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Research Article

Formulae for the Generalized Drazin Inverse of a Block Matrix in Banach Algebras

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We present some new representations for the generalized Drazin inverse of a block matrix in a Banach algebra under conditions weaker than those used in resent papers on the subject.

1. Introduction

The Drazin inverse has applications in a number of areas such as control theory, Markov chains, singular differential and difference equations, and iterative methods in numerical linear algebra. Representations for the Drazin inverse of block matrices under certain conditions were given in the literature [1–5]. Generalized inverses of block matrices have important applications in automatics, probability, statistics, mathematical programming, numerical analysis, game theory, econometrics, control, and so on [6, 7]. In 1979, Campbell and Meyer proposed the problem of finding a formula for the Drazin inverse of a 2×2 matrix in terms of its various blocks, where the blocks on the diagonal are required to be square matrices [7]. At the present time, there is no known complete solution to this problem.

In this paper, we present the representation for the Drazin inverse of $a+b$ under the conditions that $a^k b = ab$ and $ba^\pi = b$ and that $a^k b = ab$ and $ba^2 = ab$ (cf. Theorem 4). And we also give several representations for the Drazin inverse of $a + b$ under some weaker conditions.

Let $\mathscr{A}$ be a complex Banach algebra with the unit 1. The sets of all Drazin invertible and quasinilpotent elements ($\sigma(a) = \{0\}$) of $\mathscr{A}$ will be denoted by $\mathscr{A}^d$ and $\mathscr{A}^{\text{qnil}}$, respectively.

The generalized Drazin inverse of $a \in \mathscr{A}$ (introduced by Koliha in [8]) is the element $x \in \mathscr{A}$ which satisfies

$$xax = x,$$
$$ax = xa,$$
$$a - a^2 x \in \mathscr{A}^{\text{qnil}}. \tag{1}$$

If there exist the generalized Drazin inverse, then the generalized Drazin inverse of a is unique and is denoted by a^d.

Let $a \in \mathscr{A}$ and let $p \in \mathscr{A}$ be an idempotent ($p = p^2$). We denote $\overline{p} = \mathbb{1} - p$. Then we can write

$$a = pap + pa\overline{p} + \overline{p}ap + \overline{p}a\overline{p}. \tag{2}$$

Every idempotent $p \in \mathscr{A}$ induces a representation of an arbitrary element $a \in \mathscr{A}$ given by the following matrix:

$$a = \begin{bmatrix} pap & pa\overline{p} \\ \overline{p}ap & \overline{p}a\overline{p} \end{bmatrix}_p. \tag{3}$$

Let $a \in \mathscr{A}^d$ and $a^\pi = \mathbb{1} - aa^d$ be the spectral idempotent of a corresponding to $\{0\}$. It is well known that $a \in \mathscr{A}$ can be represented in the following matrix form [9, Ch. 2]:

$$a = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}_p, \tag{4}$$

where $p = aa^d$, a_1 is invertible in the algebra $p\mathscr{A}p$, a^d is its inverse in $p\mathscr{A}p$, and a_2 is quasinilpotent in the algebra $\overline{p}\mathscr{A}\overline{p}$. Thus, the generalized Drazin inverse of a can be expressed as

$$a^d = \begin{bmatrix} a_1^d & 0 \\ 0 & 0 \end{bmatrix}_p. \tag{5}$$

Obviously, if $a \in \mathscr{A}^{\mathrm{qnil}}$, then a is generalized Drazin invertible and $a^d = 0$.

2. Preliminary Results

First we start the following result which is proved in [10] for matrices, extended in [2] for a bounded linear operator and in [11] for arbitrary elements in a Banach algebra.

Lemma 1 (see [11, Theorem 2.3]). *Let $\mathscr{A}$ be a Banach algebra, $x, y \in \mathscr{A}$, and $p \in \mathscr{A}$ an idempotent. Assume that x and y are represented as*

$$x = \begin{bmatrix} a & 0 \\ c & b \end{bmatrix}_p, \qquad y = \begin{bmatrix} b & c \\ 0 & a \end{bmatrix}_p. \tag{6}$$

(i) *If $a \in (p\mathscr{A}p)^d$ and $b \in (\overline{p}\mathscr{A}\overline{p})^d$, then x and y are generalized Drazin invertible, and*

$$x^d = \begin{bmatrix} a^d & 0 \\ u & b^d \end{bmatrix}_p, \qquad y^d = \begin{bmatrix} b^d & u \\ 0 & a^d \end{bmatrix}_p, \tag{7}$$

where

$$u = \sum_{n=0}^{\infty} \left(b^d\right)^{n+2} ca^n a^\pi + \sum_{n=0}^{\infty} b^\pi b^n c \left(a^d\right)^{n+2} - b^d c a^d. \tag{8}$$

(ii) *If $x \in \mathscr{A}^d$ and $a \in (p\mathscr{A}p)^d$, then $b \in (\overline{p}\mathscr{A}\overline{p})^d$, and x^d and y^d are given by (7) and (8).*

Lemma 2. *Let $b \in \mathscr{A}^d$, $a \in \mathscr{A}^{\mathrm{qnil}}$.*

(1) *[11, Corollary 3.4] If $ab = 0$, then $a + b \in \mathscr{A}^d$ and*

$$(a+b)^d = \sum_{n=0}^{\infty} \left(b^d\right)^{n+1} a^n. \tag{9}$$

(2) *[12, Theorem 2.2] If $ba = 0$, then $a + b \in \mathscr{A}^d$ and*

$$(a+b)^d = \sum_{n=0}^{\infty} a^n \left(b^d\right)^{n+1}. \tag{10}$$

Lemma 3. *Let $\mathscr{A}$ be a Banach algebra and let $a \in \mathscr{A}^d$ have the representation (4), where $p = aa^d$. If $x \in p\mathscr{A}$ and $a_1 x = b$ for some $b \in \mathscr{A}$, then $x = a^d b$ (in particular, if $a_1 x = 0$, then $x = 0$). If $y \in \mathscr{A}p$ and $ya_1 = c$ for some $c \in \mathscr{A}$, then $y = ca^d$ (in particular, if $ya_1 = 0$, then $y = 0$).*

Proof. From $a \in \mathscr{A}^d$ we get $a_1 \in \mathscr{A}^d$, $a_1^d = a^d$, and $aa^d = a_1 a_1^d$. There exists $u \in \mathscr{A}$ such that $x = pu$. Since $b = a_1 x$, we obtain $a^d b = a^d a_1 x = aa^d x = px = ppu = pu = x$. The proof for y is similar. □

3. Main Results

In [13, Theorem 3.2] authors gave an explicit representation for $(a+b)^d$ under conditions $a^\pi b = b$ and $a^2 b a^\pi = ab^2 a^\pi = 0$. Here we replace the last condition by the condition $a^k b = ab$; we will get a much simpler expression for $(a+b)^d$.

Theorem 4. *Let $\mathscr{A}$ be a Banach algebra and let $a, b \in \mathscr{A}^d$. If there exist $k \in \mathbb{N}$, $k > 1$, such that $a^k b = ab$ and $ba^\pi = b$, then $a + b \in \mathscr{A}^d$ and*

$$\begin{aligned}(a+b)^d &= a^d + a^\pi \sum_{n=0}^{\infty} \left(b^d\right)^{n+1} a^n - a^d b \sum_{n=0}^{\infty} \left(b^d\right)^{n+1} a^n \\ &\quad + \sum_{n=0}^{\infty} \left(a^d\right)^{n+2} b (a+b)^n b^\pi \\ &\quad - \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \left(a^d\right)^{n+2} b (a+b)^n \left(b^d\right)^{k+1} a^{k+1},\end{aligned} \tag{11}$$

$$\begin{aligned}&\left\|(a+b)^d - a^d\right\| \\ &\quad \le \left[\left\|a^\pi\right\| + \left\|a^d\right\| \|b\|\right] \sum_{n=0}^{\infty} \left\|b^d\right\|^{n+1} \left\|a^n\right\| \\ &\qquad + \sum_{n=0}^{\infty} \left\|a^d\right\|^{n+2} \|b\| \|a+b\|^n \left\|b^\pi\right\| \\ &\qquad + \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \left\|a^d\right\|^{n+2} \|b\| \|a+b\|^n \left[\left\|b^d\right\|^{k+1} \|a\|^{k+1}\right].\end{aligned} \tag{12}$$

Proof. Let $p = aa^d$. We can represent a as in (4), where a_1 is invertible in the subalgebra $p\mathscr{A}p$ and a_2 is quasinilpotent. Hence,

$$a^d = \begin{bmatrix} a_1^d & 0 \\ 0 & 0 \end{bmatrix}_p. \tag{13}$$

Let us write

$$b = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}. \tag{14}$$

From $ba^\pi = b$ and

$$b\left(\mathbb{1} - a^\pi\right) = \begin{bmatrix} b_1 & 0 \\ b_3 & 0 \end{bmatrix}_p = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}_p, \tag{15}$$

we get $b_1 = b_3 = 0$. Thus, b and $a + b$ can be represented as

$$b = \begin{bmatrix} 0 & b_2 \\ 0 & b_4 \end{bmatrix}_p, \qquad a + b = \begin{bmatrix} a_1 & b_2 \\ 0 & a_2 + b_4 \end{bmatrix}_p. \tag{16}$$

By Lemma 1 and $b \in \mathscr{A}^d$, we get $b_4 \in (\overline{p}\mathscr{A}\overline{p})^d$ and

$$b^d = \begin{bmatrix} 0 & b_2\left(b_4^d\right)^2 \\ 0 & b_4^d \end{bmatrix}_p. \tag{17}$$

An elementary computation from (17) and $b^\pi = \mathbb{1} - bb^d$ leads to

$$b^\pi = \begin{bmatrix} p & -b_2 b_4^d \\ 0 & b_4^\pi \end{bmatrix}_p. \tag{18}$$

Since $a^k b = ab$,

$$\begin{aligned} a^k b &= \begin{bmatrix} 0 & a_1^k b_2 \\ 0 & a_2^k b_4 \end{bmatrix}_p, \\ ab &= \begin{bmatrix} 0 & a_1 b_2 \\ 0 & a_2 b_4 \end{bmatrix}_p, \end{aligned} \tag{19}$$

we get $a_2^k b_4 = a_2 b_4$. An induction argument proves $a_2^{r(k-1)+1} b_4 = a_2 b_4$ for any $r \in \mathbb{N}$. Let us denote $m_r = r(k-1)+1$ and observe that since $k > 1$, then $\{a_2^{m_r}\}_{r=1}^\infty$ is a subsequence of $\{a_2^r\}_{r=1}^\infty$. Since a_2 is quasinilpotent, we get

$$\begin{aligned} \left\|a_2 b_4\right\|^{1/m_r} &= \left\|a_2^{m_r} b_4\right\|^{1/m_r} \le \left\|a_2^{m_r}\right\|^{1/m_r} \left\|b_4\right\|^{1/m_r} \\ &\xrightarrow{r\to\infty} 0. \end{aligned} \tag{20}$$

Hence $a_2 b_4 = 0$. By Lemma 2, we have that $a_2 + b_4 \in \mathscr{A}^d$, because $b_4 \in \mathscr{A}^d$, $a_2 \in \mathscr{A}^{\mathrm{qnil}}$, and $a_2 b_4 = 0$. Also, Lemma 2 allows us to obtain

$$\left(a_2 + b_4\right)^d = \sum_{n=0}^\infty \left(b_4^d\right)^{n+1} a_2^n. \tag{21}$$

Again, from Lemma 1 and (16), we get that $a + b \in \mathscr{A}^d$ and

$$(a+b)^d = \begin{bmatrix} a_1^d & u \\ 0 & \left(a_2 + b_4\right)^d \end{bmatrix}, \tag{22}$$

where

$$\begin{aligned} u = &\sum_{n=0}^\infty \left(a_1^d\right)^{n+2} b_2 \left(a_2 + b_4\right)^n \left(a_2 + b_4\right)^\pi \\ &+ \sum_{n=0}^\infty a_1^\pi a_1^n b_2 \left[\left(a_2 + b_4\right)^d\right]^{n+2} - a_1^d b_2 \left(a_2 + b_4\right)^d. \end{aligned} \tag{23}$$

Observe that since $a_1 \in (p\mathscr{A}p)^{-1}$, then $a_1^\pi = 0$. Hence, the expression of u reduces to

$$\begin{aligned} u = &\sum_{n=0}^\infty \left(a_1^d\right)^{n+2} b_2 \left(a_2 + b_4\right)^n \left(a_2 + b_4\right)^\pi \\ &- a_1^d b_2 \left(a_2 + b_4\right)^d. \end{aligned} \tag{24}$$

From $a_2 b_4 = 0$ we get $a_2 b_4^d = a_2 b_4 (b_4^d)^2 = 0$. Hence

$$\begin{aligned} \left(a_2 + b_4\right)^\pi &= \overline{p} - \left(a_2 + b_4\right)\left(a_2 + b_4\right)^d \\ &= \overline{p} - \left(a_2 + b_4\right) \sum_{n=0}^\infty \left(b_4^d\right)^{n+1} a_2^n \\ &= \overline{p} - b_4\left(b_4^d + \left(b_4^d\right)^2 a_2 + \left(b_4^d\right)^3 a_2^2 + \cdots\right) \\ &= \overline{p} - b_4 b_4^d \\ &\quad - \left(b_4\left(b_4^d\right)^2 a_2 + b_4\left(b_4^d\right)^3 a_2^2 + \cdots\right) \\ &= b_4^\pi - \left(b_4^d a_2 + \left(b_4^d\right)^2 a_2^2 + \cdots\right). \end{aligned} \tag{25}$$

So, we get

$$\begin{aligned} u = &\sum_{n=0}^\infty \left(a_1^d\right)^{n+2} b_2 \left(a_2 + b_4\right)^n b_4^\pi \\ &- \sum_{n=0}^\infty \sum_{k=0}^\infty \left(a_1^d\right)^{n+2} b_2 \left(a_2 + b_4\right)^n \left(b_4^d\right)^{k+1} a_2^{k+1} \\ &- a_1^d b_2 \sum_{n=0}^\infty \left(b_4^d\right)^{n+1} a_2^n. \end{aligned} \tag{26}$$

Observe that (22) and $a_1^d = a^d$ yield

$$(a+b)^d = a^d + \left(a_2 + b_4\right)^d + u. \tag{27}$$

We will express $(a_2 + b_4)^d$ using (21) in terms only of a and b. We have from (17)

$$\begin{aligned} a^\pi \left(b^d\right)^{n+1} a^n &= \begin{bmatrix} 0 & 0 \\ 0 & \overline{p} \end{bmatrix}_p \begin{bmatrix} 0 & b_2\left(b_4^d\right)^{n+2} \\ 0 & \left(b_4^d\right)^{n+1} \end{bmatrix}_p \begin{bmatrix} a_1^n & 0 \\ 0 & a_2^n \end{bmatrix}_p \\ &= \begin{bmatrix} 0 & 0 \\ 0 & \left(b_4^d\right)^{n+1} a_2^n \end{bmatrix}_p = \left(b_4^d\right)^{n+1} a_2^n. \end{aligned} \tag{28}$$

Hence from (21) we obtain

$$\left(a_2 + b_4\right)^d = a^\pi \sum_{n=0}^\infty \left(b^d\right)^{n+1} a^n. \tag{29}$$

Also we have (we have written with an asterisk any entry whose exactly expression is not necessary)

$$\begin{aligned}
\left(a^d\right)^{n+2} b\,(a+b)^n b^\pi &= \begin{bmatrix} \left(a_1^d\right)^{n+2} & 0 \\ 0 & 0 \end{bmatrix}_p \begin{bmatrix} 0 & b_2 \\ 0 & b_4 \end{bmatrix}_p \\
&\cdot \begin{bmatrix} a_1^n & * \\ 0 & (a_2+b_4)^n \end{bmatrix}_p \begin{bmatrix} p & -b_2 b_4^d \\ 0 & b_4^\pi \end{bmatrix}_p \\
&= \begin{bmatrix} 0 & \left(a_1^d\right)^{n+2} b_2 (a_2+b_4)^n b_4^\pi \\ 0 & 0 \end{bmatrix}_p = \left(a_1^d\right)^{n+2} \\
&\cdot b_2 (a_2+b_4)^n b_4^\pi, \\
a^d b \left(b^d\right)^{n+1} a^n &= \begin{bmatrix} a_1^d & 0 \\ 0 & 0 \end{bmatrix}_p \begin{bmatrix} 0 & b_2 \\ 0 & b_4 \end{bmatrix}_p \begin{bmatrix} 0 & * \\ 0 & \left(b_4^d\right)^{n+1} \end{bmatrix}_p \\
&\cdot \begin{bmatrix} a_1^n & 0 \\ 0 & a_2^n \end{bmatrix}_p = \begin{bmatrix} 0 & a_1^d b_2 \left(b_4^d\right)^{n+1} a_2^n \\ 0 & 0 \end{bmatrix}_p \\
&= a_1^d b_2 \left(b_4^d\right)^{n+1} a_2^n, \\
\left(a_1^d\right)^{n+2} b\,(a+b)^n \left(b^d\right)^{k+1} a^{k+1} &= \begin{bmatrix} \left(a_1^d\right)^{n+2} & 0 \\ 0 & 0 \end{bmatrix}_p \\
&\cdot \begin{bmatrix} 0 & b_2 \\ 0 & b_4 \end{bmatrix}_p \begin{bmatrix} a_1^n & * \\ 0 & (a_2+b_4)^n \end{bmatrix}_p \begin{bmatrix} 0 & * \\ 0 & \left(b_4^d\right)^{k+1} \end{bmatrix}_p \\
&\cdot \begin{bmatrix} a_1^{k+1} & 0 \\ 0 & a_2^{k+1} \end{bmatrix}_p \\
&= \begin{bmatrix} 0 & \left(a_1^d\right)^{n+2} b_2 (a_2+b_4)^n \left(b_4^d\right)^{k+1} a_2^{k+1} \\ 0 & 0 \end{bmatrix}_p \\
&= \left(a_1^d\right)^{n+2} b_2 (a_2+b_4)^n \left(b_4^d\right)^{k+1} a_2^{k+1}.
\end{aligned} \tag{30}$$

From (27), (29), and (30), it follows (11). The inequality (12) trivially follows from (11). □

If $\mathscr{A}$ is a Banach algebra, then we can define another multiplication in $\mathscr{A}$ by $a \odot b = ba$. It is trivial that $(\mathscr{A}, \odot)$ is a Banach algebra. If we apply Theorem 4 to this new algebra, we can immediately establish the following result.

Theorem 5. *Let $\mathscr{A}$ be a Banach algebra and let $a, b \in \mathscr{A}^d$. If there exist $k \in \mathbb{N}$, $k > 1$, such that $ba^k = ba$ and $a^\pi b = b$, then $a + b \in \mathscr{A}^d$ and*

$$\begin{aligned}
&(a+b)^d \\
&\quad = a^d + \sum_{n=0}^{\infty} a^n \left(b^d\right)^{n+1} a^\pi - \sum_{n=0}^{\infty} a^n \left(b^d\right)^{n+1} b a^d \\
&\qquad + b^\pi \sum_{n=0}^{\infty} (a+b)^n b \left(a^d\right)^{n+2} \\
&\qquad - \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} a^{k+1} \left(b^d\right)^{k+1} (a+b)^n b \left(a^d\right)^{n+2}, \\
&\left\|(a+b)^d - a^d\right\| \\
&\quad \le \sum_{n=0}^{\infty} \left\|a^n\right\| \left\|b^d\right\|^{n+1} \left[\left\|a^\pi\right\| + \|b\| \left\|a^d\right\|\right] \\
&\qquad + \left\|b^\pi\right\| \sum_{n=0}^{\infty} \|a+b\|^n \|b\| \left\|a^d\right\|^{n+2} \\
&\qquad + \sum_{n=0}^{\infty} \left[\sum_{k=0}^{\infty} \|a\|^{k+1} \left\|b^d\right\|^{k+1}\right] \|a+b\|^n \|b\| \left\|a^d\right\|^{n+2}.
\end{aligned} \tag{31}$$

Observe that we can obtain paired results as Theorems 4 and 5. We will not write explicitly these (trivially obtained) results.

Theorem 6. *Let $\mathscr{A}$ be a Banach algebra and let $a, b \in \mathscr{A}^d$ be such that $w = (a+b)aa^d \in \mathscr{A}^d$. If there exist $k \in \mathbb{N}$, $k > 1$, such that $a^k b = ab$ and $ba = ab^2$, then $a + b \in \mathscr{A}^d$ and*

$$\begin{aligned}
(a+b)^d &= w^d + a^\pi b^d + \sum_{n=0}^{\infty} \left(w^d\right)^{n+2} b a^\pi (a+b)^n b^\pi \\
&\quad + \sum_{n=0}^{\infty} w^\pi w^n b a^\pi \left(b^d\right)^{n+2} - w^d b a^\pi b^d.
\end{aligned} \tag{32}$$

Proof. Let us consider the matrix representations of a, a^d, and b given in (4), (13), and (14) relative to the idempotent $p = aa^d$. As in the proof of Theorem 4, from $a^k b = ab$, we get $a_2 b_3 = a_2 b_4 = 0$.

From $ba = ab^2$ we have

$$\begin{aligned}
ab^2 &= \begin{bmatrix} a_1 b_1^2 + a_1 b_2 b_3 & a_1 b_1 b_2 + a_1 b_2 b_4 \\ a_2 b_3 b_1 + a_2 b_4 b_3 & a_2 b_3 b_2 + a_2 b_4^2 \end{bmatrix}_p \\
&= \begin{bmatrix} a_1 b_1^2 + a_1 b_2 b_3 & a_1 b_1 b_2 + a_1 b_2 b_4 \\ 0 & 0 \end{bmatrix}_p, \\
ba &= \begin{bmatrix} b_1 a_1 & b_2 a_2 \\ b_3 a_1 & b_4 a_2 \end{bmatrix}_p.
\end{aligned} \tag{33}$$

Therefore, $b_3 a_1 = 0$ and $a_2 b_4 = b_4 a_2 = 0$. From Lemma 3 we get $b_3 = 0$. Hence

$$b = \begin{bmatrix} b_1 & b_2 \\ 0 & b_4 \end{bmatrix}_p,$$

$$a + b = \begin{bmatrix} a_1 + b_1 & b_2 \\ 0 & a_2 + b_4 \end{bmatrix}_p,$$

$$w = (a+b)\,aa^d = \begin{bmatrix} a_1+b_1 & b_2 \\ 0 & a_2+b_4 \end{bmatrix}_p \begin{bmatrix} p & 0 \\ 0 & 0 \end{bmatrix}_p = \begin{bmatrix} a_1+b_1 & 0 \\ 0 & 0 \end{bmatrix}_p = a_1+b_1. \tag{34}$$

Lemma 2 implies that $a_2+b_4 \in \mathscr{A}^d$ because $a \in \mathscr{A}^{\mathrm{qnil}}$, $b \in \mathscr{A}^d$, and $a_2b_4 = 0$. Also, we obtain

$$(a_2+b_4)^d = \sum_{n=0}^{\infty} \left(b_4^d\right)^{n+1} a_2^n. \tag{35}$$

In this situation, we obtain

$$(a+b)^d = \begin{bmatrix} w^d & u \\ 0 & (a_2+b_4)^d \end{bmatrix}_p = w^d + u + (a_2+b_4)^d, \tag{36}$$

$$\begin{aligned} u &= \sum_{n=0}^{\infty} \left((a_1+b_1)^d\right)^{n+2} b_2 (a_2+b_4)^n (a_2+b_4)^{\pi} \\ &\quad + \sum_{n=0}^{\infty} (a_1+b_1)^{\pi} (a_1+b_1)^n b_2 \left((a_2+b_4)^d\right)^{n+2} \\ &\quad - (a_1+b_1)^d b_2 (a_2+b_4)^d. \end{aligned} \tag{37}$$

Since $a_2b_4 = b_4a_2 = 0$, we have $b_4^d a_2 = (b_4^d)^2 b_4 a_2 = 0$. Thus (35) reduces to

$$(a_2+b_4)^d = b_4^d = a^{\pi} b^d. \tag{38}$$

From $a_2b_4 = 0$ we have $a_2 b_4^d = a_2 b_4 (b_4^d)^2 = 0$. Therefore,

$$\begin{aligned} (a_2+b_4)^{\pi} &= \overline{p} - (a_2+b_4)(a_2+b_4)^d \\ &= \overline{p} - a_2 b_4^d - b_4 b_4^d = b_4^{\pi}, \end{aligned} \tag{39}$$

so we get

$$\begin{aligned} u &= \sum_{n=0}^{\infty} \left((a_1+b_1)^d\right)^{n+2} b_2 (a_2+b_4)^n b_4^{\pi} \\ &\quad + \sum_{n=0}^{\infty} (a_1+b_1)^{\pi} (a_1+b_1)^n b_2 \left(b_4^d\right)^{n+2} \\ &\quad - (a_1+b_1)^d b_2 b_4^d. \end{aligned} \tag{40}$$

We have

$$\begin{aligned} \left(w^d\right)^{n+2} ba^{\pi} (a+b)^n b^{\pi} &= \begin{bmatrix} \left((a_1+b_1)^d\right)^{n+2} & 0 \\ 0 & 0 \end{bmatrix}_p \\ &\quad \cdot \begin{bmatrix} b_1 & b_2 \\ 0 & b_4 \end{bmatrix}_p \begin{bmatrix} 0 & 0 \\ 0 & \overline{p} \end{bmatrix}_p \begin{bmatrix} b_1^n & * \\ 0 & (a_2+b_4)^n \end{bmatrix}_p \begin{bmatrix} b_1^{\pi} & * \\ 0 & b_4^{\pi} \end{bmatrix}_p \\ &= \begin{bmatrix} 0 & \left((a_1+b_1)^d\right)^{n+2} b_2 (a_2+b_4)^n b_4^{\pi} \\ 0 & 0 \end{bmatrix} \\ &= \left((a_1+b_1)^d\right)^{n+2} b_2 (a_2+b_4)^n b_4^{\pi}. \end{aligned} \tag{41}$$

Also,

$$\begin{aligned} w^{\pi} w^n ba^{\pi} \left(b^d\right)^{n+2} &= \begin{bmatrix} (a_1+b_1)^{\pi} & 0 \\ 0 & \overline{p} \end{bmatrix}_p \\ &\quad \cdot \begin{bmatrix} (a_1+b_1)^n & 0 \\ 0 & 0 \end{bmatrix}_p \begin{bmatrix} b_1 & b_2 \\ 0 & b_4 \end{bmatrix}_p \begin{bmatrix} 0 & 0 \\ 0 & \overline{p} \end{bmatrix}_p \\ &\quad \cdot \begin{bmatrix} \left(b_1^d\right)^{n+2} & * \\ 0 & \left(b_4^d\right)^{n+2} \end{bmatrix}_p \\ &= \begin{bmatrix} 0 & (a_1+b_1)^{\pi} (a_1+b_1)^n b_2 \left(b_4^d\right)^{n+2} \\ 0 & 0 \end{bmatrix}_p \\ &= (a_1+b_1)^{\pi} (a_1+b_1)^n b_2 \left(b_4^d\right)^{n+2}, \end{aligned} \tag{42}$$

$$\begin{aligned} w^d ba^{\pi} b^d &= \begin{bmatrix} (a_1+b_1)^d & 0 \\ 0 & 0 \end{bmatrix}_p \begin{bmatrix} b_1 & b_2 \\ 0 & b_4 \end{bmatrix}_p \begin{bmatrix} 0 & 0 \\ 0 & \overline{p} \end{bmatrix}_p \\ &\quad \cdot \begin{bmatrix} b_1^d & * \\ 0 & b_4^d \end{bmatrix}_p = \begin{bmatrix} 0 & (a_1+b_1)^d b_2 b_4^d \\ 0 & 0 \end{bmatrix} = (a_1+b_1)^d \\ &\quad \cdot b_2 b_4^d. \end{aligned}$$

From (36), (38), (41), and (42), it follows (32).
The proof is finished. □

As we have commented before, we can obtain a paired result by considering the Banach algebra $\mathscr{A}$ with the product $a \odot b = ba$. The key hypothesis of this new result will be $ba^k = ba$ and $b^2a = ab$.

Theorem 7. *Let $\mathscr{A}$ be a Banach algebra and let $a, b \in \mathscr{A}^d$ be such that $w = aa^d(a+b) \in \mathscr{A}^d$. If there exist $k \in \mathbb{N}$, $k > 1$, such that $ba^k = ba$ and $b^2a = ab$, then $a+b \in \mathscr{A}^d$ and*

$$\begin{aligned} (a+b)^d &= w^d + b^d a^{\pi} - b^d a^{\pi} b w^d \\ &\quad + \sum_{n=0}^{\infty} \left(b^d\right)^{n+2} a^{\pi} b w^n w^{\pi} \\ &\quad + b^{\pi} \sum_{n=0}^{\infty} (a+b)^n a^{\pi} b \left(w^d\right)^{n+2}. \end{aligned} \tag{43}$$

Theorem 8. *Let $\mathscr{A}$ be a Banach algebra and let $a, b \in \mathscr{A}^d$ be such that $w = aa^d(a+b) \in \mathscr{A}^d$. If there exist $k, n, m \in \mathbb{N}$, $k > 1$, such that $a^n b = ba^m$ and $a^k b = ab$, then $a + b \in \mathscr{A}^d$ and*

$$(a+b)^d = w^d + \sum_{n=0}^{\infty} \left(b^d\right)^{n+1} a^n a^\pi. \tag{44}$$

Proof. Let us consider the matrix representations of a, a^d, and b given in (4), (13), and (14) relative to the idempotent $p = aa^d$. From $a^n b = ba^m$, we have

$$\begin{aligned} a_1^n b_1 &= b_1 a_1^m, \\ a_1^n b_2 &= b_2 a_2^m, \\ a_2^n b_3 &= b_3 a_1^m, \\ a_2^n b_4 &= b_4 a_2^m. \end{aligned} \tag{45}$$

The second equality of (45) implies $a_1^{nk} b_2 = b_2 a_2^{mk}$ for any $k \in \mathbb{N}$. By Lemma 3 we get $b_2 = (a^d)^{nk} b_2 a_2^{mk}$. Hence

$$\|b_2\|^{1/(mk)} \le \left\|\left(a^d\right)^{nk}\right\|^{1/(mk)} \|b_2\|^{1/(mk)} \left\|a_2^{mk}\right\|^{1/(mk)}. \tag{46}$$

Therefore, if β denotes $\lim_{k\to\infty}\|b_2\|^{1/(mk)}$ and α denotes $\lim_{k\to\infty}\|(a^d)^{nk}\|^{1/(mk)}$, then $0 \le \beta \le \beta\alpha r(a_2)$. Recall that a_2 is quasinilpotent, and thus $r(a_2) = 0$. We have just proved $\beta = 0$ or, equivalently, $b_2 = 0$. Similarly we obtain $b_3 = 0$.

Thus, b and $a + b$ can be represented as

$$\begin{aligned} b &= \begin{bmatrix} b_1 & 0 \\ 0 & b_4 \end{bmatrix}_p, \\ a+b &= \begin{bmatrix} a_1 + b_1 & 0 \\ 0 & a_2 + b_4 \end{bmatrix}_p. \end{aligned} \tag{47}$$

Since $a^k b = ab$, by the proof of Theorem 4, we get $a_2 b_4 = 0$. From Lemma 2, we have $a_2 + b_4 \in \mathscr{A}^d$, because $b_4 \in \mathscr{A}^d$, $a_2 \in \mathscr{A}^{\text{qnil}}$, and $a_2 b_4 = 0$. Also, Lemma 2 allows us to obtain

$$\left(a_2 + b_4\right)^d = \sum_{n=0}^{\infty} \left(b_4^d\right)^{n+1} a_2^n. \tag{48}$$

We have $a + b \in \mathscr{A}^d$ if and only if $a_1 + b_1 \in \mathscr{A}^d$. But $w = aa^d(a+b) = a_1 + b_1$. If $w \in \mathscr{A}^d$, then

$$\begin{aligned} (a+b)^d &= \begin{bmatrix} \left(a_1 + b_1\right)^d & 0 \\ 0 & \left(a_2 + b_4\right)^d \end{bmatrix}_p \\ &= w^d + \sum_{n=0}^{\infty} \left(b^d\right)^{n+1} a^n a^\pi. \end{aligned} \tag{49}$$

The theorem is just proved. □

Theorem 9. *Let $\mathscr{A}$ be a Banach algebra and let $a, b \in \mathscr{A}^d$. If there exist $k \in \mathbb{N}$, $k > 1$, such that $a^k b = aba^\pi$ and $ba = ab^2$, then $a + b \in \mathscr{A}^d$ and*

$$(a+b)^d = a^d + a^\pi b^d + \left(a^d\right)^3 b\,(a+b), \tag{50}$$

$$\left\|(a+b)^d - a^d\right\| \le \left\|a^\pi\right\| \left\|b^d\right\| + \left\|a^d\right\|^3 \|b\| \, \|a+b\|. \tag{51}$$

Proof. Let us consider the matrix representations of a, a^d, and b given in (4), (13), and (14) relative to the idempotent $p = aa^d$. Since $a^k b = aba^\pi$,

$$\begin{aligned} a^k b &= \begin{bmatrix} a_1^k b_1 & a_1^k b_2 \\ a_2^k b_3 & a_2^k b_4 \end{bmatrix}_p, \\ aba^\pi &= \begin{bmatrix} 0 & a_1 b_2 \\ 0 & a_2 b_4 \end{bmatrix}_p, \end{aligned} \tag{52}$$

we get $a_1^k b_1 = 0$ and $a_2^k b_4 = a_2 b_4$; as in the proof of Theorem 4, we have $a_2 b_4 = 0$. From Lemma 3 we get $b_1 = 0$.

From $ba = ab^2$ we have

$$\begin{aligned} ab^2 &= \begin{bmatrix} a_1 b_1^2 + a_1 b_2 b_3 & a_1 b_1 b_2 + a_1 b_2 b_4 \\ a_2 b_3 b_1 + a_2 b_4 b_3 & a_2 b_3 b_2 + a_2 b_4^2 \end{bmatrix}_p \\ &= \begin{bmatrix} a_1 b_2 b_3 & a_1 b_2 b_4 \\ 0 & a_2 b_3 b_2 \end{bmatrix}_p, \\ ba &= \begin{bmatrix} b_1 a_1 & b_2 a_2 \\ b_3 a_1 & b_4 a_2 \end{bmatrix}_p = \begin{bmatrix} 0 & b_2 a_2 \\ b_3 a_1 & b_4 a_2 \end{bmatrix}_p. \end{aligned} \tag{53}$$

Therefore, $b_3 a_1 = 0$, $b_2 a_2 = a_1 b_2 b_4$, and $b_4 a_2 = a_2 b_3 b_2$. From Lemma 3 we get $b_3 = 0$ and $b_4 a_2 = a_2 b_3 b_2 = 0$. Hence

$$\begin{aligned} b &= \begin{bmatrix} 0 & b_2 \\ 0 & b_4 \end{bmatrix}_p, \\ a+b &= \begin{bmatrix} a_1 & b_2 \\ 0 & a_2 + b_4 \end{bmatrix}_p. \end{aligned} \tag{54}$$

Lemma 2 implies that $a_2 + b_4 \in \mathscr{A}^d$ because $a \in \mathscr{A}^{\text{qnil}}$, $b \in \mathscr{A}^d$, and $a_2 b_4 = b_4 a_2 = 0$; as in the proof of Theorem 6, we obtain

$$\begin{aligned} \left(a_2 + b_4\right)^d &= b_4^d, \\ \left(a_2 + b_4\right)^\pi &= b_4^\pi. \end{aligned} \tag{55}$$

In this situation, we obtain

$$(a+b)^d = \begin{bmatrix} a_1^d & u \\ 0 & (a_2+b_4)^d \end{bmatrix}_p = a_1^d + u + (a_2+b_4)^d, \quad (56)$$

$$\begin{aligned} u &= \sum_{n=0}^{\infty} \left(a_1^d\right)^{n+2} b_2 (a_2+b_4)^n (a_2+b_4)^\pi \\ &\quad + \sum_{n=0}^{\infty} a_1^\pi a_1^n b_2 \left[(a_2+b_4)^d\right]^{n+2} \\ &\quad - a_1^d b_2 (a_2+b_4)^d. \end{aligned} \quad (57)$$

Observe that since $a_1 \in (p\mathscr{A}p)^{-1}$, then $a_1^\pi = 0$. Hence, the expression of u reduces to

$$u = \sum_{n=0}^{\infty} \left(a_1^d\right)^{n+2} b_2 (a_2+b_4)^n b_4^\pi - a_1^d b_2 b_4^d. \quad (58)$$

From $a_2 b_4 = b_4 a_2 = 0$ and $b_2 a_2 = a_1 b_2 b_4$, we have

$$\begin{aligned} (a_2+b_4)^n &= a_2^n + b_4^n, \\ a_1^d b_2 b_4^d &= \left(a_1^d\right)^2 a_1 b_2 b_4 b_4 \left(b_4^d\right)^3 \\ &= \left(a_1^d\right)^2 b_2 a_2 b_4 \left(b_4^d\right)^2 = 0. \end{aligned} \quad (59)$$

So we get

$$\begin{aligned} &\left(a_1^d\right)^{n+2} b_2 (a_2+b_4)^n b_4 b_4^d \\ &\quad = \left(a_1^d\right)^{n+2} b_2 a_2^n b_4 b_4^d + \left(a_1^d\right)^{n+2} b_2 b_4^n b_4 b_4^d = 0, \\ &\left(a_1^d\right)^{n+2} b_2 (a_2+b_4)^n = \left(a_1^d\right)^{n+2} b_2 a_2^n + \left(a_1^d\right)^{n+2} b_2 b_4^n \\ &\quad = \left(a_1^d\right)^{n+2} b_2 a_2 a_2^{n-1} + \left(a_1^d\right)^{n+3} a_1 b_2 b_4 b_4^{n-1} \\ &\quad = \left(a_1^d\right)^{n+2} a_1 b_2 b_4 a_2^{n-1} + \left(a_1^d\right)^{n+3} b_2 a_2 b_4^{n-1} = 0, \\ &\qquad n \geq 2. \end{aligned} \quad (60)$$

Hence

$$\begin{aligned} u &= \sum_{n=0}^{\infty} \left(a_1^d\right)^{n+2} b_2 (a_2+b_4)^n b_4^\pi - a_1^d b_2 b_4^d \\ &= \sum_{n=0}^{\infty} \left(a_1^d\right)^{n+2} b_2 (a_2+b_4)^n \\ &\quad - \sum_{n=0}^{\infty} \left(a_1^d\right)^{n+2} b_2 (a_2+b_4)^n b_4 b_4^d \\ &= \left(a_1^d\right)^3 b_2 (a_2+b_4). \end{aligned} \quad (61)$$

Thus (55) reduces to

$$(a_2+b_4)^d = b_4^d = a^\pi b^d. \quad (62)$$

We have

$$\begin{aligned} &\left(a^d\right)^3 b (a+b) \\ &\quad = \begin{bmatrix} \left(a_1^d\right)^3 & 0 \\ 0 & 0 \end{bmatrix}_p \begin{bmatrix} 0 & b_2 \\ 0 & b_4 \end{bmatrix}_p \begin{bmatrix} a_1 & b_2 \\ 0 & a_2+b_4 \end{bmatrix}_p \\ &\quad = \begin{bmatrix} 0 & \left(a_1^d\right)^3 b_2 (a_2+b_4) \\ 0 & 0 \end{bmatrix} = \left(a_1^d\right)^3 b_2 (a_2+b_4). \end{aligned} \quad (63)$$

From (56), (62), and (63), it follows (50). The inequality (51) trivially follows from (50).

The proof is finished. □

As we have commented before, we can obtain a paired result by considering the Banach algebra $\mathscr{A}$ with the product $a \odot b = ba$. The key hypothesis of this new result will be $ba^k = a^\pi ba$ and $b^2 a = ab$.

Theorem 10. *Let $\mathscr{A}$ be a Banach algebra and let $a, b \in \mathscr{A}^d$. If there exist $k \in \mathbb{N}$, $k > 1$, such that $ba^k = a^\pi ba$ and $b^2 a = ab$, then $a + b \in \mathscr{A}^d$ and*

$$\begin{aligned} (a+b)^d &= a^d + b^d a^\pi + (a+b) b \left(a^d\right)^3, \\ \left\|(a+b)^d - a^d\right\| &\leq \left\|b^d\right\| \left\|a^\pi\right\| + \|a+b\| \|b\| \left\|a^d\right\|^3. \end{aligned} \quad (64)$$

In the rest of the paper, we will use some weaker conditions than in [11, Theorem 4.1]. For example, if we assume that $a^k b a^\pi = ab$ and $a^\pi b = ab^2$ instead of $aba^\pi = 0$ and $a^\pi b = 0$, we will get a much simpler expression for $(a+b)^d$.

Theorem 11. *Let $\mathscr{A}$ be a Banach algebra and let $a, b \in \mathscr{A}^d$. If there exist $k \in \mathbb{N}$, $k > 1$, such that $a^k b a^\pi = ab$ and $ab^2 = a^\pi b$, then $a + b \in \mathscr{A}^d$ and*

$$\begin{aligned} (a+b)^d &= a^d + \sum_{n=0}^{\infty} \left(a^d\right)^{n+2} b a^n, \\ \left\|(a+b)^d - a^d\right\| &\leq \sum_{n=0}^{\infty} \left\|a^d\right\|^{n+2} \|b\| \|a\|^n. \end{aligned} \quad (65)$$

Proof. Let us consider the matrix representations of a, a^d, and b given in (4), (13), and (14) relative to the idempotent $p = aa^d$. We will use the condition $a^k b a^\pi = ab$: since

$$\begin{aligned} a^k b a^\pi &= \begin{bmatrix} 0 & a_1^k b_2 \\ 0 & a_2^k b_4 \end{bmatrix}_p, \\ ab &= \begin{bmatrix} a_1 b_1 & a_1 b_2 \\ a_2 b_3 & a_2 b_4 \end{bmatrix}_p, \end{aligned} \quad (66)$$

we get $a_1 b_1 = a_2 b_3 = 0$ and $a_2^k b_4 = a_2 b_4$; as in the proof of Theorem 4, we have $a_2 b_4 = 0$. From Lemma 3 we get $b_1 = 0$.

From $ab^2 = a^\pi b$ we have

$$\begin{aligned} ab^2 &= \begin{bmatrix} a_1 b_1^2 + a_1 b_2 b_3 & a_1 b_1 b_2 + a_1 b_2 b_4 \\ a_2 b_3 b_1 + a_2 b_4 b_3 & a_2 b_3 b_2 + a_2 b_4^2 \end{bmatrix}_p \\ &= \begin{bmatrix} a_1 b_2 b_3 & a_1 b_2 b_4 \\ 0 & 0 \end{bmatrix}_p, \\ a^\pi b &= \begin{bmatrix} 0 & 0 \\ b_3 & b_4 \end{bmatrix}_p. \end{aligned} \quad (67)$$

Therefore, $b_3 = b_4 = 0$. Hence

$$b = \begin{bmatrix} 0 & b_2 \\ 0 & 0 \end{bmatrix}_p,$$
$$a + b = \begin{bmatrix} a_1 & b_2 \\ 0 & a_2 \end{bmatrix}_p, \tag{68}$$
$$(a+b)^d = \begin{bmatrix} a_1^d & u \\ 0 & 0 \end{bmatrix}_p = a_1^d + u,$$

where

$$u = \sum_{n=0}^{\infty} \left(a_1^d\right)^{n+2} b_2 a_2^n = \sum_{n=0}^{\infty} \left(a^d\right)^{n+2} b a^n. \tag{69}$$

The proof is finished. □

Theorem 12. *Let $\mathscr{A}$ be a Banach algebra and let $a, b \in \mathscr{A}^d$. If there exist $k \in \mathbb{N}$, $k > 1$, such that $a^k b a^\pi = ab$ and $a^\pi b = aba$, then $a + b \in \mathscr{A}^d$ and*

$$(a+b)^d = a^d + \left(a^d\right)^2 b,$$
$$\left\|(a+b)^d - a^d\right\| \le \left\|a^d\right\|^2 \|b\|. \tag{70}$$

Proof. Let us consider the matrix representations of a, a^d, and b given in (4), (13), and (14) relative to the idempotent $p = aa^d$. As in the proof of Theorem 11, from $a^k b a^\pi = ab$, we get $b_1 = 0$ and $a_2 b_3 = a_2 b_4 = 0$.

Since $a^\pi b = aba$,

$$ab = \begin{bmatrix} a_1 b_1 & a_1 b_2 \\ a_2 b_3 & a_2 b_4 \end{bmatrix}_p = \begin{bmatrix} 0 & a_1 b_2 \\ 0 & 0 \end{bmatrix}_p,$$
$$aba = \begin{bmatrix} 0 & a_1 b_2 \\ 0 & 0 \end{bmatrix}_p \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}_p = \begin{bmatrix} 0 & a_1 b_2 a_2 \\ 0 & 0 \end{bmatrix}_p, \tag{71}$$
$$a^\pi b = \begin{bmatrix} 0 & 0 \\ b_3 & b_4 \end{bmatrix}_p,$$

we get $a_1 b_2 a_2 = b_3 = b_4 = 0$. As in the proof of Theorem 11, we have

$$(a+b)^d = \begin{bmatrix} a_1^d & u \\ 0 & 0 \end{bmatrix}_p = a_1^d + u, \tag{72}$$

where

$$u = \sum_{n=0}^{\infty} \left(a_1^d\right)^{n+2} b_2 a_2^n. \tag{73}$$

From $a_1 b_2 a_2 = 0$, we get

$$u = \sum_{n=0}^{\infty} \left(a_1^d\right)^{n+2} b_2 a_2^n = \left(a_1^d\right)^2 b_2 = \left(a^d\right)^2 b. \tag{74}$$

The proof is finished. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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