Hindawi Publishing Corporation
Journal of Function Spaces
Volume 2015, Article ID 727312, 11 pages
http://dx.doi.org/10.1155/2015/727312

Research Article

The Spaces of Functions of Two Variables of Bounded $\kappa\Phi$-Variation in the Sense of Schramm-Korenblum

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Received 20 March 2015; Accepted 10 May 2015

Academic Editor: Kehe Zhu

The purpose of this paper is twofold. Firstly, we introduce the concept of bounded $\kappa\Phi$-variation in the sense of Schramm-Korenblum for real functions with domain in a rectangle of $\mathbb{R}^2$. Secondly, we study some properties of these functions and we prove that the space generated by these functions has a structure of Banach algebra.

1. Introduction

During the last decades, several developments, extensions, and generalizations have been considered for the classical concept of the total variation of a function. It is well known that such extensions and generalization play significant role and find many applications in different areas of mathematics. In this paper we introduce a new definition of variation for functions defined on a nonempty rectangle subset of the plane, and we examine the algebra of functions of two variables of bounded generalized variation one obtains from this new definition. Fourier series were introduced by Joseph Fourier (1768–1830) for the purpose of solving the heat equation in a metal plate. The development of the theory of Fourier series in mathematical analysis began in the 19th century and it has been a source of new ideas for analysis during the last two centuries and is likely to be so in years to come. The first exactly proved result was published in Dirichlet's paper in 1829. That theorem concerns the convergence of Fourier series of piecewise monotonic functions. According to Lakatos [1], functions of bounded variation were discovered by Jordan through a "critical reexamination" of Dirichlet's famous flawed proof that arbitrary functions can be represented by Fourier series. Jordan [2] proved that if a continuous function has bounded variation, then its Fourier series converges uniformly on a closed bounded set [3]. Jordan gave the characterization of such functions as differences of increasing functions. It is well known that the space of functions of bounded variation on a compact interval $[a, b] \subset \mathbb{R}$ is a commutative Banach algebra with respect to pointwise multiplications [4–6]. Functions of bounded variation of one variable are of great interest and usefulness because of their valuable properties. Such properties, particularly with respect to additivity, decomposability into monotone functions, continuity, differentiability, measurability, integrability, and so on, have been much studied. It is largely to the possession of these properties that functions of bounded variation owe their important role in the study of rectifiable curves, Fourier series, Walsh-Fourier series, and other series, Stieltjes integrals, Henstock-Kurzweil integral, and other integrals, and the calculus of variations [7]. Consequently, the study of notions of generalized bounded variation forms an important direction in the field of mathematical analysis [8, 9]. Two well-known generalizations are the functions of bounded p-variation and the functions of bounded Φ-variation, due to N. Wiener and L. C. Young, respectively. In 1924 Wiener showed that the Fourier series of function in one variable of finite p-variation converges almost everywhere. In 1938 L. C. Young developed an integration theory with respect to functions of finite φ-variation and showed that the Fourier series of such functions converges everywhere. In 1972 Waterman [10] studied a class of bounded Λ-variation. Combining the notion of bounded Λ-variation with that of bounded Φ-variation, Leindler [11] introduced the class $\Lambda_\Phi BV$ of functions of bounded Λ_Φ-variation, and both classes of bounded Λ-variation and bounded Φ-variation are its special cases. In 1980 Shiba [12] introduced the class $\Lambda BV^{(p)}$ expanding a fundamental concept of bounded Λ-variation formulated by Waterman.

In 1986, S. K. Kim and J. Kim [13] introduced the notion of functions of bounded $\kappa\Phi$-variation on compact interval $[a,b] \subset \mathbb{R}$ which is a combination of concepts of bounded κ-variation and bounded ϕ-variation in the sense of Schramm [14]. In [15, 16] Castillo et al. introduce the notion of bounded κ-variation in the sense of Riesz-Korenblum, which is a combination of the notions of bounded p-variation in the sense of Riesz and bounded κ-variation in the sense of Korenblum. In the year 2014 Guerrero et al. [17] have introduced the space $\kappa BV(I_a^b,\mathbb{R})$ of the functions of two variables of bounded κ-variation in the sense of Hardy-Vitali-Korenblum and showed that the space $\kappa BV(I_a^b,\mathbb{R})$ is a Banach space.

Soon after Jordan's work, many mathematicians began to study notions of bounded variation for functions of several variables. There is no uniquely suitable way to extend the notion of variation to function of more than one variable. Proposers of definitions of bounded variation for functions of two variables have been actuated mainly by the desire to single out for attention a class of functions having properties analogous to some particular properties of a function of one variable of bounded variation. Clarkson and Adams [18] study six such generalizations, and Adams and Clarkson [19] mention two more. Two of these definitions are relevant to our purpose. Clarkson and Adams attribute the first to Vitali, Lebesgue, Fréchet, and De la Vallée Poussin and the second to Hardy [20] and Krause [21]. We will refer to them as Vitali variation and Hardy-Krause variation, respectively. Owen [22] provides a very helpful discussion of the concepts of Vitali and Hardy-Krause variation. Another useful reference is Hobson [23]. At the beginning of the past century Hardy [20] generalized the Jordan criterion to the double Fourier series and he proved that if a continuous function of two variables has bounded variation (in the sense of Hardy), then its Fourier series converges uniformly.

Motivated by [13, 17] we introduce for functions of two variables the concept of bounded $\kappa\Phi$-variation in the sense of Schramm-Korenblum, which is a suitable combination of the notions of bounded Φ-variation in the sense of Schramm and bounded κ-variation in the sense of Korenblum for real functions defined on a rectangle of the plane. Our paper is structured as follows. Section 2 provides a review of the notion of Vitali and Hardy-Krause variations for multivariate functions. We recall some notions of variation and introduce for functions of two variables the definitions of bounded $\kappa\Phi$-variation in the sense of Schramm-Korenblum. In Section 3 we state and prove our main result: the linear space generated by the class of all bounded $\kappa\Phi$-variation functions is a Banach algebra.

2. Preliminaries, Background, and Notations

We begin with some general notation and definitions systematically used throughout the paper.

As usual if A and B are nonempty sets the symbol A^B denotes the family of functions $f : B \to A$. We denote by S_n the set of all permutations σ of set $\{1,\ldots,n\}$ n positive integer.

Let $I = [a,b] \subset \mathbb{R}$ be any nonempty interval. For $f \in \mathbb{R}^I$, we define $\|f\|_\infty := \sup_{x\in B}|f(x)|$, and if $\|f\|_\infty < \infty$ we say that f is bounded. We denote by $\mathscr{P}(I)$ the class of all partitions of I. Let $f \in \mathbb{R}^{[a,b]}$ and $\xi = \{t_i\}_{i=1}^k \in \mathscr{P}(I)$, and the following notations are used frequently:

$$\begin{aligned} \Delta t_{i+1} &= t_{i+1} - t_i, \\ \Delta f(t_{i+1}) &= f(t_{i+1}) - f(t_i), \\ & \qquad i = 1,\ldots,k-1. \end{aligned} \tag{1}$$

We will establish the following $a = (a_1,b_1)$, $b = (a_2,b_2) \in \mathbb{R}^2$, $I = [a_1,a_2]$, $J = [b_1,b_2]$, $I_a^b = I \times J$, and $k_1,k_2 \geq 2$ integer. If $f \in \mathbb{R}^{I_a^b}$, $\xi = \{t_i\}_{i=1}^{k_1}$ and $\eta = \{s_j\}_{j=1}^{k_2}$ are partitions of the intervals I, J, respectively, denoted by

$$\begin{aligned} \Delta_{10} f(t_{i+1},s_{j+1}) &:= f(t_{i+1},s_{j+1}) - f(t_i,s_{j+1}), \\ \Delta_{01} f(t_{i+1},s_{j+1}) &:= f(t_{i+1},s_{j+1}) - f(t_{i+1},s_j), \\ \Delta_{11} f(t_{i+1},s_{j+1}) &:= f(t_i,s_j) - f(t_i,s_{j+1}) \\ &\quad - f(t_{i+1},s_j) + f(t_{i+1},s_{j+1}). \end{aligned} \tag{2}$$

Definition 1. A function $\kappa : [0,1] \to [0,1]$ is called distortion function if κ is continuous, increasing, and concave and satisfies $\kappa(0) = 0$, $\kappa(1) = 1$ and $\lim_{t\to 0^+}(\kappa(t)/t) = \infty$.

The set of all distortion functions will be denoted by $\mathscr{K}$. A distortion function is always subadditive (see [6, Section 2.5 page 170]) in the sense that

$$\kappa(s+t) \leq \kappa(s) + \kappa(t) \quad (0 \leq s,t \leq 1). \tag{3}$$

Important special cases for choices of κ are $\kappa_0(x) = \kappa(1-\log x)$ and $\kappa_\alpha(x) = x^\alpha$, $0 < \alpha < 1$ (see [24], [25, Section 5], and [26]).

Definition 2 (see [14]). A $\phi = \{\phi_n\}_{n\geq 1}$ is a Φ-sequence if ϕ is a sequence of increasing convex functions, defined on $\mathbb{R}_+ = [0,\infty)$ such that $\phi_n(0) = 0$, $\phi_{n+1}(t) \leq \phi_n(t)$ for all n and t, and $\sum_n \phi_n(t)$ diverges for $t > 0$.

Throughout the paper the double sequence $\phi = \{\phi_{i,j}\}_{i,j\geq 1}$ will be a Φ-sequence if for i or j fixed $\phi = \{\phi_{i,j}\}_{i,j\geq 1}$ is a Φ-sequence. It is worthy to recall that the initial works on double sequences can be found in [27, 28].

In what follows we recall different notions of generalized bounded variation.

The notion of variation was introduced by Jordan in 1881 in the one-dimensional case and Vitali and Hardy, which generalized the notion given by Jordan, in 1904–1906 (see [20, 29]). This generalization is for functions of two variables.

Definition 3 (see [2]). The function $f \in \mathbb{R}^I$ is of bounded variation if

$$V(f;[a,b]) = V(f) := \sup_{\mathscr{P}(I)} \sum_{i=1}^{k-1} |\Delta f(t_{i+1})| < \infty, \quad (4)$$

where the supremum is taken over all partitions $\{t_i\}_{i=1}^{k_1} \in \mathscr{P}(I)$. We denote by $BV[a,b]$ the space of all functions of bounded variation and it is known that $BV[a,b]$ is a Banach algebra with respect to the norm $\|f\| = |f(a)| + V(f)$, $f \in BV[a,b]$.

Definition 4 (see [20, 29]). Let $f \in \mathbb{R}^{I_a^b}$ and $y \in J$ be fixed. The Jordan variation of the function $f(\cdot,y) \in \mathbb{R}^I$ is denoted by

$$V(f(\cdot,y)) = V_I(f(\cdot,y)) = \sup_{\mathscr{P}(I)} \sum_{i=1}^{k_1-1} |\Delta_{10} f(t_{i+1},y)|, \quad (5)$$

where the supremum is taken over all partitions $\{t_i\}_{i=1}^{k_1} \in \mathscr{P}(I)$.

For $x \in I$ is fixed, the variation of Jordan of function $f(x,\cdot) \in \mathbb{R}^J$ is defined by

$$V(f(x,\cdot)) = V_J(f(x,\cdot)) = \sup_{\mathscr{P}(J)} \sum_{j=1}^{k_2-1} \left|\Delta_{01} f\left(x,s_{j+1}\right)\right|, \quad (6)$$

where the supremum is taken over all partitions $\{s_j\}_{j=1}^{k_2} \in \mathscr{P}(J)$.

The variation of f in the sense of Hardy-Vitali in the rectangle I_a^b is defined by

$$V_{I_a^b}(f) := \sup_{\mathscr{P}(I),\mathscr{P}(J)} \sum_{i=1}^{k_1-1} \sum_{j=1}^{k_2-1} \left|\Delta_{11} f\left(t_{i+1},s_{j+1}\right)\right|, \quad (7)$$

where the supremum is taken over all partitions $\{t_i\}_{i=1}^{k_1} \in \mathscr{P}(I)$ and $\{s_j\}_{j=1}^{k_2} \in \mathscr{P}(J)$.

The total variation of the function f is defined by

$$\begin{aligned} TV(f) &= TV\left(f, I_a^b\right) \\ &:= V_I(f(\cdot,b_2)) + V_J(f(a_1,\cdot)) + V_{I_a^b}(f). \end{aligned} \quad (8)$$

We denote by $BV(I_a^b)$ the space of all functions having bounded total variation finite.

The notion of κ-variation was introduced by Korenblum in 1975 (see [24]) in one-dimensional case and Guerrero et al. in 2015 (see [17]) in two-dimension case.

Definition 5 (see [24]). Let $\kappa \in \mathscr{K}$, and the function $f \in \mathbb{R}^{[a,b]}$ is of bounded κ-variation if

$$\begin{aligned} \kappa V(f;[a,b]) = \kappa V(f) &:= \sup_{\mathscr{P}[a,b]} \frac{\sum_{i=1}^{k-1} |\Delta f(t_{i+1})|}{\sum_{i=1}^{k-1} \kappa(\Delta t_{i+1}/(b-a))} \\ &< \infty, \end{aligned} \quad (9)$$

where the supremum is taken over all partitions $\{t_i\}_{i=1}^{k} \in \mathscr{P}[a,b]$. We denote by $\kappa BV[a,b]$ the space of functions of bounded κ-variation on $[a,b]$.

Definition 6 (see [17]). Let $\kappa \in \mathscr{K}$, $f \in \mathbb{R}^{I_a^b}$, and $y \in J$ be fixed, and the Jordan κ-variation of the function $f(\cdot,y) \in \mathbb{R}^I$ is denoted by

$$\begin{aligned} \kappa V(f(\cdot,y)) &= \kappa V_I(f(\cdot,y)) \\ &= \sup_{\mathscr{P}(I)} \frac{\sum_{i=1}^{k_1-1} |\Delta_{10} f(t_{i+1},y)|}{\sum_{i=1}^{k_1-1} \kappa(\Delta t_{i+1}/(a_2-a_1))}, \end{aligned} \quad (10)$$

where the supremum is taken over all partitions $\{t_i\}_{i=1}^{k_1} \in \mathscr{P}(I)$.

For $x \in I$ is fixed, the Jordan κ-variation of the function $f(x,\cdot) \in \mathbb{R}^J$ is defined by

$$\begin{aligned} \kappa V(f(x,\cdot)) &= \kappa V_J(f(x,\cdot)) \\ &= \sup_{\mathscr{P}(J)} \frac{\sum_{j=1}^{k_2-1} \left|\Delta_{01} f\left(x,s_{j+1}\right)\right|}{\sum_{j=1}^{k_2-1} \kappa\left(\Delta s_{j+1}/(b_2-b_1)\right)}, \end{aligned} \quad (11)$$

where the supremum is taken over all partitions $\{s_j\}_{j=1}^{k_2} \in \mathscr{P}(J)$.

The two-dimensional Hardy-Vitali κ-variation of f in the rectangle I_a^b is defined by

$$\begin{aligned} &\kappa V_{I_a^b}(f) \\ &:= \sup_{\mathscr{P}(I),\mathscr{P}(J)} \frac{\sum_{i=1}^{k_1-1} \sum_{j=1}^{k_2-1} \left|\Delta_{11} f\left(t_{i+1},s_{j+1}\right)\right|}{\sum_{i=1}^{k_1-1} \sum_{j=1}^{k_2-1} \kappa(\Delta t_{i+1}\Delta s_{i+1}/(a_2-a_1)(b_2-b_1))}, \end{aligned} \quad (12)$$

where the supremum is taken over all partitions $\{t_i\}_{i=1}^{k_1} \in \mathscr{P}(I)$ and $\{s_j\}_{j=1}^{k_2} \in \mathscr{P}(J)$.

The total κ-variation of the function f is defined by

$$\begin{aligned} \kappa TV(f) &= \kappa TV\left(f, I_a^b\right) \\ &:= \kappa V_I(f(\cdot,b_2)) + \kappa V_J(f(a_1,\cdot)) \\ &\quad + \kappa V_{I_a^b}(f). \end{aligned} \quad (13)$$

We denote by $\kappa BV(I_a^b)$ the space of all functions having bounded κ-variation total.

The notion of Φ-variation was introduced by Schramm in 1985 (see [14]) in one-dimensional case and Ereú et al. in 2010 (see [30]) in two-dimensional case.

Definition 7 (see [14]). Let $\phi = \{\phi_n\}_{n\geq 1}$ be a Φ-sequence and $\sigma \in S_{k-1}$. The function $f \in \mathbb{R}^{[a,b]}$ is of bounded Φ-variation if

$$\begin{aligned} V_\phi^S(f;[a,b]) = V_\phi^S(f) &:= \sup_{\mathscr{P}[a,b]} \sum_{i=1}^{k-1} \phi_i\left(f\left(\Delta t_{\sigma(i)+1}\right)\right) \\ &< \infty, \end{aligned} \quad (14)$$

where the supremum is taken over all partitions $\{t_i\}_{i=1}^k \in \mathscr{P}[a,b]$ and $\sigma \in S_{k-1}$. The class of functions with bounded Φ-variation is denoted by $V_\phi^S[a,b]$ and the space generated by this class is denoted by $BV_\phi^S[a,b]$.

Definition 8 (see [30]). Let $\phi = \{\phi_{i,j}\}_{i,j\geq 1}$ be a Φ-sequence and let $f \in \mathbb{R}^{I_a^b}$ and $y \in J$ be fixed, and the Φ-variation in the Schramm sense of the function $f(\cdot, y) \in \mathbb{R}^I$ is defined by

$$V_\phi^S(f(\cdot,y)) = V_{\phi,I}^S(f(\cdot,y)) = \sup_{\mathscr{P}(I)} \sum_{i=1}^{k_1-1} \phi_{i,1}\left(\left|\Delta_{10} f\left(t_{\sigma_1(i)+1}, y\right)\right|\right), \tag{15}$$

where the supremum is taken over all partitions $\{t_i\}_{i=1}^{k_1} \in \mathscr{P}(I)$ and $\sigma_1 \in S_{k_1-1}$.

For $x \in I$ is fixed, the Φ-variation, in the Schramm sense of the function $f(x,\cdot) \in \mathbb{R}^J$, is defined by

$$V_\phi^S(f(x,\cdot)) = V_{\phi,J}^S(f(x,\cdot)) = \sup_{\mathscr{P}(J)} \sum_{j=1}^{k_2-1} \phi_{1,j}\left(\left|\Delta_{01} f\left(x, s_{\sigma_2(j)+1}\right)\right|\right), \tag{16}$$

where the supremum is taken over all partitions $\{s_j\}_{j=1}^{k_2} \in \mathscr{P}(J)$ and $\sigma_2 \in S_{k_2-1}$.

The bidimensional variation in the sense of Schramm of the function f in the rectangle I_a^b is defined by

$$V_{\phi,I_a^b}^S(f) := \sup_{\mathscr{P}(I),\mathscr{P}(J)} \sum_{i=1}^{k_1-1}\sum_{j=1}^{k_2-1} \phi_{i,j}\left(\left|\Delta_{11} f\left(t_{\sigma_1(i)+1}, s_{\sigma_2(j)+1}\right)\right|\right), \tag{17}$$

where the supremum is taken over all partitions $\{t_i\}_{i=1}^{k_1} \in \mathscr{P}(I)$ and $\{s_j\}_{j=1}^{k_2} \in \mathscr{P}(J)$ of the intervals I, J, respectively, and $\sigma_1 \in S_{k_1+1}$, $\sigma_2 \in S_{k_2+1}$.

The total Φ-variation of the function f is defined by

$$TV_{\phi,I_a^b}^S(f) = TV_\phi^S\left(f, I_a^b\right) := V_{\phi,I}^S(f(\cdot,b_2)) + V_{\phi,J}^S(f(a_1,\cdot)) + V_{\phi,I_a^b}^S(f). \tag{18}$$

The class of function with total bounded Φ-variation is denoted by $V_\phi^S(I_a^b)$ and the space generated by this class is denoted by $BV_\Phi^S(I_a^b)$.

In 1986 S. K. Kim and J. Kim (see [13]) combined the concepts of κ-variation and Φ-variation introduced by Korenblum and Schramm, respectively, to create the concept $\kappa\Phi$-variation in the Schramm-Korenblum sense.

Definition 9 (see [13]). Let $\phi = \{\phi_n\}_{n\geq 1}$ be a Φ-sequence, $\kappa \in \mathscr{K}$, and $\sigma \in S_{k-1}$. The function $f \in \mathbb{R}^{[a,b]}$ is of bounded $\kappa\Phi$-variation if

$$\kappa V_\phi(f;[a,b]) := \sup_{\mathscr{P}[a,b]} \frac{\sum_{i=1}^{k-1} \phi_i\left(f\left(\Delta t_{\sigma(i)+1}\right)\right)}{\sum_{i=1}^{k-1} \kappa\left(\Delta t_{i+1}/(b-a)\right)} < \infty, \tag{19}$$

where the supremum is taken over all partitions $\{t_i\}_{i=1}^k \in \mathscr{P}[a,b]$ and $\sigma \in S_{k-1}$. The vectorial space generated by this class of functions is denoted by $\kappa BV_\phi[a,b]$.

3. Main Results

In this section we present the main result of this paper, we generalize the concept of $\kappa\Phi$-variation in $[a,b]$, presented by S. K. Kim and J. Kim in [13], to the two-dimensional total $\kappa\Phi$-variation in I_a^b in the sense of Schramm-Korenblum, and we prove that the space $\kappa BV_\phi^S(I_a^b)$ is a Banach algebra.

Definition 10. Let $\phi = \{\phi_{i,j}\}_{i,j\geq 1}$ be a Φ-sequence and let $\kappa \in \mathscr{K}$, $f \in \mathbb{R}^{I_a^b}$, and $y \in J$ be fixed. The variation of the function f in the sense of Schramm-Korenblum of the function $f(\cdot,y) \in \mathbb{R}^I$ is denoted by

$$\kappa V_\phi^S(f(\cdot,y)) = \kappa V_{\phi,I}^S(f(\cdot,y)) = \sup_{\mathscr{P}(I)} \frac{\sum_{i=1}^{k_1-1} \phi_{i,1}\left(\left|\Delta_{10} f\left(t_{\sigma_1(i)+1}, y\right)\right|\right)}{\sum_{i=1}^{k_1-1} \kappa\left(\Delta t_{i+1}/(a_2-a_1)\right)}, \tag{20}$$

where the supremum is taken over all partitions $\{t_i\}_{i=1}^{k_1} \in \mathscr{P}(I)$ and $\sigma_1 \in S_{k_1-1}$.

For $x \in I$ is fixed, the variation of Schramm-Korenblum of the function $f(x,\cdot) \in \mathbb{R}^J$ is defined by

$$\kappa V_\phi^S(f(x,\cdot)) = \kappa V_{\phi,J}^S(f(x,\cdot)) = \sup_{\mathscr{P}(J)} \frac{\sum_{j=1}^{k_2-1} \phi_{1,j}\left(\left|\Delta_{01} f\left(x, s_{\sigma_2(j)+1}\right)\right|\right)}{\sum_{j=1}^{k_2-1} \kappa\left(\Delta S_{j+1}/(b_2-b_1)\right)}, \tag{21}$$

where the supremum is taken over all partitions $\{s_j\}_{j=1}^{k_2} \in \mathscr{P}(J)$ and $\sigma_2 \in S_{k_2-1}$.

The two-dimensional variation in the sense of Schramm-Korenblum of f in the rectangle I_a^b is defined by

$$\kappa V_{\phi,I_a^b}^S(f) := \sup_{\mathscr{P}(I),\mathscr{P}(J)} \frac{\sum_{i=1}^{k_1-1}\sum_{j=1}^{k_2-1} \phi_{i,j}\left(\left|\Delta_{11} f\left(t_{\sigma_1(i)+1}, s_{\sigma_2(j)+1}\right)\right|\right)}{\sum_{i=1}^{k_1-1}\sum_{j=1}^{k_2-1} \kappa\left(\Delta t_{i+1}\Delta s_{j+1}/(a_2-a_1)(b_2-b_1)\right)}, \tag{22}$$

where the supremum is taken over all partitions $\{t_i\}_{i=1}^{k_1} \in \mathscr{P}(I)$ and $\{s_j\}_{j=1}^{k_2} \in \mathscr{P}(J)$ of the intervals I, J, respectively, and $\sigma_1 \in S_{k_1-1}$, $\sigma_2 \in S_{k_2-1}$.

The total bidimensional $\kappa\Phi$-variation in the sense of Schramm-Korenblum of the function $f \in \mathbb{R}^{I_a^b}$ is defined by

$$\kappa TV_{\phi,I_a^b}^{S}(f) = \kappa TV_{\phi}^{S}(f, I_a^b) := \kappa V_{\phi,I}^{S}(f(\cdot,b_2)) + \kappa V_{\phi,J}^{S}(f(a_1,\cdot)) + \kappa V_{\phi,I_a^b}^{S}(f). \quad (23)$$

The class of functions $f \in \mathbb{R}^{I_a^b}$ with bounded $\kappa\Phi$-variation in the sense of Schramm-Korenblum is denoted by $\kappa V_{\phi}^{S}(I_a^b)$ and the space generated by this class is denoted by $\kappa BV_{\phi}^{S}(I_a^b)$.

Remark 11. (1) If $\phi_{i,j}(t) = t$, $t \geq 0$, $i, j \geq 1$, then the $\kappa\Phi$-variation in the sense of Schramm-Korenblum coincides with the Hardy-Vitali-Korenblum variation studied by Guerrero et al. in [17].

(2) If $\phi_{i,j} = \varphi$, $i, j \geq 1$, where φ is a Young function, then the Schramm-Korenblum $\kappa\Phi$-variation is a combination of the concepts of Wiener φ-variation and Korenblum-Hardy-Vitali bidimensional variation.

(3) For $\phi_{i,j}(t) = t/\lambda_i\lambda_j$, $t \geq 0$, $i, j \geq 1$, where $\{\lambda_i\}_{i\geq 1}$ is a Λ-sequence in the sense of Waterman, our notion is a combination of the Waterman bidimensional variation introduced by Sahakyan and also studied by Sablim (see [31, 32]) with Korenblum variation [24].

The following comprehensive type results give us interesting properties of the space $\kappa BV_{\phi}^{S}(I_a^b)$.

Theorem 12. *Let $f \in \mathbb{R}^{I_a^b}$. Then*

(1) *$\kappa TV_{\phi}^{S}(f, I_a^b) \geq 0$ and $\kappa TV_{\phi}^{S}(f) = 0$ if and only if $f =$ ctte;*

(2) *$\kappa V_{\phi}^{S}(I_a^b)$ is convex and symmetry set;*

(3) *$\kappa BV_{\phi}^{S}(I_a^b) = \{f \in \mathbb{R}^{I_a^b} : \exists\lambda > 0, \kappa V_{\phi}^{S}(\lambda f) < \infty\}$;*

(4) *if $\kappa_1, \kappa_2 \in \mathcal{K}$ and $\kappa_1 \leq \kappa_2$, then $\kappa_1 V_{\phi}^{S}(I_a^b) \subset \kappa_2 V_{\phi}^{S}(I_a^b)$. In particular $\kappa_1 BV_{\phi}^{S}(I_a^b) \subset \kappa_2 BV_{\phi}^{S}(I_a^b)$;*

(5) *$BV(I_a^b) \subset V_{\phi}^{s}(I_a^b) \subset \kappa V_{\phi}^{s}(I_a^b)$ and therefore $BV(I_a^b) \subset BV_{\phi}^{S}(I_a^b) \subset \kappa BV_{\phi}^{S}(I_a^b)$;*

(6) *if $\kappa TV_{\phi}^{S}(f, I_a^b) < \infty$, then f is bounded and*

$$\|f\|_{\infty} \leq |f(a_1,b_1)| + 3\phi_{1,1}^{-1}\left(4\kappa TV_{\phi,I_a^b}^{S}(f)\right). \quad (24)$$

Proof. (1) Suppose that $\kappa TV_{\phi}^{S}(f) = 0$; then $\kappa V_{\phi,I}^{S}(f) = \kappa V_{\phi,J}^{S}(f) = 0$. In particular $f(t,b_1) - f(a_1,b_1) = 0$ and $f(t,s) - f(t,b_1) = 0$, $(s,t) \in I \times J$; therefore

$$f(t,s) = f(t,s) - f(t,b_1) + f(t,b_1) = f(a_1,b_1). \quad (25)$$

The "only if" part is immediate.

Parts (2) and (4) follow from Definition 10.

(3) is consequences of (2).

(5) Let $f \in BV(I_a^b)$ and $\sigma \in S_{k-1}$. Consider a partition $\{t_i\}_{i=1}^{k} \in \mathcal{P}(I)$ and let us establish the set

$$A := \{i : \Delta_{10} f(t_{\sigma(i)+1}, b_2) \leq 1\}. \quad (26)$$

Since the functions $\phi_{i,j}$, $i, j \geq 1$, are convex and $V_{\phi,I}^{S} f(\cdot,b_2) < \infty$, then

$$\begin{aligned}\sum_{i=1}^{k-1}&\phi_{i,1}(|\Delta_{10} f(t_{\sigma(i)+1},b_2)|) \\ &= \sum_{i\in A}\phi_{i,1}(|\Delta_{10} f(t_{\sigma(i)+1},b_2)|) + \sum_{i\notin A}\phi_{i,1}(|\Delta_{10} f(t_{\sigma(i)+1},b_2)|) \\ &\leq \sum_{i\in A}\phi_{1,1}(1)|\Delta_{10} f(t_{\sigma(i)+1},b_2)| + \sum_{i\notin A}\phi_{i,1}(|\Delta_{10} f(t_{\sigma(i)+1},b_2)|) \\ &\leq \phi_{1,1}(1) V_I f(\cdot,b_2) + \sum_{i\notin A}\phi_{i,1}(|\Delta_{10} f(t_{\sigma(i)+1},b_2)|).\end{aligned} \quad (27)$$

The last sum contains at most $[V_{\phi,I}^{S} f(\cdot,b_2)]$ terms, because otherwise there would exist at least $[V_{\phi,I}^{S} f(\cdot,b_2)] + 1$ terms, so

$$V_{\phi,I}^{S}(f(\cdot,b_2)) \geq \sum_{i\notin A}\Delta_{10} f(t_{\sigma(i)+1},b_2) \geq \left[V_{\phi,I}^{S}(f(\cdot,b_2))\right] + 1, \quad (28)$$

which is absurd. Therefore

$$\sum_{i\notin A}\phi_{i,1}(|\Delta_{10} f(t_{\sigma(i)+1},b_2)|) \leq \sum_{i\notin A}\phi_{i,1}(2\|f\|_{\infty}) \leq \phi_{1,1}(4\|f\|_{\infty})[V_I(f(\cdot,b_2))], \quad (29)$$

so

$$V_{\phi}^{S} f(\cdot,b_1) \leq (\phi_{1,1}(1) + \phi_{1,1}(4\|f\|_{\infty}))\left[V_{\phi,I}^{S}(f(\cdot,b_2))\right]. \quad (30)$$

Similarly

$$V_{\phi}^{S} f(a_1,\cdot) \leq (\phi_{1,1}(1) + \phi_{1,1}(4\|f\|_{\infty}))\left[V_{\phi,J}^{S}(f(a_1,\cdot))\right], \quad (31)$$

$$V_{\phi,I_a^b}^{S}(f) \leq (\phi_{1,1}(1) + \phi_{1,1}(4\|f\|_{\infty})) V_{I_a^b}(f).$$

Finally

$$TV_{\phi,I_a^b}^{S}(f) \leq (\phi_{1,1}(1) + \phi_{1,1}(4\|f\|_{\infty})) V_{I_a^b}(f), \quad (32)$$

so the first inclusion $BV(I_a^b) \subset V_{\phi}^{S}(I_a^b)$ holds.

The other inclusion $V_\phi^S(I_a^b) \subset \kappa V_\phi^S(I_a^b)$ is a consequence of the subadditive of the function κ.

(6) Let $(x, y) \in I_a^b$, and then

$$\begin{aligned}&\left|f(x,y)-f(a_1,b_2)\right| \\ &\quad\le \left|f(x,y)-f(a_1,y)-f(x,b_2)+f(a_1,b_2)\right| \\ &\qquad+\left|f(a_1,y)+f(x,b_2)-2f(a_1,b_2)\right| \\ &\quad\le \left|f(x,y)-f(a_1,y)-f(x,b_2)+f(a_1,b_2)\right| \\ &\qquad+\left|f(a_1,y)-f(a_1,b_2)\right| \\ &\qquad+\left|f(x,b_2)-f(a_1,b_2)\right|.\end{aligned} \tag{33}$$

On the other hand

$$\begin{aligned}&\frac{\phi_{1,1}\left(\left|f(x,b_2)-f(a_1,b_2)\right|\right)}{\kappa\left(\left|x-a_1\right|/(a_2-a_1)\right)+\kappa\left(\left|a_2-x\right|/(a_2-a_1)\right)} \\ &\quad\le \kappa V_{\phi,I}^S\left(f(\cdot,b_2)\right).\end{aligned} \tag{34}$$

As $0 \le \kappa(t) \le 1$, it follows that

$$\phi_{1,1}\left(\left|f(x,b_2)-f(a_1,b_2)\right|\right) \le 2\kappa V_{\phi,I}^S\left(f(\cdot,b_2)\right). \tag{35}$$

Therefore

$$\left|f(x,b_2)-f(a_1,b_2)\right| \le \phi_{1,1}^{-1}\left(2\kappa V_{\phi,I}^S\left(f(\cdot,b_2)\right)\right). \tag{36}$$

Similarly

$$\begin{aligned}&\left|f(a_1,y)-f(a_1,b_2)\right| \le \phi_{1,1}^{-1}\left(2\kappa V_{\phi,J}^S\left(f(a_1,\cdot)\right)\right), \\ &\left|f(x,y)-f(a_1,y)-f(x,b_2)+f(a_1,b_2)\right| \\ &\quad\le \phi_{1,1}^{-1}\left(4\kappa V_{\phi,I_a^b}^S(f)\right).\end{aligned} \tag{37}$$

By inequalities (36) and (37) and the definition of total bidimensional $\kappa\Phi$-variation in the sense of Schramm-Korenblum, it follows that

$$\|f\|_\infty \le \left|f(a_1,b_2)\right| + 3\phi_{1,1}^{-1}\left(4\kappa TV_{\phi,I_a^b}^S(f)\right). \tag{38}$$

□

The following proposition allows us to calculate the total bidimensional $\kappa\Phi$-variation in the sense of Schramm-Korenblum of a sum of monotone functions.

Proposition 13. *Let $\phi = \{\phi_{i,j}\}_{i,j\ge1}$ be a Φ-sequence, $\kappa \in \mathscr{K}$, $I_a^b = I \times J$, and $f \in \mathbb{R}^{I_a^b}$, such that $f(t,s) = g(t) + h(s)$, $(t,s) \in I\times J$, where $g \in \mathbb{R}^I$ and $h \in \mathbb{R}^J$ are monotone functions. Then*

(a) $\kappa V_\phi^S f(\cdot,y) = \phi_{1,1}(|g(a_2)-g(a_1)|)$, $y \in J$;

(b) $\kappa V_\phi^S f(x,\cdot) = \phi_{1,1}(|h(b_2)-h(b_1)|)$, $x \in J$;

(c) $\kappa V_{\phi,I_a^b}^S(f) = 0$;

(d) $\kappa TV_{\phi,I_a^b}^S(f) = \phi_{1,1}(|g(a_2)-g(a_1)|)+\phi_{1,1}(|h(b_2)-h(b_1)|)$.

Proof. Since the functions $\phi_{i,j}$, $i,j \ge 1$, are convex then the function $t \to \phi_{i,j}(t)/t$, $t > 0$, $i,j \ge 1$, is increasing.

(a) Let $y \in J$ and $\sigma \in S_{k_1-1}$ and let $\{t_i\}_{i=1}^{k_1}$ and $\{s_j\}_{j=1}^{k_2}$ be partitions of the intervals I, J, respectively. Then

$$\begin{aligned}&\frac{\sum_{i=1}^{k_1-1}\phi_{i,1}\left(\left|\Delta_{10}f\left(t_{\sigma(i)+1},y\right)\right|\right)}{\sum_{i=1}^{k_1-1}\kappa\left(\Delta t_{i+1}/(a_2-a_1)\right)} \\ &= \frac{\sum_{i=1}^{k_1-1}\left(\phi_{1,1}\left(\left|\Delta g\left(t_{\sigma(i)+1}\right)\right|\right)\left|\Delta g\left(t_{\sigma(i)+1}\right)\right|/\left|\Delta g\left(t_{\sigma(i)+1}\right)\right|\right)}{\sum_{i=1}^{k_1-1}\kappa\left(\Delta t_{i+1}/(a_2-a_1)\right)}.\end{aligned} \tag{39}$$

Since g is monotone and κ is subadditive, we get

$$\begin{aligned}&\frac{\sum_{i=1}^{k_1-1}\phi_{1,1}\left(\left|\Delta g\left(t_{\sigma(i)+1}\right)\right|\right)}{\sum_{i=1}^{k_1-1}\kappa\left(\Delta t_{i+1}/(a_2-a_1)\right)} \\ &\quad\le \frac{\phi_{1,1}\left(\left|g(a_2)-g(a_1)\right|\right)}{\left|g(a_2)-g(a_1)\right|}\sum_{i=1}^{k_1-1}\left|\Delta g\left(t_{\sigma(i)+1}\right)\right| \\ &\quad= \phi_{1,1}\left(\left|g(a_2)-g(a_1)\right|\right).\end{aligned} \tag{40}$$

Since g is monotone and κ is subadditive then

$$\begin{aligned}&\sum_{i=1}^{k_1-1}\phi_{1,1}\left(\left|\Delta g\left(t_{\sigma(i)+1}\right)\right|\right) = \phi_{1,1}\left(\left|g(a_2)-g(a_1)\right|\right), \\ &\kappa(1) \le \sum_{i=1}^{k_1-1}\kappa\left(\frac{\Delta t_{i+1}}{a_2-a_1}\right).\end{aligned} \tag{41}$$

Therefore

$$\kappa V_\phi^S\left(f(\cdot,y)\right) \le \phi_{1,1}\left(\left|g(a_2)-g(a_1)\right|\right), \quad y \in J. \tag{42}$$

Reciprocally

$$\begin{aligned}&\phi_{1,1}\left(\left|g(a_2)-g(a_1)\right|\right) \\ &\quad= \phi_{1,1}\left(f(a_2,y)-f(a_1,y)\right) \le \kappa V_\phi^S f(\cdot,y), \\ &\qquad y \in J.\end{aligned} \tag{43}$$

Thus

$$\kappa V_\phi^S\left(f(\cdot,y)\right) = \phi_{1,1}\left(\left|g(a_2)-g(a_1)\right|\right), \quad y \in J. \tag{44}$$

(b) With a similar reasoning as in part (a) we get

$$\kappa V_\phi^S\left(f(x,\cdot)\right) = \phi_{1,1}\left(\left|h(b_2)-h(b_1)\right|\right), \quad x \in I. \tag{45}$$

(c) If $\sigma_1 \in S_{k_1-1}$ $\sigma_2 \in S_{k_2-1}$, then

$$\begin{aligned}&\Delta_{11}f\left(t_{\sigma_1(i)+1},s_{\sigma_2(j)+1}\right) \\ &\quad= f\left(t_{\sigma_1(i)},s_{\sigma_2(j)}\right)-f\left(t_{\sigma_1(i)},s_{\sigma_2(j)+1}\right) \\ &\qquad+ f\left(t_{\sigma_1(i)+1},s_{\sigma_2(j)+1}\right)-f\left(t_{\sigma_1(i)+1},s_{\sigma_2(j)}\right) = 0,\end{aligned} \tag{46}$$

$i \in \{1,\ldots,k_1-1\}$, $j \in \{1,\ldots,k_2-1\}$. Therefore $V_{\phi,I_a^b}^S(f) = 0$.

(d) Is a consequence of Definition 10. □

Example 14. Particular cases are obtained when $f : [0,1] \times [0,1] \to \mathbb{R}$ is a function given by the following:

(1) $f(t,s) = -5t+9s^2+7$; then $\kappa TV_\phi^S(f) = \phi_{1,1}(5)+\phi_{1,1}(9)$;

(2) $f(t,s) = 3\sqrt{t} + \ln(t+1) - 5\sqrt[3]{s^2}$; then $\kappa TV_\phi^S(f) = \phi_{1,1}(3\ln 2) + \phi_{1,1}(5)$.

Below we present two lemmas that play an important role in what follows.

Lemma 15. *Let $\phi = \{\phi_{i,j}\}_{i,j\geq 1}$ be a Φ-sequence and $k \in \mathscr{K}$. Then the operator $\kappa TV_\phi^S : \kappa V_\phi^S(I_a^b) \to [0,\infty)$, defined by $\kappa TV_\phi^S(f) := \kappa TV_{\phi,I_b^a}^S(f)$, is convex and the set*

$$\kappa A_\phi^S\left(I_a^b\right) := \left\{f \in \kappa BV_\phi^S\left(I_a^b\right) : \kappa TV_{\phi,I_a^b}^S\left(f\right) \leq 1\right\} \tag{47}$$

is convex, symmetric, and absorbent.

Proof. The proof is straightforward; see [33]. □

The Minkowski functional associated with $\kappa A_\phi^S(I_a^b)$, $\kappa\mu_\phi^S : \kappa BV_\phi^S(I_a^b) \to [0,\infty)$, defined by

$$\kappa\mu_\phi^S\left(f\right) := \inf\left\{\lambda > 0 : \kappa TV_{\phi,I_a^b}^S\left(\frac{f}{\lambda}\right) < 1\right\}, \quad f \in \kappa BV_\phi^S\left(I_a^b\right), \tag{48}$$

is a seminorm.

Lemma 16. *Let $f \in \kappa BV_\phi^S(I_a^b)$. Then there exist $\lambda > 0$, such that $\kappa\mu_\phi^S(f) \leq \lambda$ if and only if $\kappa TV_{\phi,I_a^b}^S(f/\lambda) < 1$.*

Proof. If $\kappa\mu_\phi^S(f) < \lambda$, $\lambda > 0$, then there exist $0 < k < \lambda$, such that $\kappa TV_{\phi,I_a^b}^S(f/k) \leq 1$. Then

$$\kappa TV_{\phi,I_a^b}^S\left(\frac{f}{\lambda}\right) = \kappa TV_{\phi,I_a^b}^S\left(\frac{k}{\lambda}\frac{f}{k}\right) \leq \frac{k}{\lambda}\kappa TV_{\phi,I_a^b}^S\left(\frac{f}{k}\right) \leq 1. \tag{49}$$

If $\kappa\mu_\phi^S(f) = \lambda$, $\lambda > 0$, then $\kappa\mu_\phi^S(f) < \lambda_n$. Therefore there exists a sequence $\{\lambda_n\}_{n\geq 1}$, such that $\lambda_n \downarrow \lambda$; then

$$\kappa TV_{\phi,I_a^b}^S\left(\frac{f}{\lambda_n}\right) < 1. \tag{50}$$

Taking limit as $n \to \infty$ and the result follow. The other part is a consequence of the definition of $\kappa TV_{\phi,I_a^b}^S(\cdot)$ and the continuity of the functions $\phi_{i,j}$, $i, j \geq 1$, and taking limit as $n \to \infty$ we get desired result.

The "only if" part is consequence of the definition of $\kappa\mu_\phi^S$. □

In the next theorem we will prove that the functional $\kappa\|\cdot\|_\phi^s : \kappa BV_\phi^S(I_a^b) \to [0,\infty)$, defined by

$$\kappa\left\|f\right\|_\phi^s := \left|f\left(a_1,b_2\right)\right| + \kappa\mu_\phi^S\left(f\right), \tag{51}$$

is a norm.

Theorem 17. *$(\kappa BV_\phi^S(I_a^b), \kappa\|\cdot\|_\phi^s)$ is a normed space.*

Proof. As $\kappa\|\cdot\|_\phi^s$ is a seminorm we only need to show that if $\kappa\|f\|_\phi^s = 0$ then $f \equiv 0$.

If $\kappa\mu_\phi^S(f) = 0$, then $f(a_1,b_2) = 0$ and $\kappa\mu_\phi^S(f) < 1/n$, for all $n \geq 1$. Accordingly

$$n\kappa TV_{\phi,I_a^b}^S\left(f\right) \leq \kappa TV_{\phi,I_a^b}^S\left(nf\right) < 1, \quad n \geq 1. \tag{52}$$

Hence $\kappa TV_{\phi,I_a^b}^S(f) = 0$. By part (a) of Theorem 12, $f = $ *ctte*. As $f(a_1,b_2) = 0$, we obtain $f \equiv 0$. □

We will use the next lemma at several places in this work.

Lemma 18. *Let $\{t_i\}_{i=1}^{k_1} \in \mathscr{P}(I)$, $\{s_j\}_{j=1}^{k_2} \in \mathscr{P}(J)$, $\lambda > 0$, and $f, g : I_a^b \to \mathbb{R}$ be bounded functions. Then*

(a) *$\Delta_{10}(fg)(t_{i+1},y) \leq \|g\|_\infty\Delta_{10}f(t_{i+1},y) + \|f\|_\infty\Delta_{10}g(t_{i+1},y)$, $y \in J$, $i = 1,\dots,k_1-1$;*

(b) *$\Delta_{01}(fg)(x,s_{j+1}) \leq \|g\|_\infty\Delta_{01}f(x,s_{j+1})+\|f\|_\infty\Delta_{01}g(x,s_{j+1})$, $x \in I$, $j = 1,\dots,k_2-1$;*

(c) *$\Delta_{11}(fg)(t_{i+1},s_{j+1}) \leq \|g\|_\infty\Delta_{11}f(t_{i+1},s_{j+1}) + \|f\|_\infty\Delta_{11}g(t_{i+1},s_{j+1})$, $i = 1,\dots,k_1-1$, $j = 1,\dots,k_2-1$.*

Proof. (a) Consider

$$\begin{aligned}
\Delta_{10}\left(fg\right)\left(t_{i+1},y\right) &= \left[\left(fg\right)\left(t_{i+1},y\right) - \left(fg\right)\left(t_i,y\right)\right] \\
&= \left[f\left(t_{i+1},y\right)g\left(t_{i+1},y\right) - f\left(t_i,y\right)g\left(t_i,y\right)\right] \\
&= \left[f\left(t_{i+1},y\right)g\left(t_{i+1},y\right) - f\left(t_i,y\right)g\left(t_i,y\right)\right. \\
&\quad \left.+ f\left(t_i,y\right)g\left(t_{i+1},y\right) - f\left(t_i,y\right)g\left(t_{i+1},y\right)\right] \\
&= \left[\left(f\left(t_{i+1},y\right) - f\left(t_i,y\right)\right)g\left(t_{i+1},y\right)\right. \\
&\quad \left.+ \left(g\left(t_{i+1},y\right) - g\left(t_i,y\right)\right)f\left(t_i,y\right)\right] \\
&= \left[g\left(t_{i+1},y\right)\Delta_{10}f\left(t_{i+1},y\right)\right. \\
&\quad \left.+ f\left(t_i,y\right)\Delta_{10}g\left(t_{i+1},y\right)\right] \leq \left\|g\right\|_\infty\Delta_{10}f\left(t_{i+1},y\right) \\
&\quad + \left\|f\right\|_\infty\Delta_{10}g\left(t_{i+1},y\right).
\end{aligned} \tag{53}$$

(b) It follows by reasoning as in part (a).

(c) For i, j are fixed, we obtain

$$\begin{aligned}
\Delta_{11}(fg)(t_{i+1},s_{j+1}) &= [(fg)(t_i,s_j)-(fg)(t_i,s_{j+1}) \\
&\quad -(fg)(t_{i+1},s_j)+(fg)(t_{i+1},s_{j+1})] \\
&= [f(t_i,s_j)g(t_i,s_j)-f(t_i,s_{j+1})g(t_i,s_{j+1}) \\
&\quad -f(t_{i+1},s_j)g(t_{i+1},s_j) \\
&\quad +f(t_{i+1},s_{j+1})g(t_{i+1},s_{j+1})] = [f(t_i,s_j)g(t_i,s_j) \\
&\quad -f(t_i,s_{j+1})g(t_i,s_{j+1})+f(t_{i+1},s_j)g(t_i,s_j) \\
&\quad -f(t_{i+1},s_j)g(t_i,s_j)+f(t_i,s_{j+1})g(t_{i+1},s_{j+1}) \\
&\quad -f(t_i,s_{j+1})g(t_{i+1},s_{j+1})-f(t_{i+1},s_j)g(t_{i+1},s_j) \\
&\quad +f(t_{i+1},s_{j+1})g(t_{i+1},s_{j+1})] \\
&= [(f(t_i,s_j)-f(t_{i+1},s_j))g(t_i,s_j) \\
&\quad +(f(t_{i+1},s_{j+1})-f(t_i,s_{j+1}))g(t_{i+1},s_{j+1}) \\
&\quad +(g(t_{i+1},s_{j+1})-g(t_i,s_{j+1}))f(t_i,s_{j+1}) \\
&\quad +(g(t_i,s_j)-g(t_{i+1},s_j))f(t_{i+1},s_j)] \\
&\le [(f(t_i,s_j)-f(t_{i+1},s_j))\|g\|_\infty \\
&\quad +(f(t_{i+1},s_{j+1})-f(t_i,s_{j+1}))\|g\|_\infty \\
&\quad +(g(t_{i+1},s_{j+1})-g(t_i,s_{j+1}))\|f\|_\infty \\
&\quad +(g(t_i,s_j)-g(t_{i+1},s_j))\|f\|_\infty] \\
&= [\Delta_{11}f(t_{i+1},s_{j+1})\|g\|_\infty \\
&\quad +\Delta_{11}g(t_{i+1},s_{j+1})\|f\|_\infty].
\end{aligned} \tag{54}$$

□

Theorem 19. *$\kappa BV_\phi^S(I_a^b)$ is an algebra.*

Proof. Let $f,g\in\kappa BV_\phi^S(I_a^b)$ (f and g not identically zero) and $\phi=\{\phi_{i,j}\}_{i,j\ge1}$, a Φ-sequence; then there exist $\alpha>0$ and $\beta>0$ such that

$$\begin{aligned}
&\kappa TV_{\phi,I_a^b}^S(\alpha f)<\infty, \\
&\kappa TV_{\phi,I_a^b}^S(\beta g)<\infty.
\end{aligned} \tag{55}$$

We set

$$\begin{aligned}
\lambda &:= \frac{\alpha\beta}{\alpha\|f\|_\infty+\beta\|g\|_\infty}, \\
\lambda_1 &:= \frac{\beta\|g\|_\infty}{\alpha\|f\|_\infty+\beta\|g\|_\infty}, \\
\lambda_2 &:= \frac{\alpha\|f\|_\infty}{\alpha\|f\|_\infty+\beta\|g\|_\infty}.
\end{aligned} \tag{56}$$

Now, $x\in I$, $y\in J$ are fixed, $\{t_i\}_{i=1}^{k_1}\in\mathcal{P}(I)$, $\{s_j\}_{j=1}^{k_2}\in\mathcal{P}(J)$, $\sigma_1\in S_{k_1-1}$, and $\sigma_2\in S_{k_2-1}$. Then by Lemma 18 we have

$$\begin{aligned}
&\Delta_{10}(\lambda fg)(t_{\sigma_1(i)+1},y) \\
&\quad\le \lambda\|g\|_\infty\Delta_{10}f(t_{\sigma_1(i)+1},y) \\
&\qquad +\lambda\|f\|_\infty\Delta_{10}g(t_{\sigma_1(i)+1},y) \\
&\quad= \frac{\beta\|g\|_\infty}{\alpha\|f\|_\infty+\beta\|g\|_\infty}\alpha\Delta_{10}f(t_{\sigma_1(i)+1},y) \\
&\qquad +\frac{\alpha\|f\|_\infty}{\alpha\|f\|_\infty+\beta\|g\|_\infty}\beta\Delta_{10}g(t_{\sigma_1(i)+1},y);
\end{aligned} \tag{57}$$

therefore

$$\begin{aligned}
&\Delta_{10}(\lambda fg)(t_{\sigma_1(i)+1},y) \\
&\quad\le \lambda_1\alpha\Delta_{10}f(t_{\sigma_1(i)+1},y)+\lambda_2\beta\Delta_{10}g(t_{\sigma_1(i)+1},y).
\end{aligned} \tag{58}$$

Similarly

$$\begin{aligned}
&\Delta_{01}(\lambda fg)(x,s_{\sigma_2(j)+1})\le\lambda_1\alpha\Delta_{01}f(x,s_{\sigma_2(j)+1}) \\
&\quad +\lambda_2\beta\Delta_{01}g(x,s_{\sigma_2(j)+1}),
\end{aligned} \tag{59}$$

$$\begin{aligned}
&\Delta_{11}(\lambda fg)(t_{\sigma_1(i)+1},s_{\sigma_2(j)+1}) \\
&\quad= \lambda[\Delta_{11}f(t_{\sigma_1(i)+1},s_{\sigma_2(j)+1})\|g\|_\infty \\
&\quad +\Delta_{11}g(t_{\sigma_1(i)+1},s_{\sigma_2(j)+1})\|f\|_\infty] \\
&\quad= \frac{\beta\|g\|_\infty}{\alpha\|f\|_\infty+\beta\|g\|_\infty}\alpha\Delta_{11}f(t_{\sigma_1(i)+1},s_{\sigma_2(j)+1}) \\
&\quad +\frac{\alpha\|f\|_\infty}{\alpha\|f\|_\infty+\beta\|g\|_\infty}\beta\Delta_{11}g(t_{\sigma_1(i)+1},s_{\sigma_2(j)+1});
\end{aligned} \tag{60}$$

we get

$$\begin{aligned}&\Delta_{11}(\lambda fg)\left(t_{\sigma_1(i)+1}, s_{\sigma_2(j)+1}\right)\\&\quad\leq \lambda_1\alpha\Delta_{11}f\left(t_{\sigma_1(i)+1}, s_{\sigma_2(j)+1}\right)\\&\qquad+\lambda_2\beta\Delta_{11}g\left(t_{\sigma_1(i)+1}, s_{\sigma_2(j)+1}\right).\end{aligned} \tag{61}$$

Next, combining (58), (59), and (61) with the fact that the functions $\phi_{i,j}$, $i, j \geq 1$, are increasing and convex and $\lambda_1, \lambda_2 > 0$, $\lambda_1 + \lambda_2 = 1$, we obtain

$$\begin{aligned}\kappa TV_{\phi,I_a^b}(\lambda fg) &\leq \lambda_1\kappa V_{\phi,I}^S(\alpha f)+\lambda_2\kappa V_{\phi,I}^S(\beta g)\\&\quad+\lambda_1\kappa V_{\phi,J}^S(\alpha f)+\lambda_2\kappa V_{\phi,J}^S(\beta g)\\&\quad+\lambda_1\kappa V_{\phi,I_a^b}^S(\alpha f)+\lambda_2\kappa V_{\phi,I_a^b}^S(\beta g)\\&=\lambda_1\kappa TV_{\phi,I}^S(\alpha f)+\lambda_2\kappa TV_{\phi,I}^S(\beta g)\\&<\infty;\end{aligned} \tag{62}$$

therefore $fg \in \kappa BV_\phi^S(I_a^b)$. □

The main result of our paper now reads as follows.

Theorem 20. *$(\kappa BV_\phi^S(I_a^b), \kappa\|\cdot\|_\phi^s)$ is a Banach space.*

Proof. Let $\{f_n\}_{n\geq 1}$ be a Cauchy sequence in $\kappa BV_\phi^S(I_a^b)$; then, for each $\varepsilon > 0$, there exists N_ε, such that if $n, m \geq N_\varepsilon$, then

$$\left|f_n(a_1, b_2) - f_m(a_1, b_2)\right| < \varepsilon, \tag{63}$$

and by Lemma 16 we get

$$\begin{aligned}\max\Bigg\{&\kappa V_{\phi,I}^S\left(\left(\frac{f_n-f_m}{\varepsilon}\right)(\cdot, b_2)\right),\\&\kappa V_{\phi,J}^S\left(\left(\frac{f_n-f_m}{\varepsilon}\right)(a_1,\cdot)\right),\kappa V_{\phi,I_a^b}^S\left(\frac{f_n-f_m}{\varepsilon}\right)\Bigg\}<\varepsilon.\end{aligned} \tag{64}$$

By the definition of $\kappa V_{\phi,I}^S(((f_n - f_m)/\varepsilon)(\cdot, b_2))$ and considering $a_1 < x \leq a_2$ we get

$$\begin{aligned}&\frac{\phi_{1,1}\left(\left|(f_n-f_m)(x,b_2)-(f_n-f_m)(a_1,b_2)\right|/\varepsilon\right)}{\kappa((x-a_1)/(a_2-a_1))+\kappa((a_2-x)/(a_2-a_1))}<1,\\&\qquad n,m\geq N_\varepsilon,\ a_1<x\leq a_2.\end{aligned} \tag{65}$$

Due to $0 \leq \kappa(t) \leq 1$, $t \in [0, 1]$, and (63), we obtain

$$\begin{aligned}&\left|(f_n-f_m)(x,b_2)\right|<\left(\phi_{1,1}^{-1}(2)+1\right)\varepsilon,\\&\qquad n,m\geq N_\varepsilon,\ x\in I.\end{aligned} \tag{66}$$

Similarly, we can consider the definition of $\kappa V_{\phi,J}^S(((f_n - f_m)/\varepsilon)(a_1,\cdot))$ and get

$$\begin{aligned}&\left|(f_n-f_m)(a_1,y)\right|<\left(\phi_{1,1}^{-1}(2)+1\right)\varepsilon,\\&\qquad n,m\geq N_\varepsilon,\ y\in J.\end{aligned} \tag{67}$$

By the definition of $\kappa V_{\phi,I_a^b}^S((f_n - f_m)/\varepsilon)$

$$\begin{aligned}&\frac{\phi_{1,1}\left(\left|(f_n-f_m)(a_1,y)-(f_n-f_m)(x,y)-(f_n-f_m)(a_1,b_2)+(f_n-f_m)(x,b_2)\right|/\varepsilon\right)}{\kappa((x-a_1)(y-b_1)/(a_2-a_1)(b_2-b_1))+\kappa((x-a_1)(b_2-y)/(a_2-a_1)(b_2-b_1))+\kappa((a_2-x)(y-b_1)/(a_2-a_1)(b_2-b_1))+\kappa((a_2-x)(b_2-y)/(a_2-a_1)(b_2-b_1))}\\&<1,\end{aligned} \tag{68}$$

$n, m \geq N_\varepsilon$, $(t, s) \in I \times J$, and $a_1 < x \leq a_2$. Consider

$$\begin{aligned}&\left|(f_n-f_m)(x,y)\right|<3\left(\phi_{1,1}^{-1}(4)+1\right)\varepsilon,\\&\qquad n,m>N_\varepsilon,\ (x,y)\in I\times J.\end{aligned} \tag{69}$$

Therefore $\{f_n\}_{n\geq 1}$ is a uniformly Cauchy sequence in $I \times J$, so there exist $f \in \mathbb{R}^{I_a^b}$, such that $f_n(x, y) \to f(x, y)$, as $n \to \infty$. By the continuity of the functions $\phi_{i,j}$, $i, j \geq 1$, and Definition 10 we conclude that $\kappa TV_{\phi_{i,j},I_a^b}^S(f) < \infty$, as $n \to \infty$.

With a similar argument

$$\begin{aligned}&\lim_{n\to\infty}\lim_{m\to\infty}\kappa TV_{\phi_{i,j},I_a^b}^S(f_n-f_m)\\&\quad=\lim_{n\to\infty}\kappa TV_{\phi_{i,j},I_a^b}^S(f_n-f)=0.\end{aligned} \tag{70}$$

□

Theorem 21. *The space $\kappa BV_\phi^S(I_a^b)$ equipped with the norm $\kappa\|\cdot\|_\phi^s$ is a Banach algebra.*

Proof. Let $f, g \in \kappa BV_\phi^S(I_a^b)$ such that they are not identically zero or constant functions and

$$\lambda := \kappa\mu_\phi^s(f)\|g\|_\infty + \kappa\mu_\phi^s(g)\|f\|_\infty. \tag{71}$$

By Lemma 16, $\kappa\mu_\phi^S(f/\kappa\|f\|_\phi^s) < 1$. By Lemma 18

$$\begin{aligned}&\Delta_{10}\left(\frac{fg}{\lambda}\right)\left(t_{\sigma_1(i)+1}, y\right)\\&\quad\leq\frac{\|g\|_\infty}{\lambda}\Delta_{10}f\left(t_{\sigma_1(i)+1}, y\right)+\frac{\|f\|_\infty}{\lambda}\Delta_{10}g\left(t_{\sigma_1(i)+1}, y\right)\\&\quad=\frac{\kappa\mu_\phi^s(f)\|g\|_\infty}{\lambda}\frac{\Delta_{10}f\left(t_{\sigma_1(i)+1}, y\right)}{\kappa\mu_\phi^s(f)}\\&\qquad+\frac{\kappa\mu_\phi^s(g)\|f\|_\infty}{\lambda}\frac{\Delta_{10}g\left(t_{\sigma_1(i)+1}, y\right)}{\kappa\mu_\phi^s(g)}.\end{aligned} \tag{72}$$

Similarly

$$\Delta_{01}\left(\frac{fg}{\lambda}\right)\left(x, s_{\sigma_2(j)+1}\right) \le \frac{\kappa\mu_\phi^s(f)\|g\|_\infty}{\lambda}\frac{\Delta_{01}f\left(x, s_{\sigma_2(j)+1}\right)}{\kappa\mu_\phi^s(f)} + \frac{\kappa\mu_\phi^s(g)\|f\|_\infty}{\lambda}\frac{\Delta_{01}g\left(x, s_{\sigma_2(j)+1}\right)}{\kappa\mu_\phi^s(g)},$$

$$\Delta_{11}\left(\frac{fg}{\lambda}\right)\left(t_{\sigma_1(i)+1}, s_{\sigma_2(j)+1}\right) \le \frac{\kappa\mu_\phi^s(f)\|g\|_\infty}{\lambda}\frac{\Delta_{11}f\left(t_{\sigma_1(i)+1}, s_{\sigma_2(j)+1}\right)}{\kappa\mu_\phi^s(f)} + \frac{\kappa\mu_\phi^s(g)\|f\|_\infty}{\lambda}\frac{\Delta_{11}g\left(t_{\sigma_1(i)+1}, s_{\sigma_2(j)+1}\right)}{\kappa\mu_\phi^s(g)}. \quad (73)$$

As

$$\frac{\kappa\mu_\phi^s(f)\|g\|_\infty}{\lambda} + \frac{\kappa\mu_\phi^s(g)\|f\|_\infty}{\lambda} = 1. \quad (74)$$

By inequalities (72) and (73), Definition 1, the convexity of functions $\phi_{i,j}$, $i, j \ge 1$, and the convexity of operator $\kappa TV^S_{\phi,I_a^b}$ we have

$$\kappa TV^S_{\phi,I_a^b}\left(\frac{fg}{\lambda}\right) \le \frac{\kappa\mu_\phi^s(f)\|g\|_\infty}{\lambda}\kappa TV^S_{\phi,I_a^b}\left(\frac{f}{\kappa\mu_\phi^S(f)}\right) + \frac{\kappa\mu_\phi^s(g)\|f\|_\infty}{\lambda}\kappa TV^s_{\phi,I_a^b}\left(\frac{g}{\kappa\mu_\phi^s(g)}\right). \quad (75)$$

We also have by Lemma 16

$$\kappa TV^S_{\phi,I_a^b}\left(\frac{f}{\kappa\mu_\phi^S(f)}\right) < 1, \qquad \kappa TV^S_{\phi,I_a^b}\left(\frac{g}{\kappa\mu_\phi^S(g)}\right) < 1. \quad (76)$$

Thus $\kappa TV^S_{\phi,I_a^b}(fg/\lambda) < 1$. Using Lemma 16 again

$$\begin{aligned} \kappa\mu_\phi^s(fg) &\le \lambda = \kappa\mu_\phi^s(f)\|g\|_\infty + \kappa\mu_\phi^s(g)\|f\|_\infty, \\ \kappa\|fg\|_\phi^s &\le 2\left|f(a_1,b_2)\right|\left|g(a_1,b_2)\right| + \kappa\mu_\phi^s(f)\|g\|_\infty + \kappa\mu_\phi^s(g)\|f\|_\infty \\ &\le \left|f(a_1,b_2)\right|\|g\|_\infty + \left|g(a_1,b_2)\right|\|f\|_\infty + \kappa\mu_\phi^s(f)\|g\|_\infty + \kappa\mu_\phi^s(g)\|f\|_\infty \\ &= \kappa\|f\|_\phi^s\|g\|_\infty + \kappa\|g\|_\phi^s\|f\|_\infty. \end{aligned} \quad (77)$$

Clearly, all the conditions in Section 4.1 of [4] are satisfied. Hence the conditions in Section 4.1 of [4] ensure that the space $\kappa BV^S_\phi(I_a^b)$ is a Banach algebra equipped with the norm $\|\cdot\|_\infty + \kappa\|\cdot\|_\phi^s$ and by part (6) of Theorem 12 and Lemma 18, this norm is equivalent to norm $\kappa\|\cdot\|_\phi^s$. This completes the proof. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

This research has been partially supported by the Central Bank of Venezuela. The authors want to give thanks to the library staff of B.C.V for compiling the references.

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