right)ds + A_1\left[|\beta|\int_0^{\eta} e^{-\lambda(\eta-s)}\left(\int_0^s u^{\alpha-2}(s-u)\,du\right)ds + \int_0^1 e^{-\lambda(1-s)}\left(\int_0^s u^{\alpha-2}(s-u)\,du\right)ds\right]\right\} \\
&\le \frac{Lr+M}{\alpha(\alpha-1)}\left\{\sup_{t\in[0,1]}\int_0^t e^{-\lambda(t-s)}s^{\alpha}ds + A_1\left(|\beta|\int_0^{\eta} e^{-\lambda(\eta-s)}s^{\alpha}ds + \int_0^1 e^{-\lambda(1-s)}s^{\alpha}ds\right)\right\} \\
&\le (Lr+M)\,\frac{1+A_1\left(|\beta|\eta^{\alpha}\left(1-e^{-\lambda\eta}\right)+1-e^{-\lambda}\right)}{\lambda\alpha(\alpha-1)} \\
&= (Lr+M)\,B\le r.
\end{aligned} \tag{33}$$

Now, for $x,y\in\mathcal{C}$ and for each $t\in[0,1]$, we obtain

$$\begin{aligned}
&|(\mathcal{T}x)(t)-(\mathcal{T}y)(t)| \\
&= \sup_{t\in[0,1]}|(\mathcal{T}x)(t)-(\mathcal{T}y)(t)| \\
&\le \sup_{t\in[0,1]}\left\{\int_0^t e^{-\lambda(t-s)}\left(\int_0^s|f(u,x(u))-f(u,y(u))|\cdot u^{\alpha-2}(s-u)\,du\right)ds\right.
\end{aligned}$$

$$
\begin{aligned}
&+ A(t)\left[\beta\int_0^{\eta} e^{-\lambda(\eta-s)}\right.\\
&\quad\cdot\left(\int_0^s \left|f(u,x(u)) - f(u,y(u))\right| u^{\alpha-2}\right.\\
&\qquad\left.\left.\cdot (s-u)\,du\right)ds\\
&\quad+\int_0^1 e^{-\lambda(1-s)}\\
&\quad\cdot\left(\int_0^s \left|f(u,x(u)) - f(u,y(u))\right|\right.\\
&\qquad\left.\left.\cdot u^{\alpha-2}(s-u)\,du\right)ds\right]\Bigg\}\\
&\le L\|x-y\|\\
&\quad\cdot\left\{\sup_{t\in[0,1]}\int_0^t e^{-\lambda(t-s)}\left(\int_0^s u^{\alpha-2}(s-u)\,du\right)ds\right.\\
&\qquad+\sup_{t\in[0,1]} A(t)\left[\beta\int_0^{\eta} e^{-\lambda(\eta-s)}\left(\int_0^s u^{\alpha-2}(s-u)\,du\right)ds\right.\\
&\qquad\left.\left.+\int_0^1 e^{-\lambda(1-s)}\left(\int_0^s u^{\alpha-2}(s-u)\,du\right)ds\right]\right\}\\
&\le L\frac{1+A_1\left(|\beta|\eta^{\alpha}\left(1-e^{-\lambda\eta}\right)+1-e^{-\lambda}\right)}{\lambda\alpha(\alpha-1)}\|x-y\|\\
&= BL\|x-y\|.
\end{aligned}
\tag{34}
$$

As $B < 1/L$, we conclude that $\mathscr{T}$ is a contraction. Thus, the statement of the theorem follows by the classical Banach fixed point theorem. This concludes the proof. □

Now we recall a known result due to Krasnoselskii (see [9, Theorem 4.4.1]) which we will use to prove existence of at least one solution to (1)-(2).

Theorem 13. *Let N be a closed, convex, and nonempty subset of a Banach space X. Let $\mathscr{T}_1$ and $\mathscr{T}_2$ be operators such that*

(i) $\mathscr{T}_1 x + \mathscr{T}_2 y \in N$ *whenever* $x, y \in N$,

(ii) $\mathscr{T}_1$ *is compact and continuous,*

(iii) $\mathscr{T}_2$ *is a contraction mapping.*

Then there exists $z \in N$ such that $z = \mathscr{T}_1 z + \mathscr{T}_2 z$.

Theorem 14. *Let $f : [0,1]\times\mathbb{R} \to \mathbb{R}$ be a jointly continuous function satisfying the following conditions:*

(H1) $|f(t,v) - f(t,w)| \le L|v-w|$ *for all* $t \in [0,1]$, $v, w \in \mathbb{R}$,

(H2) $|f(t,v)| \le \mu(t)$ *for all* $(t,v) \in [0,1]\times\mathbb{R}$ *with* $\mu \in \mathscr{C}$.

Then, boundary value problem (1)-(2) has at least one solution on $\mathscr{C}$ if

$$
A_1\frac{|\beta|\eta^{\alpha}\left(1-e^{-\lambda\eta}\right)+\left(1+e^{-\lambda}\right)}{\lambda\alpha(\alpha-1)} < 1. \tag{35}
$$

Proof. Letting $\|\mu\| = \sup_{t\in[0,1]}|\mu(t)|$, we fix

$$
r \ge \frac{1+A_1\left[|\beta|\eta^{\alpha}\left(1-e^{-\lambda\eta}\right)+\left(1-e^{-\lambda}\right)\right]}{\lambda\alpha(\alpha-1)}\|\mu\|, \tag{36}
$$

and we consider $\mathscr{B}_r$ as in Theorem 12.

Define the operators $\mathscr{T}_1$ and $\mathscr{T}_2$ as

$$
\begin{aligned}
&(\mathscr{T}_1 x)(t) = \int_0^t e^{-\lambda(t-s)}\left(\int_0^s f(u,x(u))\,u^{\alpha-2}(s-u)\,du\right)ds,\\
&(\mathscr{T}_2 y)(t)\\
&= A(t)\left[\beta\int_0^{\eta} e^{-\lambda(\eta-s)}\left(\int_0^s f(u,y(u))\,u^{\alpha-2}(s-u)\,du\right)ds\right.\\
&\qquad\left.+\int_0^1 e^{-\lambda(1-s)}\left(\int_0^s f(u,y(u))\,u^{\alpha-2}(s-u)\,du\right)ds\right].
\end{aligned}
\tag{37}
$$

For $x, y \in \mathscr{B}_r$, it follows by (36) that

$$
\begin{aligned}
&\|\mathscr{T}_1 x + \mathscr{T}_2 y\|\\
&\quad\le \frac{1+A_1\left[|\beta|\eta^{\alpha}\left(1-e^{-\lambda\eta}\right)+\left(1-e^{-\lambda}\right)\right]}{\lambda\alpha(\alpha-1)}\|\mu\| \le r.
\end{aligned}
\tag{38}
$$

Thus, $\mathscr{T}_1 x + \mathscr{T}_2 y \in \mathscr{B}_r$. In view of condition (35), we have that $\mathscr{T}_2$ is a contraction mapping.

Now we show that $\mathscr{T}_1$ is compact and continuous. The continuity of f implies that the operator $\mathscr{T}_1$ is continuous. In addition, $\mathscr{T}_1$ is uniformly bounded on $\mathscr{B}_r$ as

$$
\begin{aligned}
&\|\mathscr{T}_1 x\|\\
&\quad= \sup_{t\in[0,1]}\left|\int_0^t e^{-\lambda(t-s)}\left(\int_0^s f(u,x(u))\,u^{\alpha-2}(s-u)\,du\right)ds\right|\\
&\quad\le \|\mu\|\sup_{t\in[0,1]}\int_0^t e^{-\lambda(t-s)}\left(\int_0^s u^{\alpha-2}(s-u)\,du\right)ds\\
&\quad= \frac{\|\mu\|}{\alpha(\alpha-1)}\int_0^t e^{-\lambda(t-s)}s^{\alpha}ds\\
&\quad\le \|\mu\|\frac{1-e^{-\lambda}}{\lambda\alpha(\alpha-1)}.
\end{aligned}
\tag{39}
$$

Setting $\Omega = [0,1] \times \mathscr{B}_r$, we define $M_r = \sup_{(t,x)\in\Omega}|f(t,x)|$, and consequently we have that, for $0 \le t_2 < t_1 \le 1$,

$$\begin{aligned}
&\left|(\mathscr{T}_1 x)(t_1) - (\mathscr{T}_1 x)(t_2)\right| \\
&= \left|\int_0^{t_1} e^{-\lambda(t_1-s)}\left(\int_0^s f(u,x(u))\,u^{\alpha-2}(s-u)\,du\right)ds\right. \\
&\qquad \left. - \int_0^{t_2} e^{-\lambda(t_2-s)}\left(\int_0^s f(u,x(u))\,u^{\alpha-2}(s-u)\,du\right)ds\right| \\
&\le \frac{M_r}{\alpha(\alpha-1)}\left(e^{\lambda t_1} - e^{\lambda t_2}\right)\left(e^{-\lambda t_1}{t_1}^{\alpha} - e^{-\lambda t_2}{t_2}^{\alpha}\right),
\end{aligned} \tag{40}$$

which is independent of x and tends to zero as $t_2 \to t_1$. This shows that $\mathscr{T}_1$ is relatively compact on $\mathscr{B}_r$. Hence, by the Ascoli-Arzelá Theorem, $\mathscr{T}_1$ is compact on $\mathscr{B}_r$. Thus, all the hypotheses of Theorem 13 are satisfied and the conclusion of Theorem 13 implies that the boundary value problem (1)-(2) has at least one solution on $\mathscr{B}_r \subset \mathscr{C}$, with r satisfying (36). This completes the proof. □

5. Example

Consider the boundary value problem over the interval $[0,1]$, given by:

$$\begin{aligned}
&\mathrm{D}^{3/2}(\mathrm{D}+4)\,x(t) = L\left(t^2 + \cos t + \arctan x(t)\right), \\
&x(0) = 0, \qquad x'(0) = 0, \qquad x(1) = x\left(\frac{1}{2}\right).
\end{aligned} \tag{41}$$

Here, $f(t,v) = L(t^2 + \cos t + \arctan v)$, $\lambda = 4$, $\beta = 1$, $\eta = 1/2$. Clearly,

$$\begin{aligned}
&\left|f(t,v) - f(t,w)\right| \le L\left|\arctan v - \arctan w\right| \le L\left|v - w\right|, \\
&A_1 = \frac{4 + e^{-4} - 1}{4 + e^{-4} - 1 - \left(2 + e^{-2} - 1\right)} \approx 1.6029, \\
&B = \frac{1 + A_1\left(1/2^{3/2}\left(1 - e^{-2}\right) + 1 - e^{-4}\right)}{3} \approx 1.0212.
\end{aligned} \tag{42}$$

For $L < 0.9792$, it follows, by Theorem 12, that boundary value problem (41) has a unique solution.

Remark 15. Authors of [5] obtained a similar result considering fractional Caputo derivative instead of fractional conformable derivative in (41).

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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