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Research Article

Some Classes of Continuous Operators on Spaces of Bounded Vector-Valued Continuous Functions with the Strict Topology

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Let X be a completely regular Hausdorff space and let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be Banach spaces. Let $C_b(X, E)$ be the space of all E-valued bounded, continuous functions on X, equipped with the strict topology β_σ. We study the relationship between important classes of $(\beta_\sigma, \|\cdot\|_F)$-continuous linear operators $T : C_b(X, E) \to F$ (strongly bounded, unconditionally converging, weakly completely continuous, completely continuous, weakly compact, nuclear, and strictly singular) and the corresponding operator measures given by Riesz representing theorems. Some applications concerning the coincidence among these classes of operators are derived.

1. Introduction and Terminology

Throughout the paper let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be real Banach spaces and let E' and F' denote the Banach duals of E and F, respectively. By $B_{F'}$ and B_E we denote the closed unit ball in F' and E, respectively. By $\mathscr{L}(E, F)$ we denote the space of all bounded linear operators from E to F. Given a locally convex space (L, ξ) by $(L, \xi)'$ or L'_ξ we will denote its topological dual. We denote by $\sigma(L, K)$ the weak topology on L with respect to a dual pair $\langle L, K\rangle$. Let $F(\mathbb{N})$ stand for the collection of all finite subsets of the set $\mathbb{N}$ of all natural numbers.

Assume that X is a completely regular Hausdorff space. By $\mathscr{Z}$ (resp., $\mathscr{P}$) we will denote the family of all zero sets (resp., of cozero sets) in X, respectively. Let $C_b(X, E)$ stand for the Banach space of all bounded continuous functions $f : X \to E$, equipped with the uniform norm $\|\cdot\|$. We write $C_b(X)$ instead of $C_b(X, \mathbb{R})$. By $C_b(X, E)'$ we denote the Banach dual of $C_b(X, E)$. For $f \in C_b(X, E)$ let $\tilde{f}(t) = \|f(t)\|_E$ for $t \in X$.

Let $\mathscr{B}$ (resp., $\mathscr{B}a$) stand for the algebra (resp., σ-algebra) of Baire sets in X, respectively. Let $B(\mathscr{B}, E)$ (resp., $B(\mathscr{B}a, E)$) stand for the Banach space of all totally $\mathscr{B}$-measurable (resp., totally $\mathscr{B}a$-measurable) functions $f : X \to E$ (see [1, 2]).

The strict topology β_σ (called also a superstrict topology and denoted by β_1) on $C_b(X)$ and $C_b(X, E)$ is of importance in the topological measure theory (see [3–9] for definitions and more details). $C_b(X, E)'_{\beta_\sigma}$ is a closed subspace of the Banach space $C_b(X, E)'$ and β_σ-bounded sets in $C_b(X, E)$ are $\|\cdot\|$-bounded. It is known that $C_b(X) \otimes E$ is β_σ-dense in $C_b(X, E)$ if one of the following conditions holds (see [6, Theorems 5.1 and 5.2]):

(i) X has a σ-compact dense subset (e.g., X separable).

(ii) X is a D-space (see [10]).

(iii) E is a D-space.

Remark 1. Throughout the paper we will assume that $C_b(X) \otimes E$ is β_σ-dense in $C_b(X, E)$.

For X being a locally compact Hausdorff space, by $C_o(X, E)$ we denote the Banach space of all continuous functions $f : X \to E$ tending to zero at infinity, equipped with the uniform norm. If X is a compact Hausdorff space, then β_σ coincides with the uniform norm topology on $C_b(X, E)$. In this case we write simply $C(X, E)$ instead of $C_b(X, E)$.

Let $M(X)$ stand for the Banach lattice of all Baire measures on $\mathscr{B}$, provided with the norm $\|\nu\| = |\nu|(X)$ (= the total variation of ν). Due to the Alexandrov representation theorem $C_b(X)'$ can be identified with $M(X)$ through the lattice isomorphism $M(X) \ni \nu \mapsto \varphi_\nu \in C_b(X)'$, where $\varphi_\nu(u) = \int_X u\, d\nu$ for $u \in C_b(X)$, and $\|\varphi_\nu\| = \|\nu\|$ (see [4, Theorem 5.1]).

By $M(X, E')$ we denote the set of all finitely additive measures $\mu : \mathscr{B} \to E'$ with the following properties:

(i) for each $x \in E$, the function $\mu_x : \mathscr{B} \to \mathbb{R}$ defined by $\mu_x(A) = \mu(A)(x)$ belongs to $M(X)$;

(ii) $|\mu|(X) < \infty$, where $|\mu|(A)$ stands for the variation of μ on $A \in \mathscr{B}$.

Let $C_{rc}(X, E)$ denote the Banach space of all continuous functions $h : X \to E$ such that $h(X)$ is a relatively compact set in E, equipped with the uniform norm $\|\cdot\|$. Then $C_b(X) \otimes E \subset C_{rc}(X, E) \subset B(\mathscr{B}, E)$. In view of [11, Theorem 2.5] $C_{rc}(X, E)'$ can be identified with $M(X, E')$ through the linear mapping $M(X, E') \ni \mu \mapsto \Phi_\mu \in C_{rc}(X, E)'$, where $\Phi_\mu(h) = \int_X h\, d\mu$ for $h \in C_{rc}(X, E)$ and $\|\Phi_\mu\| = |\mu|(X)$. Then one can embed $B(\mathscr{B}, E)$ into $C_{rc}(X, E)''$ by the mapping $\pi : B(\mathscr{B}, E) \to C_{rc}(X, E)''$, where, for $g \in B(\mathscr{B}, E)$,

$$\pi(g)(\Phi_\mu) := \int_X g\, d\mu \quad \text{for } \mu \in M(X, E'). \tag{1}$$

Assume that $T : C_b(X, E) \to F$ is a bounded linear operator. Then we can define the corresponding operator measure $m : \mathscr{B} \to \mathscr{L}(E, F'')$ (called the *representing measure* of T) by setting

$$m(A)(x) := \left(\left(T|_{C_{rc}(X,E)}\right)'' \circ \pi\right)(\mathbb{1}_A \otimes x) \quad \text{for } A \in \mathscr{B},\ x \in E. \tag{2}$$

Here $(T|_{C_{rc}(X,E)})'' : C_{rc}(X, E)'' \to F''$ stand for the biconjugate of $T|_{C_{rc}(X,E)}$. Then $\widetilde{m}(X) < \infty$, where the semivariation $\widetilde{m}(A)$ of m on $A \in \mathscr{B}$ is defined by $\widetilde{m}(A) := \sup \|\Sigma m(A_i)(x_i)\|_{F''}$, where the supremum is taken over all finite $\mathscr{B}$-partitions (A_i) of A and $x_i \in B_E$ for each i. For $y' \in F'$ let us put

$$m_{y'}(A)(x) := (m(A)(x))(y') \quad \text{for } A \in \mathscr{B},\ x \in E. \tag{3}$$

Let $|m_{y'}|(A)$ stand for the variation of $m_{y'}$ on A. Then (see [1, §4, Proposition 5])

$$\widetilde{m}(A) = \sup\left\{\left|m_{y'}\right|(A) : y' \in B_{F'}\right\}. \tag{4}$$

By $M(X, \mathscr{L}(E, F''))$ we denote the set of all operator measures $m : \mathscr{B} \to \mathscr{L}(E, F'')$ such that $\widetilde{m}(X) < \infty$ and $m_{y'} \in M(X, E')$ for each $y' \in F'$.

Let $i_F : F \to F''$ denote the canonical embedding; that is, $i_F(y)(y') = y'(y)$ for $y \in F$, $y' \in F'$. Moreover, let $j_F : i_F(F) \to F$ stand for the left inverse of i_F; that is, $j_F \circ i_F = id_F$.

For $x \in E$ define

$$T_x(u) := T(u \otimes x) \quad \text{for } u \in C_b(X), \qquad m_x(A) := m(A)(x) \quad \text{for } A \in \mathscr{B}. \tag{5}$$

The following Bartle-Dunfor-Schwartz type theorem will be useful (see [12, Theorem 2], [13, Theorem 5, pages 153-154]).

Theorem 2. *Let $T : C_b(X, E) \to F$ be a bounded linear operator and $M(X, \mathscr{L}(E, F''))$ be its representing measure. Then for each $x \in E$ the following statements are equivalent:*

(i) *$T_x : C_b(X) \to F$ is weakly compact.*

(ii) *$m(A)(x) \in i_F(F)$ for each $A \in \mathscr{B}$ and $\{j_F(m(A)(x)) : A \in \mathscr{B}\}$ is a relatively weakly compact set in F.*

(iii) *$m_x : \mathscr{B} \to F''$ is strongly bounded.*

Following [14–16] we have the following definition.

Definition 3. A bounded linear operator $T : C_b(X, E) \to F$ is said to be *strongly bounded* if its representing measure $m \in M(X, \mathscr{L}(E, F''))$ is strongly bounded; that is, $\widetilde{m}(A_n) \to 0$ whenever (A_n) is a pairwise disjoint sequence in $\mathscr{B}$.

Note that $m \in M(X, \mathscr{L}(E, F''))$ is strongly bounded if and only if the family $\{|m_{y'}| : y' \in B_{F'}\}$ is uniformly strongly additive.

For each $x \in E$, $\|m_x(A)\|_{F''} \le \widetilde{m}(A)\|x\|_E$ for $A \in \mathscr{B}$. It follows that if $T : C_b(X, E) \to F$ is strongly bounded, then $T_x : C_b(X) \to F$ is weakly compact, and hence $m(A)(x) \in i_F(F)$ for $A \in \mathscr{B}$ (see Theorem 2).

For X being a compact Hausdorff space (resp., a locally compact Hausdorff space) different classes of bounded linera operators $T : C_b(X, E) \to F$ (resp., $T : C_0(X, E) \to F$) have been studied intensively; see [14–33]. The study of the relationship between operators $T : C(X, E) \to F$ (resp., $T : C_0(X, E) \to F$) and their representing operator-valued measures is a central problem in the theory. The main aim of the present paper is to extend to "the completely regular setting" some classical results concerning various classes of bounded operators $T : C(X, E) \to F$ (resp., $T : C_0(X, E) \to F$), where X is a compact Hausdorff space (resp., a locally compact Hausdorff space). In [12] using the device of embedding the space $B(\mathscr{B}, E)$ into $C_{rc}(X, E)''$ we establish general Riesz representation theorems for $(\beta_\sigma, \|\cdot\|_F)$-continuous linear operators $T : C_b(X, E) \to F$ with respect to the representing measures $m : \mathscr{B} \to \mathscr{L}(E, F'')$ (see Theorems 6 and 8 below). In Section 3 we show that if $T : C_b(X, E) \to F$ is $(\beta_\sigma, \|\cdot\|_F)$-continuous and strongly bounded, then its representing measure $m : \mathscr{B} \to \mathscr{L}(E, F'')$ has its values in $\mathscr{L}(E, F)$ and possesses a unique extension $\overline{m} : \mathscr{B}a \to \mathscr{L}(E, F)$ that is variationally semiregular; that is, the set $\{|\overline{m}_{y'}| : y' \in B_{F'}\}$ is uniformly countably additive (see Theorem 11 below). In Sections 4–9 we study the folowing classes of $(\beta_\sigma, \|\cdot\|_F)$-continuous linear operators $T : C_b(X, E) \to F$: unconditionally converging, weakly completely continuous, completely continuous, weakly compact, nuclear, and strongly singular. We show that if a $(\beta_\sigma, \|\cdot\|_F)$-continuous linear operator $T : C_b(X, E) \to F$ belongs to any of these classes of operators, then T is strongly bounded and, for each $A \in \mathscr{B}a$, the operator $\overline{m}(A) : E \to F$ shares the property of T (see Theorems 17, 23, 26, 29, 34, and 36 below). We derive some applications concerning to the coincidence among these classes of $(\beta_\sigma, \|\cdot\|)$-continuous operators (see Corollary 13, Theorems 18 and 19, Corollary 27, Theorem 29).

2. Integral Representation of Continuous Operators on $C_b(X,E)$

The space of all σ-aditive members of $M(X)$ will be denoted by $M_\sigma(X)$ (see [3, 4]). Then $(C_b(X),\beta_\sigma)' = \{\varphi_\nu : \nu \in M_\sigma(X)\}$. Let

$$M_\sigma\left(X,E'\right) := \left\{\mu \in M\left(X,E'\right) : \mu_x \in M_\sigma(X) \text{ for each } x \in E\right\}. \tag{6}$$

Then $|\mu| \in M_\sigma(X)$ if $\mu \in M_\sigma(X,E')$ (see [5, Proposition 3.9]).

For the integration theory of functions $f \in C_b(X,E)$ with respect to $\mu \in M_\sigma(X,E')$ we refer the reader to [6, page 197], [5]. The following result will be of importance (see [6, Theorem 5.3]).

Theorem 4. *The following statements hold:*

(i) *for $\Phi \in C_b(X,E)'$ the following conditions are equivalent:*

(a) *Φ is β_σ-continuous;*

(b) *there exists a unique $\mu \in M_\sigma(X,E')$ such that*

$$\Phi(f) = \Phi_\mu(f) = \int_X f\,d\mu \quad \text{for } f \in C_b(X,E), \tag{7}$$

and $\|\Phi_\mu\| = |\mu|(X)$;

(ii) *for $\mu \in M_\sigma(X,E')$, $|\int_X f\,d\mu| \le \int_X \tilde{f}\,d|\mu|$ for $f \in C_b(X,E)$.*

In view of [9, Corollary 5] we have the following characterization of convergence in $(C_b(X,E),\sigma(C_b(X,E),C_b(X,E)'_{\beta_\sigma}))$.

Theorem 5. *For a sequence (f_n) in $C_b(X,E)$ the following statements are equivalent:*

(i) *$f_n \to 0$ for $\sigma(C_b(X,E),M_\sigma(X,E'))$;*

(ii) *$\sup_n\|f_n\| < \infty$ and $f_n(t) \to 0$ in $\sigma(E,E')$ for each $t \in X$.*

The following theorem gives a characterization of $(\beta_\sigma,\|\cdot\|_F)$-continuous operators $T : C_b(X,E) \to F$ in terms of the corresponding operator measures $m : \mathscr{B} \to \mathscr{L}(E,F'')$ (see [12, Theorem 9 and Corollary 7]).

Theorem 6. *Let $T : C_b(X,E) \to F$ be a $(\beta_\sigma,\|\cdot\|_F)$-continuous linear operator and $m \in M(X,\mathscr{L}(E,F''))$ be the representing measure of T. Then the following statements hold.*

(i) *$m \in M_\sigma(X,\mathscr{L}(E,F''))$.*

(ii) *For each $y' \in F'$, $y'(T(f)) = \int_X f\,dm_{y'}$ for $f \in C_b(X,E)$.*

(iii) *For each $f \in C_b(X,E)$ and $A \in \mathscr{B}$ there exists a unique vector in F'', denoted by $\int_A f dm$, such that $(\int_A f\,dm)(y') = \int_A f\,dm_{y'}$ for each $y' \in F'$.*

(iv) *For each $A \in \mathscr{B}$, the mapping $C_b(X,E) \ni f \mapsto \int_A f\,dm \in F''$ is a $(\beta_\sigma,\|\cdot\|_{F''})$-continuous linear operator.*

(v) *For $f \in C_b(X,E)$, $\int_X f\,dm \in i_F(F)$ and $T(f) = j_F(\int_X f\,dm)$.*

(vi) *$\|T\| = \widetilde{m}(X)$.*

(vii) *For $U \in \mathscr{P}$ and $y' \in F'$; we have*

$$\left|m_{y'}\right|(U) = \sup\left\{\left|\int_U h\,dm_{y'}\right| : h \in C_b(X) \otimes E,\ \|h\| = 1,\ \operatorname{supp} h \subset U\right\}. \tag{8}$$

Following [34] by $M_\sigma(\mathscr{B}a)$ $(= ca(\mathscr{B}a))$, we denote the space of all bounded countably additive, real-valued, regular (with respect to zero sets) measures on $\mathscr{B}a$.

We define $M_\sigma(\mathscr{B}a,E')$ to be the set of all measures $\mu : \mathscr{B}a \to E'$ such that the following two conditions are satisfied.

(i) For each $x \in E$, the function $\mu_x : \mathscr{B}a \to \mathbb{R}$ defined by $\mu_x(A) = \mu(A)(x)$ for $A \in \mathscr{B}a$, belongs to $M_\sigma(\mathscr{B}a)$.

(ii) $|\mu|(X) < \infty$, where, for each $A \in \mathscr{B}a$, we define $|\mu|(A) = \sup|\sum \mu(A_i)(x_i)|$, where the supremum is taken over all finite $\mathscr{B}a$-partitions (A_i) of A and all finite collections $x_i \in B_E$.

It is known that if $\mu \in M_\sigma(\mathscr{B}a,E')$, then $|\mu| \in M_\sigma(\mathscr{B}a)$ (see [34, Lemma 2.1]).

The following result will be of importance (see [34, Theorem 2.5]).

Theorem 7. *Let $\mu \in M_\sigma(X,E')$. Then μ possesses a unique extension $\overline{\mu} \in M_\sigma(\mathscr{B}a,E')$ and $|\overline{\mu}|(X) = |\mu|(X)$.*

From Theorem 7 and [13, Corollary 10, page 4] it follows that if $\mu \in M_\sigma(X,E')$, then $|\overline{\mu}|(A) = |\mu|(A)$ for $A \in \mathscr{B}$.

By $M_\sigma(X,\mathscr{L}(E,F))$ we will denote the space of all operator measures $m : \mathscr{B} \to \mathscr{L}(E,F)$ such that $\widetilde{m}(X) < \infty$ and $m_{y'} \in M_\sigma(X,E')$ for each $y' \in F'$. By $M_\sigma(\mathscr{B}a,\mathscr{L}(E,F))$ we will denote the space of all operator measures $m : \mathscr{B}a \to \mathscr{L}(E,F)$ with $\widetilde{m}(X) < \infty$ such that $m_{y'} \in M_\sigma(\mathscr{B}a,E')$ for each $y' \in F'$.

The following theorem characterizes $(\beta_\sigma,\|\cdot\|_F)$-continuous linear operators $T : C_b(X,E) \to F$ such that $T_x : C_b(X) \to F$ are weakly compact for each $x \in E$ (see [12, Theorem 14 and Lemma 11]).

Theorem 8. *Let $T : C_b(X,E) \to F$ be a $(\beta_\sigma,\|\cdot\|_F)$-continuous linear operator such that $T_x : C_b(X) \to F$ is weakly compact for each $x \in E$, and let $m \in M_\sigma(X,\mathscr{L}(E,F''))$ be the representing measure of T. Then the following statements hold.*

(i) *$m(A)(x) \in i_F(F)$ for each $A \in \mathscr{B}$, $x \in E$ and the measure $m_F : \mathscr{B} \to \mathscr{L}(E,F)$ defined by $m_F(A)(x) := j_F(m(A)(x))$ for $A \in \mathscr{B}$, $x \in E$, belongs* *to $M_\sigma(X,\mathscr{L}(E,F))$ and possesses a unique extension $\overline{m} \in M_\sigma(\mathscr{B}a,\mathscr{L}(E,F))$ with $\widetilde{\overline{m}}(X) = \widetilde{m}(X)$ which is countably additive both in the strong operator topology and the weak star operator topology. Moreover, $\overline{m}_{y'} = \overline{m_{y'}}$ for $y' \in F'$.*

(ii) *For every $f \in C_b(X,E)$ and $A \in \mathscr{B}a$ there exists a unique vector in F, denoted by $\int_A f\, d\overline{m}$, such that for each $y' \in F'$, $y'(\int_A f\, d\overline{m}) = \int_A f\, d\overline{m}_{y'}$ and*

$$\left|\int_A f\, d\overline{m}_{y'}\right| \leq \int_A f\, d\left|\overline{m}_{y'}\right|. \tag{9}$$

(iii) *For each $A \in \mathscr{B}a$, the mapping $T_A : C_b(X,E) \to F$ defined by $T_A(f) = \int_A f d\overline{m}$ is a $(\beta_\sigma, \|\cdot\|_F)$-continuous linear operator.*

(iv) *$T(f) = T_X(f) = \int_X f d\overline{m}$ for $f \in C_b(X,E)$.*

Remark 9. As a consequence of Theorem 8 (for $F = \mathbb{R}$) we have

$$\left(C_b(X,E),\beta_\sigma\right)' = \left\{\Phi_\mu : \mu \in M_\sigma\left(\mathscr{B}a,E'\right)\right\}, \tag{10}$$

where for $\mu \in M_\sigma(\mathscr{B}a,E')$, $\Phi_\mu(f) = \int_X f d\mu$ for $f \in C_b(X,E)$ and $\|\Phi_\mu\| = |\mu|(X)$.

3. Strongly Bounded Operators on $C_b(X,E)$

Making use of [35, Theorem 8] we can state the following analogue (for Baire measures on a completely regular Hausdorff space) of the celebrated Dieudonné-Grothendieck's criterion on weak compactness in the space of Borel measures on a compact Hausdorff space (see [36, Theorem 2], [37, Theorem 14, pages 98–103]), which will play a crucial role in the study of different classes of operators on $C_b(X,E)$.

By $\mathscr{T}_s$ we denote the topology of simple convergence in $ca(\mathscr{B}a)$. Then $\mathscr{T}_s$ is generated by the family $\{p_A : A \in \mathscr{B}a\}$ of seminorms, where $p_A(\nu) = |\nu(A)|$ for $\nu \in ca(\mathscr{B}a)$.

A completely regular Hausdorff space X is said to be an *z-space* if a subset which meets every zero-set in a zero-set must be a zero-set. One can note that every metrizable space is a z-space.

From now on we will assume that X is a z-space.

Theorem 10. *Assume that $\mathscr{M}$ is a subset of $ca^+(\mathscr{B}a)$ such that $\sup_{\nu\in\mathscr{M}}\nu(X) < \infty$. Then the following statements are equivalent.*

(i) *$\mathscr{M}$ is relatively $\mathscr{T}_s$-compact subset of $ca(\mathscr{B}a)$.*

(ii) *$\mathscr{M}$ is uniformly countably additive, that is, $\sup_{\nu\in\mathscr{M}}\nu(A_n) \to 0$ whenever $A_n \downarrow \emptyset$, $(A_n) \subset \mathscr{B}a$.*

(iii) *$\mathscr{M}$ is uniformly strongly additive, that is, $\sup_{\nu\in\mathscr{M}}\nu(A_n) \to 0$ whenever (A_n) is pairwise disjoint in $\mathscr{B}a$.*

(iv) *$\sup_{\nu\in\mathscr{M}}\nu(U_n) \to 0$ for every pairwise disjoint sequence (U_n) in $\mathscr{P}$.*

Proof. (i)⇔(ii) See [38, Theorem 7].

(ii)⇔(iii) See [37, Theorem 10, pages 88-89].

(iv)⇔(i) See [35, Theorem 8]. □

Now we can state a characterization of $(\beta_\sigma, \|\cdot\|_F)$-continuous strongly bounded operators $T : C_b(X,E) \to F$.

Theorem 11. *Let $T : C_b(X,E) \to F$ be a $(\beta_\sigma, \|\cdot\|_F)$-continuous linear operator and let $m \in M_\sigma(X,\mathscr{L}(E,F''))$ be its representing measure. Then the following statements are equivalent.*

(i) *For each $x \in E$, $T_x : C_b(X) \to F$ is weakly compact and $\overline{m}$ is variationally semiregular; that is, $\sup\{|\overline{m}_{y'}|(A_n) : y' \in B_{F'}\} \to 0$ whenever $A_n \downarrow \emptyset$, $(A_n) \subset \mathscr{B}a$.*

(ii) *T is strongly bounded.*

(iii) *$T(f_n) \to 0$ whenever (f_n) is a uniformly bounded sequence in $C_b(X,E)$ such that $f_n(t) \to 0$ in E for each $t \in X$.*

(iv) *$T(f_n) \to 0$ whenever (f_n) is a uniformly bounded sequence in $C_b(X,E)$ such that $\operatorname{supp} f_n \cap \operatorname{supp} f_m = \emptyset$ for $n \neq m$.*

Proof. (i)⇔(ii) It follows from Theorem 8 and [12, Theorem 16].

(ii)⇒(iii) It follows from [12, Theorem 17].

(iii)⇒(iv) It is obvious.

(iv)⇒(i) Assume that (iv) holds. First we shall show that for each $x \in E$, $T_x : C_b(X) \to F$ is weakly compact. Assume on the contrary that $T_{x_o} : C_b(X) \to F$ is not weakly compact for some $x_o \in E$. This means that $m_{x_o} : \mathscr{B} \to F''$ is not strongly bounded. Since for $A \in \mathscr{B}$, $\|m_{x_0}(A)\|_{F''} = \sup\{|m_{x_0,y'}(A)| : y' \in B_{F'}\}$, we obtain that the family $\{m_{x_o,y'} : y' \in B_{F'}\}$ is not uniformly strongly additive. Hence the family $\{\overline{m_{x_o,y'}} : y' \in B_{F'}\}$ is not uniformly countably additive. It follows that the family $\{|\overline{m_{x_o,y'}}| : y' \in B_{F'}\}$ is not uniformly countably additive. In view of Theorem 10 there exist $\varepsilon_0 > 0$, a sequence (y'_n) in $B_{F'}$ and a pairwise disjoint sequence (U_n) in $\mathscr{P}$ such that for $n \in \mathbb{N}$, $|m_{x_o,y'}|(U_n) \geq \varepsilon_0$. Note that

$$\begin{aligned}\left|m_{x_o,y'_n}\right|(U_n) = \sup\Bigg\{&\left|\int_{U_n} u\, dm_{x_o,y'_n}\right| : u \in C_b(X), \\ &\|u\| = 1,\ \operatorname{supp} u \subset U_n\Bigg\}.\end{aligned} \tag{11}$$

Hence there exists a sequence (u_n) in $C_b(X)$ such that $\|u_n\| = 1$, $\operatorname{supp} u_n \subset U_n$ and

$$\begin{aligned}\left|y'_n\left(T_{x_0}(u_n)\right)\right| &= \left|\int_X u_n dm_{x_0,y'_n}\right| \\ &= \left|\int_{U_n} u_n dm_{x_o,y'_n}\right| \\ &\geq \left|m_{x_o,y'_n}\right|(U_n) - \frac{\varepsilon_0}{2} \geq \frac{\varepsilon_0}{2}.\end{aligned} \tag{12}$$

Let $f_n = u_n \otimes x_0$ for $n \in \mathbb{N}$. Then $\operatorname{supp} f_n \cap \operatorname{supp} f_n = \emptyset$ for $n \neq m$ and by (iv), $\|T(f_n)\|_F \to 0$, which contradics (12). This means that $T_x : C_b(X) \to F$ is weakly compact for each $x \in E$, as desired.

In view of Theorem 8 m can be uniquely extended to a measure $\overline{m} : \mathscr{B}a \to \mathscr{L}(E,F)$. Assume that $\overline{m}$ is not variationally semiregular. Then by Theorem 10 there exist $\varepsilon_0 > 0$, a pairwise disjoint sequence (U_n) in $\mathscr{P}$ and a sequence (y_n') in $B_{F'}$ such that $|m_{y_n'}|(U_n) > \varepsilon_0$. Hence by Theorem 7 there exists a sequence (h_n) in $C_b(X) \otimes E$ and $\|h_n\| = 1$ with $\operatorname{supp} h_n \subset U_n$ for $n \in \mathbb{N}$ such that

$$\left|\int_{U_n} h_n dm_{y_n'}\right| \geq |m_{y_n'}|(U_n) - \frac{\varepsilon_0}{2} > \frac{\varepsilon_0}{2}. \quad (13)$$

Then, for $n \in \mathbb{N}$,

$$\begin{aligned} \|T(h_n)\|_F &= \sup\left\{|y'(T(h_n))| : y' \in B_{F'}\right\} \\ &= \sup\left\{\left|\int_X h_n dm_{y'}\right| : y' \in B_{F'}\right\} \\ &= \sup\left\{\left|\int_{U_n} h_n dm_{y'}\right| : y' \in B_{F'}\right\} \\ &\geq \left|\int_{U_n} h_n dm_{y_n'}\right| > \frac{\varepsilon_0}{2}. \end{aligned} \quad (14)$$

On the other hand, since $\operatorname{supp} h_n \cap \operatorname{supp} h_m = \emptyset$ for $n \neq m$, by (iv), $\|T(h_n)\|_F \to 0$. This contradiction establishes that (i) holds. □

Corollary 12. *Let $T : C_b(X,E) \to F$ be a $(\beta_\sigma, \|\cdot\|_F)$-continuous and strongly bounded linear operator and let $m \in M_\sigma(X, \mathscr{L}(E,F''))$ be its representing measure. Then the set $\{|\overline{m}_{y'}| : y' \in B_{F'}\}$ is uniformly regular on $\mathscr{B}a$; that is, for each $A \in \mathscr{B}a$ and $\varepsilon > 0$, there exist $Z \in \mathscr{Z}$ with $Z \subset A$ and $U \in \mathscr{P}$ with $A \subset U$ such that*

$$\sup\left\{|\overline{m}_{y'}|(B) : B \in \mathscr{B}a,\ B \subset U \setminus Z,\ y' \in B_{F'}\right\} \leq \varepsilon. \quad (15)$$

Proof. In view of Theorem 11 the family $\{|\overline{m}_{y'}| : y' \in B_{F'}\}$ is uniformly countably additive. Let $\lambda \in ca^+(\mathscr{B}a)$ be a control measure for $\{|\overline{m}_{y'}| : y' \in B_{F'}\}$ and let $A \in \mathscr{B}a$ and $\varepsilon > 0$ be given. Then there is $\delta > 0$ such that $\sup\{|\overline{m}_{y'}|(B) : y' \in B_{F'}\} \leq \varepsilon$ whenever $B \in \mathscr{B}a$ and $\lambda(B) \leq \delta$. By the regularity of λ there exists $Z \in \mathscr{Z}$ with $Z \subset A$ and $U \in \mathscr{P}$ with $A \subset U$ such that $\lambda(U \setminus Z) \leq \delta$. Hence we get $\sup\{|\overline{m}_{y'}|(B) : B \in \mathscr{B}a, B \subset U \setminus Z, y' \in B_{F'}\} \leq \varepsilon$. □

Corollary 13. *Assume that $T : C_b(X,E) \to F$ is a $(\beta_\sigma, \|\cdot\|_F)$-continuous linear operator and F contains no isomorphic copy of c_0. Then T is strongly bounded.*

Proof. Let $m \in M_\sigma(X, \mathscr{L}(E,F''))$ stand for the representing measure of T. We shall first show that $T_x : C_b(X) \to F$ is weakly compact for each $x \in E$. Assume on the contrary that $T_{x_0} : C_b(X) \to F$ is not weakly compact for some $x_0 \in E$. Then by the proof of implication (iv)⇒(i) of Theorem 11 there exist $\varepsilon_0 > 0$, a sequence (y_n') in $B_{F'}$, and a pairwise disjoint sequence (U_n) in $\mathscr{P}$ such that $|m_{x_0,y_n'}|(U_n) \geq \varepsilon_0$ for $n \in \mathbb{N}$. By the Rosenthall lemma (see [13, Lemma 1, page 18]) the sequence (U_n) in $\mathscr{P}$ and (y_n') in $B_{F'}$ can be chosen such that for $n \in \mathbb{N}$,

$$|m_{x_0,y_n'}|(U_n) \geq \varepsilon_0, \qquad |m_{x_0,y_n'}|\left(\bigcup_{m\neq n} U_m\right) < \frac{\varepsilon_0}{2}. \quad (16)$$

Since, for $n \in \mathbb{N}$,

$$\begin{aligned} |m_{x_0,y_n'}|(U_n) = \sup\Bigg\{\left|\int_{U_n} u\, dm_{x_0,y_n'}\right| : u \in C_b(X), \\ \|u\| = 1 \text{ with } \operatorname{supp} u \subset U_n\Bigg\}, \end{aligned} \quad (17)$$

there exists a sequence (u_n) in $C_b(X)$ such that $\|u_n\| = 1$ with $\operatorname{supp} u_n \subset U_n$ and

$$|y_n'(T_{x_0}(u_n))| = \left|\int_X u_n dm_{x_0,y_n'}\right| = \left|\int_{U_n} u_n dm_{x_0,y_n'}\right| > \varepsilon_0. \quad (18)$$

Let $Y = \{\sum_{n=1}^\infty a_n u_n : (a_n) \in c_0\}$. We see that Y is an isomorphic copy of c_0. Assume that $u = \sum_{n=1}^\infty a_n u_n$ for some sequence (a_n) in c_0. Then for $n \in \mathbb{N}$ we have

$$\begin{aligned} |y_n'(T_{x_0}(u))| &= \left|\int_X u\, dm_{x_0,y_n'}\right| \\ &= \left|a_n \int_{U_n} u_n dm_{x_0,y_n'} + \int_{\bigcup_{m\neq n} U_m} u\, dm_{x_0,y_n'}\right| \\ &\geq |a_n|\,\varepsilon_0 - \int_{\bigcup_{m\neq n} U_m} |u|\, d|m_{x_0,y_n'}| \\ &\geq |a_n|\,\varepsilon_0 - |m_{x_0,y_n'}|\left(\bigcup_{m\neq n} U_m\right)\|u\| \\ &\geq |a_n|\,\varepsilon_0 - \frac{\varepsilon_0}{2}\|u\|. \end{aligned} \quad (19)$$

But $\|u\| = \sup_n |a_n|$, so

$$\|T_{x_0}(u)\|_F \geq \sup_n |y_n'(T_{x_0}(u))| \geq \varepsilon_0\|u\| - \frac{\varepsilon_0}{2}\|u\| = \frac{\varepsilon_0}{2}\|u\|. \quad (20)$$

This means that $T_{x_0} : C_b(X) \to F$ is an isomorphism on Y, so F contains an isomorphic copy of c_0, which contradicts our assumption on F. This means that T_x is weakly compact for each $x \in E$. Hence in view of Theorem 8 $\overline{m} : \mathscr{B}a \to \mathscr{L}(E,F)$ is countably additive in the weak star operator topology and by [19, Remark 7, page 923] and Theorem 11 we derive that T is strongly bounded, as desired. □

Remark 14. If X is a compact Hausdorff space, the equivalence (ii)⇔(iii) of Theorem 11 was obtained by Brooks and Lewis (see [16, Theorem 2.1]).

Let $\mathscr{L}^\infty(\mathscr{B}a, E)$ stand for the Banach space of all bounded strongly $\mathscr{B}a$-measurable functions $g : X \to E$, equipped with the uniform norm $\|\cdot\|$. Assume that $m : \mathscr{B} \to \mathscr{L}(E, F)$ with $\widetilde{m}(X) < \infty$ is variationally semiregular. Then every $g \in \mathscr{L}^\infty(\mathscr{B}a, E)$ is m-integrable (see [39, Definition 2, page 523 and Theorem 5, page 524]) and $\int_X g_n dm \to 0$ whenever (g_n) is a uniformly bounded sequence in $\mathscr{L}^\infty(\mathscr{B}a, E)$ converging pointwise to 0 (see [40, Proposition 2.2]).

Note that if $f \in C_b(X, E)$ then $y' \circ f$ is $\mathscr{B}a$-measurable. Hence if E is assumed to be separable then f is strongly $\mathscr{B}a$-measurable; that is, $f \in \mathscr{L}^\infty(\mathscr{B}a, E)$ (see [2, Proposition 21, page 9]).

Recall that a function $g : X \to E'$ is weak*-measurable if for each $x \in E$ the function $X \ni t \mapsto \langle x, g(t)\rangle \in \mathbb{R}$ is $\mathscr{B}a$-measurable. For $\lambda \in ca^+(\mathscr{B}a)$ by $\mathscr{L}^1_{w^*}(\lambda, E')$ we denote the vector space of all weak*-measurable functions $g : X \to E'$ for which there exists $u \in \mathscr{L}^1(\lambda)$ such that $\|g(t)\|_{E'} \le u(t)$ λ-a.e. on X (see [41, page 26]).

Following [40] we can distinguish an important class of operators on $\mathscr{L}^\infty(\mathscr{B}a, E)$.

Definition 15. A bounded linear operator $S : \mathscr{L}^\infty(\mathscr{B}a, E) \to F$ is said to be *σ-smooth* if $S(g_n) \to 0$ whenever (g_n) is a uniformly bounded sequence in $\mathscr{L}^\infty(\mathscr{B}a, E)$ such that $g_n(t) \to 0$ for each $t \in X$.

Proposition 16. *Assume that E is separable. Let $T : C_b(X, E) \to F$ be a $(\beta_\sigma, \|\cdot\|_F)$-continuous and strongly bounded linear operator, and let $m \in M_\sigma(X, \mathscr{L}(E, F''))$ be its representing measure. Then for each $y' \in F'$ there exists $g_{y'} \in \mathscr{L}^1_{w^*}(\lambda, E')$ such that*

$$y'(T(f)) = \int_X \langle f, g_{y'}\rangle\, d\lambda \quad \text{for } f \in C_b(X, E), \tag{21}$$

where $\lambda \in ca^+(\mathscr{B}a)$ is a control measure for $\{|\overline{m}_{y'}| : y' \in B_{F'}\}$.

Proof. Since E is supposed to be separable, $C_b(X, E) \subset \mathscr{L}^\infty(\mathscr{B}a, F)$. Moreover, since $\overline{m} : \mathscr{B}a \to \mathscr{L}(E, F)$ is variationally semiregular (see Theorem 11), the corresponding integration operator $S_{\overline{m}} : \mathscr{L}^\infty(\mathscr{B}a, E) \to F$ is σ-smooth and for $y' \in F'$ we have (see [40, Proposition 2.2])

$$y'(S_{\overline{m}}(f)) = \int_X f\, d\overline{m}_{y'} = y'(T(f)) \quad \forall f \in C_b(X, E). \tag{22}$$

It follows that $S_{\overline{m}}(f) = T(f)$ for each $f \in C_b(X, E)$.

Let $y' \in F'$. Then $y' \circ S_{\overline{m}}$ is a σ-smooth functional on $\mathscr{L}^\infty(\mathscr{B}a, E)$, and $\overline{m}_{y'}$ is λ-absolutely continuous; that is, $\overline{m}_{y'} \in cabv_\lambda(\mathscr{B}a, E')$. According to the Radon-Nikodym type theorem (see [41, Theorem 1.5.3]) there exists a weak*-measurable function $g_{y'} : X \to E'$ which satisfies the following conditions.

(1) The function $X \ni t \mapsto \|g_{y'}(t)\|_{E'} \in \mathbb{R}$ is $\mathscr{B}a$-measurable and λ-integrable; that is, $\|g_{y'}(\cdot)\|_{E'} \in \mathscr{L}^1(\lambda)$.

(2) For every $x \in E$ and $A \in \mathscr{B}a$,

$$\begin{aligned} \overline{m}_{y'}(A)(x) &= \int_A \langle x, g_{y'}\rangle\, d\lambda, \\ |\overline{m}_{y'}|(A) &= \int_A \|g_{y'}(\cdot)\|_{E'}\, d\lambda. \end{aligned} \tag{23}$$

It follows that $g_{y'} \in \mathscr{L}^1_{w^*}(\lambda, E')$. Note that for every $s = \sum_{i=1}^n (\mathbb{1}_{A_i} \otimes x_i) \in \mathscr{S}(\mathscr{B}a, E)$ the mapping $\langle s, g_{y'}\rangle : X \ni t \mapsto \langle s(t), g_{y'}(t)\rangle \in \mathbb{R}$ is $\mathscr{B}a$-measurable and using (2) we get

$$\begin{aligned} y' \circ S_{\overline{m}}(s) &= \int_X s\, d\overline{m}_{y'} = \sum_{i=1}^n \overline{m}_{y'}(A_i)(x_i) \\ &= \sum_{i=1}^n \int_X \langle \mathbb{1}_{A_i} \otimes x_i, g_{y'}\rangle\, d\lambda \\ &= \int_X \left(\sum_{i=1}^n \langle \mathbb{1}_{A_i} \otimes x_i, g_{y'}\rangle\right) d\lambda \\ &= \int_X \left\langle \sum_{i=1}^n (\mathbb{1}_{A_i} \otimes x_i), g_{y'}\right\rangle d\lambda \\ &= \int_X \langle s, g_{y'}\rangle\, d\lambda. \end{aligned} \tag{24}$$

Now let $f \in C_b(X, E) \subset \mathscr{L}^\infty(\mathscr{B}a, E)$. Then there exists a sequence (s_n) in $\mathscr{S}(\mathscr{B}a, E)$ such that $\|s_n(t) - f(t)\|_E \to 0$ and $\|s_n(t)\|_E \le \|f(t)\|_E$ for each $t \in X$ and $n \in \mathbb{N}$ (see [2, Theorem 1.6, page 4]). Then the mapping $\langle f, g_{y'}\rangle : X \ni t \mapsto \langle f(t), g_{y'}(t)\rangle \in \mathbb{R}$ is $\mathscr{B}a$-measurable. Using the Lebesgue dominated convergence theorem we have

$$\begin{aligned} &\left|\int_X \langle s_n, g_{y'}\rangle\, d\lambda - \int_X \langle f, g_{y'}\rangle\, d\lambda\right| \\ &= \left|\int_X \langle s_n - f, g_{y'}\rangle\, d\lambda\right| \\ &\le \int_X \left|\langle s_n - f, g_{y'}\rangle\right| d\lambda \\ &\le \int_X \|(s_n - f)(t)\|_E \cdot \|g_{y'}(t)\|_{E'}\, d\lambda \longrightarrow 0. \end{aligned} \tag{25}$$

It follows that

$$\begin{aligned} y'(T(f)) &= (y' \circ S_{\overline{m}})(f) = \lim_n (y' \circ S_{\overline{m}})(s_n) \\ &= \lim_n \int_X \langle s_n, g_{y'}\rangle\, d\lambda = \int_X \langle f, g_{y'}\rangle\, d\lambda. \end{aligned} \tag{26}$$

□

4. Unconditionally Converging Operators on $C_b(X,E)$

Recall that a series $\sum_{i=1}^\infty z_i$ in a Banach space G is called *weakly unconditionally Cauchy* (wuc) if, for each $z' \in G'$,

$\sum_{i=1}^{\infty}|z'(z_i)| < \infty$. We say that a bounded linear operator $T : G \to F$ is *unconditionally converging* if, for every weakly unconditionally Cauchy series $\sum_{i=1}^{\infty} z_i$ in G, the series $\sum_{i=1}^{\infty} T(z_i)$ converges unconditionally in a Banach space F.

If X is a compact Hausdorff space, Swartz [33] proved that every unconditionally converging operator $T : C(X,E) \to F$ is strongly bounded. Dobrakov (see [28, Theorem 3]) showed that if X is a locally compact Hausdorff space, then every unconditionally converging operator $T : C_0(X,E) \to F$ is strongly bounded and for every Borel set A in X, the operator $m(A) : E \to F$ is unconditionally converging. Moreover, Brooks and Lewis [27, Theorem 5.2] showed that if E contains no isomorphic copy of c_0, then every strongly bounded operator $T : C_0(X,E) \to F$ is unconditionally converging. We will extend these results to the setting when $T : C_b(X,E) \to F$ is a $(\beta_\sigma, \|\cdot\|_F)$-continuous linear operator and X is a completely regular Hausdorff space.

Theorem 17. *Let $T : C_b(X,E) \to F$ be a $(\beta_\sigma, \|\cdot\|_F)$-continuous and unconditionally converging linear operator, and $m \in M_\sigma(X, \mathscr{L}(E, F''))$ stand for the representing measure of T. Then the following statements hold.*

(i) *T is strongly bounded.*

(ii) *For each $A \in \mathscr{B}a$, $\overline{m}(A) : E \to F$ is an unconditionally converging operator.*

Proof. (i) Assume that (f_n) is a uniformly bounded sequence in $C_b(X,E)$ such that $\operatorname{supp} f_n \cap \operatorname{supp} f_m = \emptyset$ for $n \neq m$. Then $\{\sum_{n\in M} f_n : M \in F(\mathbb{N})\}$ is bounded in $C_b(X,E)$ and, since T is unconditionally converging, we obtain that $T(f_n) \to 0$. Hence by Theorem 11 T is strongly bounded.

(ii) Let $A \in \mathscr{B}a$ and assume that $\sum_{n=1}^{\infty} x_n$ is *wuc* in E. Then $\sup\{\|\sum_{i\in M} x_i\|_E : M \in F(\mathbb{N})\} \leq r$. In view of Theorem 11 $\{|\overline{m}_{y'}| : y' \in B_{F'}\}$ is uniformly countably additive and let $\lambda \in ca^+(\mathscr{B}a)$ stand for the control measure of $\{|\overline{m}_{y'}| : y' \in B_{F'}\}$ (see Corollary 12). Let $\varepsilon > 0$ be given. Then there is $\delta > 0$ such that $\sup\{|\overline{m}_{y'}|(B) : y' \in B_{F'}\} \leq \varepsilon/r$ whenever $B \in \mathscr{B}a$, $\lambda(B) \leq \delta$. Then there exist $Z \in \mathscr{Z}$ with $Z \subset A$ and $U \in \mathscr{P}$ with $A \subset U$ such that $\lambda(U \setminus Z) \leq \delta$. Hence

$$\sup\left\{\left|m_{y'}\right|(U \setminus Z) : y' \in B_{F'}\right\} \leq \varepsilon. \tag{27}$$

Then one can choose $u \in C_b(X)$ with $0 \leq u \leq \mathbb{1}_X$, $u|_Z \equiv 1$, and $u|_{X\setminus U} \equiv 0$. Define $T_u(x) := T(u \otimes x)$ for $x \in E$. We shall show that $T_u : E \to F$ is unconditionally converging. Indeed, for $M \in F(\mathbb{N})$, $\|\sum_{i\in M}(u \otimes x_i)\| \leq \|\sum_{i\in M} x_i\|_E \leq r$. Hence the series $\sum_{n=1}^{\infty} T(u \otimes x_n)$ is unconditionally convergent; that is, T_u is unconditionally converging, as desired. Then for each $x \in B_E$, we have

$$\begin{aligned}
&\left\|T_u(x) - \overline{m}(A)(x)\right\|_F \\
&= \left\|\int_X \left((u - \mathbb{1}_A) \otimes x\right) d\overline{m}\right\|_F \\
&= \sup\left\{\left|y'\left(\int_X \left((u - \mathbb{1}_A) \otimes x\right) d\overline{m}\right)\right| : y' \in B_{F'}\right\} \\
&\leq \sup\left\{\int_X \left|u - \mathbb{1}_A\right| d\left|\overline{m}_{y'}\right| : y' \in B_{F'}\right\} \\
&\leq \sup\left\{\int_{U\setminus Z} \mathbb{1}_X d\left|m_{y'}\right| : y' \in B_{F'}\right\} \\
&\leq \sup\left\{\left|m_{y'}\right|(U \setminus Z) : y' \in B_{F'}\right\} \leq \varepsilon.
\end{aligned} \tag{28}$$

Hence $\|T_u - \overline{m}(A)\| \leq \varepsilon$ and since the class of all unconditionally converging operators from E to F is a closed linear subspace of $(\mathscr{L}(E,F), \|\cdot\|)$ (see [28, page 20]), we derive that $\overline{m}(A)$ is unconditionally converging. □

Theorem 18. *Assume that E is separable and contains no isomorphic copy of c_0. Then for a $(\beta_\sigma, \|\cdot\|_F)$-continuous linear operator $T : C_b(X,E) \to F$ the following statements are equivalent.*

(i) *T is unconditionally converging.*

(ii) *T is strongly bounded.*

Proof. (i)⇒(ii) See Theorem 17.

(ii)⇒(i) See [12, Corollary 18]. □

Recall that a subset P of a Banach space G is said to be weakly precompact if every bounded sequence (z_n) in P contains a subsequence (z_{k_n}) so that $z'(z_{k_n})$ converges for each $z' \in G'$. An operator $T : G \to F$ is said to be weakly precompact if $T(B_G)$ is weakly precompact in a Banach space F.

Abbott et al. [17, Theorem 2.8] discussed the relationship between strongly bounded and unconditionally converging operators $T : C(X,E) \to F$ whenever X is a compact Hausdorff space. They showed that if E' contains no isomorphic copy of ℓ^1 and E' has the RNP, then the classes of strongly bounded and unconditionally converging operators $T : C(X,E) \to F$ coincide. Now we state an analogue of Theorem 2.8 of [17] for $(\beta_\sigma, \|\cdot\|_F)$-continuous linear operator $T : C_b(X,E) \to F$, where X is a completely regular Hausdorff space.

Theorem 19. *Assume that E' contains no isomorphic copy of ℓ^1 and E' has the RNP. Then for a $(\beta_\sigma, \|\cdot\|_F)$-continuous linear operator $T : C_b(X,E) \to F$ the following statements are equivalent.*

(i) *$T' : F' \to C_b(X,E)'$ is weakly precompact.*

(ii) *T is unconditionally converging.*

(iii) *T is strongly bounded.*

Proof. (i)⇒(ii) See [17, Theorem 2.7].

(ii)⇒(iii) See Theorem 17.

(iii)⇒(i) Assume that T is strongly bounded. Since $\{y' \circ T : y' \in B_{F'}\} \subset C_b(X,E)'_{\beta_\sigma}$, we have to show that $\{y' \circ T : y' \in B_{F'}\}$ is a weakly precompact subset of the Banach space $(C_b(X,E)'_{\beta_\sigma}, \|\cdot\|)$. By Theorem 11 $\{|\overline{m}_{y'}| : y' \in B_{F'}\}$ is uniformly countably additive, and let $\lambda \in ca^+(\mathscr{B}a)$ be a control measure for $\{|\overline{m}_{y'}| : y' \in B_{F'}\}$. Since E' is supposed to have the RNP, for each $y' \in B_{F'}$ there exists $g_{y'} \in L^1(\lambda, E')$ such that $\overline{m}_{y'}(A) = \int_A g_{y'} d\lambda$ and $|\overline{m}_{y'}|(A) = \int_A \|g_{y'}(\cdot)\|_{E'} d\lambda$ for $A \in \mathscr{B}a$. It follows that $\{\|g_{y'}(\cdot)\|_{E'} : y' \in B_{F'}\}$ is a uniformly integrable subset of $L^1(\lambda)$ and since E' contains no isomorphic copy ℓ^1, $\{g_{y'} : y' \in B'_F\}$ is a weakly precompact subset of $L^1(\lambda, E')$ (see [42]). Since $\{\overline{m}_{y'} : y' \in B_{F'}\} \subset cabv_\lambda(\mathscr{B}a, E')$ (= the Banach space of all λ-continuous members of $cabv(\mathscr{B}a, E')$) and the Radon-Nikodym theorem establishes the isometry between $cabv_\lambda(\mathscr{B}a, E')$ and $L^1(\lambda, E')$, we obtain that $\{y' \circ T : y' \in B_{F'}\}$ is a weakly precompact subset of $C_b(X, E)'_{\beta_\sigma}$ because $(y' \circ T)(f) = \int_X f\, d\overline{m}_{y'}$ for $f \in C_b(X, E)$. □

5. Weakly Completely Continuous Operators on $C_b(X,E)$

Recall that a bounded linear operator T from a Banach space G to a Banach space F is said to be a *Dieudonné operator* if T maps $\sigma(G, G')$-Cauchy sequences in G into weakly convergent sequences in F.

If X is a compact Hausdorff space, then Dieudonné operators from the Banach space $C(X, E)$ to F were studied by Bombal and Cembranos [23] and Abbott et al. (see [17, Theorems 3.1, 3.5 and Theorem, page 334]).

Definition 20. A bounded linear operator $T : C_b(X, E) \to F$ is said to be *weakly completely continuous* if $T(f_n)$ is $\sigma(F, F')$-convergent in F whenever (f_n) is a uniformly boundd sequence in $C_b(X, E)$ such that $(f_n(t))$ is a $\sigma(E, E')$-Cauchy sequence in E for each $t \in X$.

Proposition 21. *Let $T : C_b(X, E) \to F$ be a bounded linear operator. Then the following statements are equivalent.*

(i) *T is weakly completely continuous.*

(ii) *T maps $\sigma(C_b(X, E), C_b(X, E)'_{\beta_\sigma})$-Cauchy sequences in $C_b(X, E)$ onto $\sigma(F, F')$-convergent sequences in F.*

Proof. (i)⇒(ii) Assume that T is weakly completely continuous and (f_n) is a $\sigma(C_b(X, E), C_b(X, E)'_{\beta_\sigma})$-Cauchy sequence in $C_b(X, E)$. Then for each $t \in X$, $(f_n(t))$ is a $\sigma(E, E')$-Cauchy sequence in E because $\Phi_{t,x'} \in C_b(X, E)'_{\beta_\sigma}$, where $\Phi_{t,x'}(f) = x'(f(t))$ for $f \in C_b(X, E)$. Since (f_n) is β_σ-bounded, we get $\sup \|f_n\| < \infty$. It follows that $(T(f_n))$ is $\sigma(F, F')$-convergent.

(ii)⇒(i) Assume that (ii) holds and (f_n) is a uniformly bounded sequence in $C_b(X, E)$ such that $(f_n(t))$ is a $\sigma(E, E')$-Cauchy sequence in E for each $t \in X$. We shall show that (f_n) is a $\sigma(C_b(X, E), C_b(X, E)'_{\beta_\sigma})$-Cauchy sequence. Assume on the contrary that (f_n) is not a $\sigma(C_b(X, E), C_b(X, E)'_{\beta_\sigma})$-Cauchy sequence. Then there exist $\Phi_0 \in C_b(X, E)'_{\beta_\sigma}$ and $\varepsilon_0 > 0$ and a subsequence (g_n) of (f_n) satisfying $|\Phi_0(g_{2n} - g_{2n})| \geq \varepsilon_0$ for $n \in \mathbb{N}$. Since $g_{2n}(t) - g_{2n-1}(t) \to 0$ for each $t \in X$, by Theorem 5 $g_{2n} - g_{2n-1} \to 0$ for $\sigma(C_b(X, E), C_b(X, E)'_{\beta_\sigma})$. Hence $\Phi_0(g_{2n} - g_{2n-1}) \to 0$. This contradiction establishes that (f_n) is a $\sigma(C_b(X, E), C_b(X, E)'_{\beta_\sigma})$-Cauchy sequence, and it follows that a sequence $(T(f_n))$ is $\sigma(F, F')$-convergent in F. □

From Proposition 21 it follows that every weakly completely continuous operator $T : C_b(X, E) \to F$ is a Dieudonné operator. As a consequence, we get the following result (see [37, Problem 8, page 54]).

Corollary 22. *Assume that $T : C_b(X, E) \to F$ is a weakly completely continuous operator. Then T is unconditionally converging.*

Theorem 23. *Let $T : C_b(X, E) \to F$ be a $(\beta_\sigma, \|\cdot\|_F)$-continuous and weakly completely continuous linear operator and $m \in M_\sigma(X, \mathscr{L}(E, F''))$ stand for its representing measure. Then the following statements hold.*

(i) *T is strongly bounded.*

(ii) *For each $A \in \mathscr{B}a$, $\overline{m}(A) : E \to F$ is a Dieudonné operator.*

Proof. (i) It follows from Corollary 22 and Theorem 17.

(ii) Let $A \in \mathscr{B}a$ and assume that (x_n) is a $\sigma(E, E')$-Cauchy sequence in E. Hence $\sup_n \|x_n\|_E < \infty$. Since T is strongly bounded, arguing as in the proof of Theorem 17 for a given $\varepsilon > 0$ there exist $Z \in \mathscr{Z}$ with $Z \subset A$ and $U \in \mathscr{P}$ with $A \subset U$ such that

$$\sup \left\{ \left| m_{y'} \right| (U \setminus Z) : y' \in B_{F'} \right\} \leq \varepsilon. \tag{29}$$

Then we can choose $u \in C_b(X)$ with $0 \leq u \leq \mathbb{1}_X$, $u|_Z \equiv 1$, and $u|_{X \setminus U} \equiv 0$. Define $T_u(x) := T(u \otimes x)$ for $x \in E$. We shall show that $T_u : E \to F$ is a Dieudonné operator. Let $h_n = u \otimes x_n$ for $n \in \mathbb{N}$. Then $\sup_n \|h_n\| \leq \sup_n \|x_n\|_E < \infty$ and $(h_n(t))$ is a $\sigma(E, E')$-Cauchy sequence in E for each $t \in X$. Hence $(T(h_n))$ is $\sigma(F, F')$-convergent in F and this means that T_u is a Dieudonné operator. Then arguing as in the proof of Theorem 17, we obtain that $\|T_u - \overline{m}(A)\| \leq \varepsilon$ and since the class of all Dieudonné operators from E to F is a closed linear subspace of $(\mathscr{L}(E, F), \|\cdot\|)$ (see [17, Theorem 3.5]), we derive that $\overline{m}(A)$ is a Dieudonné operator. □

6. Completely Continuous Operators on $C_b(X,E)$

Recall that a bounded linear operator T from a Banach space G to a Banach space F is said to be a Dunford-Pettis operator if $z_n \to 0$ in G for $\sigma(G, G')$ implies $\|T(z_n)\|_F \to 0$ (see [43, Section 19]).

Definition 24. A bounded linear operator $T : C_b(X, E) \to F$ is said to be *completely continuous* if $\|T(f_n)\|_F \to 0$ whenever (f_n) is a uniformly bounded sequence in $C_b(X, E)$ such that $f_n(t) \to 0$ in $\sigma(E, E')$ for each $t \in X$.

Using Theorem 5 one can get the following result.

Proposition 25. *Let $T : C_b(X,E) \to F$ be a bounded linear operator. Then the following statements are equivalent.*

(i) *T is completely continuous.*

(ii) *$\|T(f_n)\|_F \to 0$ whenever $f_n \to 0$ in $\sigma(C_b(X,E), C_b(X,E)'_{\beta_\sigma})$.*

Theorem 26. *Let $T : C_b(X,E) \to F$ be a $(\beta_\sigma, \|\cdot\|_F)$-continuous and completely continuous operator and $m \in M_\sigma(X, \mathscr{L}(E,F''))$ its representing measure. Then the following statements hold.*

(i) *T is strongly bounded.*

(ii) *For each $A \in \mathscr{B}a$, $\overline{m}(A) : E \to F$ is a Dunford-Pettis operator.*

Proof. (i) In view of [43, Theorem 19.1] and Proposition 25 T maps $\sigma(C_b(X,E), C_b(X,E)')$ Cauchy sequences in $C_b(X,E)$ onto norm convergent sequences in F. It follows that T is a Dieudonné operator and hence T is unconditionally converging. Thus T is strongly bounded (see Theorem 17).

(ii) Let $A \in \mathscr{B}a$ and assume that $x_n \to 0$ in E for $\sigma(E,E')$. Then $\sup\|x_n\|_E < \infty$. Since T is strongly bounded, arguing as in the proof of Theorem 17 for a given $\varepsilon > 0$ there exist $Z \in \mathscr{Z}$ with $Z \subset A$ and $U \in \mathscr{P}$ with $A \subset U$ such that

$$\sup\left\{\left|m_{y'}\right|(U \setminus Z) : y' \in B_{F'}\right\} \le \varepsilon. \tag{30}$$

Then we can choose $u \in C_b(X)$ with $0 \le u \le \mathbb{1}_X$, $u|_Z \equiv 1$, and $u|_{X\setminus U} \equiv 0$. Define $T_u(x) := T(u \otimes x)$ for $x \in E$. We shall show that $T_u : E \to F$ is a Dunford-Pettis operator. Let $h_n = u \otimes x_n$ for $n \in \mathbb{N}$. Then $h_n(t) \to 0$ in $\sigma(E,E')$ for each $t \in X$ and $\sup_n\|h_n\| < \infty$. It follows that $\|T_u(x_n)\|_F = \|T(h_n)\|_F \to 0$ and this means that $T_u : E \to F$ is a Dunford-Pettis operator (see [43, Theorem 19.1]). Then arguing as in the proof of (ii) of Theorem 17, we obtain that $\|T_u - \overline{m}(A)\| \le \varepsilon$. Since the class of all Dunford-Pettis operators from E to F is a closed linear subspace of $(\mathscr{L}(E,F), \|\cdot\|)$ (see [28, page 27]), we derive that $\overline{m}(A)$ is a Dunford-Pettis operator. □

Corollary 27. *Assume that E is a Schur space. Let $T : C_b(X,E) \to F$ be a $(\beta_\sigma, \|\cdot\|)$-continuous linear operator. The the following statements are equivalent.*

(i) *T is strongly bounded.*

(ii) *T is completely continuous.*

(iii) *T is weakly completely continuous.*

(iv) *T is unconditionally converging.*

(v) *$\sum_{n=1}^{\infty} T(f_n)$ converges unconditionally whenever (f_n) is a uniformly bounded sequence in $C_b(X,E)$ such that $\operatorname{supp} f_n \cap \operatorname{supp} f_m = \emptyset$ for $n \ne m$.*

Proof. (i)⇒(ii) Assume that T is strongly bounded and (f_n) is a uniformly bounded sequence in $C_b(X,E)$ such that $f_n(t) \to 0$ in $\sigma(E,E')$ for each $t \in X$. It follows that $\|f_n(t)\|_E \to 0$ because E is supposed to be a Schur space. Hence by Theorem 11 $\|T(f_n)\|_F \to 0$, as desired.

(ii)⇒(iii) It is obvious.

(iii)⇒(iv) See Proposition 21.

(iv)⇒(v) Assume that (iv) hold and let (f_n) be a uniformly bounded sequence in $C_b(X,E)$ such that $\operatorname{supp} f_n \cap \operatorname{supp} f_m = \emptyset$ for $n \ne m$. Let $C = \sup_n\|f_n\|$ and $(a_n) \in \ell^\infty$. Then

$$\sup_n \left\|\sum_{i=1}^{n} a_i f_i\right\| \le C \sup_n \left|a_n\right| \tag{31}$$

and it follows that $\sum_{n=1}^{\infty} f_n$ is *wuc* in $C_b(X,E)$ (see [44]). Hence $\sum_{n=1}^{\infty} T(f_n)$ converges unconditionally in F.

(v)⇒(i) It follows from Theorem 11. □

Theorem 28. *Assume that E is separable. Let $T : C_b(X,E) \to F$ be a $(\beta_\sigma, \|\cdot\|_F)$-continuous and strongly bounded operator and let $m \in M_\sigma(X, \mathscr{L}(E,F''))$ be its representing measure. Then the following statements are equivalent.*

(i) *T is completely continuous.*

(ii) *$\lim_n \int_X \langle f_n, g_{y'_n}\rangle d\lambda = 0$ whenever (f_n) is a uniformly bounded sequence in $C_b(X,E)$ such that $f_n(t) \to 0$ in $\sigma(E,E')$ for $t \in X$ and (y'_n) is a sequence in $B_{F'}$.*

Here $\lambda \in ca^+(\mathscr{B}a)$ is a control measure for $\{|\overline{m}_{y'}| : y' \in B_{F'}\}$ and for $n \in \mathbb{N}$, $g_{y'_n}$ is an element of $\mathscr{L}^1_{w^*}(\lambda, E')$ corresponding to $\overline{m}_{y'_n}$ (see Proposition 16).

Proof. (i)⇒(ii) Assume that T is completely continuous and let (f_n) be a uniformly bounded sequence in $C_b(X,E)$ such that $f_n(t) \to 0$ in $\sigma(E,E')$ for each $t \in X$ and (y'_n) is a sequence in $B_{F'}$. Then, by Proposition 16,

$$\left|\int_X \left\langle f_n, g_{y'_n}\right\rangle d\lambda\right| = \left|y'_n\left(T\left(f_n\right)\right)\right| \le \left\|T(f_n)\right\|_F \longrightarrow 0. \tag{32}$$

(ii)⇒(i) Assume that (ii) holds. Let (f_n) be a uniformly bounded sequence in $C_b(X,E)$ such that $f_n(t) \to 0$ in $\sigma(E,E')$ for each $t \in X$. Choose a sequence (y'_n) in $B_{F'}$ such that $|y'_n(T(f_n))| \ge (1/2)\|T(f_n)\|_F$. Hence, by Proposition 16,

$$\left\|T(f_n)\right\|_F \le 2\left|y'_n\left(T\left(f_n\right)\right)\right| = 2\left|\int_X \left\langle f_n, g_{y'_n}\right\rangle d\lambda\right| \longrightarrow 0, \tag{33}$$

so T is completely continuous. □

7. Weakly Compact Operators on $C_b(X,E)$

If X is a compact Hausdorff space (resp., X is a locally compact Hausdorff space), weakly compact operators $T : C(X,E) \to F$ (resp., $T : C_o(X,E) \to F$) have been studied intensively by Batt and Berg [19, 20], Brooks and Lewis [27], Bombal [24], and Saab [29]. The aim of this section is to extend a characterization of weakly compact operators $T : C_o(X,E) \to F$ of [27, Theorem 4.1] to $(\beta_\sigma, \|\cdot\|_F)$-continuous and weakly compact operators $T : C_b(X,E) \to F$.

Theorem 29. *Let $T : C_b(X,E) \to F$ be a $(\beta_\sigma, \|\cdot\|_F)$-continuous linear operator and let $m \in M_\sigma(X, \mathscr{L}(E,F''))$ be its representing measure. Then the following statements hold.*

(i) *Assume that T is weakly compact. Then T is strongly bounded and for each $A \in \mathscr{B}a$, $\overline{m}(A) : E \to F$ is a weakly compact operator.*

(ii) *Assume that E' and E'' have the RNP and T is strongly bounded and for each $A \in \mathscr{B}a$, $\overline{m}(A) : E \to F$ is a weakly compact operator. Then T is weakly compact.*

Proof. (i) In view of [45, Corollary 9.3.2.] the conjugate operator $T' : F' \to C_b(X,E)'_{\beta_\sigma}$ maps $B_{F'}$ onto a relatively weakly compact subset of $(C_b(X,E)'_{\beta_\sigma}, \|\cdot\|)$, where $(y' \circ T)(f) = \int_X f\, d\overline{m}_{y'}$ for $f \in C_b(X,E)$. Hence $\{\overline{m}_{y'} : y' \in B_{F'}\}$ is a relatively weakly compact subset of the Banach space $cabv(\mathscr{B}a, E')$, equipped with the total variation norm. Making use of the Bartle-Dunford-Schwartz theorem [13, Theorem 5, pages 105-106], we obtain that the set $\{|\overline{m}_{y'}| : y' \in B_{F'}\}$ is uniformly countably additive and, for each $A \in \mathscr{B}a$, the set $\{\overline{m}_{y'}(A) : y' \in B_{F'}\}$ is relatively weakly compact in E'. Thus by Theorem 11 T is strongly bounded and, since $\overline{m}(A)'(y') = \overline{m}_{y'}(A)$, we derive that $\overline{m}(A) : E \to F$ is weakly compact.

(ii) By Theorem 11 $\{|\overline{m}|_{y'} : y' \in B_{F'}\}$ is uniformly countably additive. Moreover, for each $A \in \mathscr{B}a$, $\{\overline{m}_{y'} : y' \in B_{F'}\}$ is relatively weakly compact in E'. This means that $\{\overline{m}_{y'} : y' \in B_{F'}\}$ is relatively weakly compact subset of $M_\sigma(\mathscr{B}a, E')$ (see [13, Theorem 5, pages 105-106]). Since $C_b(X,E)'_{\beta_\sigma} = \{\Phi_\mu : \mu \in M_\sigma(\mathscr{B}a, E')\}$, $\{\Phi_{\overline{m}_{y'}} : y' \in B_{F'}\}$ is a relatively weakly compact subset of $C_b(X,E)'_{\beta_\sigma}$. Hence according to [45, Corollary 9.3.2] T is weakly compact. □

Corollary 30. *Assume that E is reflexive. Then for a $(\beta_\sigma, \|\cdot\|_F)$-continuous linear operator $T : C_b(X,E) \to F$ the following statements are equivalent.*

(i) *T is weakly compact.*

(ii) *T is strongly bounded.*

As a consequence of Corollaries 13 and 30 we can state a generalization of the well known theorem due to Batt and Berg telling us that if X is a compact Hausdorff space, E is reflexive, and F contains no isomorphic copy of c_o, then every bounded linear operator $T : C(X,E) \to F$ is weakly compact (see [20, Theorem 9]).

Corollary 31. *Assume that E is reflexive and F contains no isomorphic copy of c_o. Then every $(\beta_\sigma, \|\cdot\|_F)$-continuous linear operator $T : C_b(X,E) \to F$ is weakly compact.*

8. Nuclear Operators on $C_b(X,E)$

Following [46, Ch. 3, §7] we have the following definition.

Definition 32. A $(\beta_\sigma, \|\cdot\|_F)$-continuous linear operator $T : C_b(X,E) \to F$ is said to be *nuclear* if it can be represented as

$$T(f) = \sum_{n=1}^{\infty} \lambda_n \Phi_n(f)\, y_n \quad \text{for each } f \in C_b(X,E), \tag{34}$$

where (Φ_n) is a β_σ-equicontinuous sequence in $C_b(X,E)'_{\beta_\sigma}$, (y_n) is a bounded sequence in F, and (λ_n) is a sequence in $\mathbb{R}$ such that $\sum_{n=1}^{\infty} |\lambda_n| < \infty$.

In particular, an operator $L \in \mathscr{L}(E,F)$ is said to be *nuclear* if there exist sequences (x'_n) in E' and (y_n) in F such that L is of the form

$$L(x) = \sum_{n=1}^{\infty} x'_n(x)\, y_n \quad \text{for each } x \in E, \tag{35}$$

and $\sum_{n=1}^{\infty} \|x'_n\|_{E'} \cdot \|y_n\|_F < \infty$. Then we say that $\sum_{n=1}^{\infty} (x'_n \otimes y_n)$ represents a nuclear operator L. The *nuclear norm* of a nuclear operator $L : E \to F$ is defined by

$$\|L\|_{\text{nuc}} := \inf \sum_{n=1}^{\infty} \left\|x'_n\right\|_{E'} \cdot \left\|y_n\right\|_F, \tag{36}$$

where the infimum is taken over all sequences (x'_n) and (y_n) such that $L(x) = \sum_{n=1}^{\infty} x'_n(x) y_n$ holds for each $x \in E$. The nuclear operators $L : E \to F$ form a normed space under the nuclear norm $\|\cdot\|_{\text{nuc}}$, which we shall denote by $\mathscr{N}(E,F)$ (see [13, Proposition 2, page 170]).

If X is a compact Hausdorff space, then nuclear operators from the Banach space $C(X,E)$ to F have been studied by Saab and Smith [31]. In this section we extend Proposition 1 of [31] to the completely regular setting.

Let $C_b(X,E)''_{\beta_\sigma}$ stand for bidual of $(C_b(X,E), \beta_\sigma)$. Note that $C_b(X,E)''_{\beta_\sigma} = (C_b(X,E)'_{\beta_\sigma}, \|\cdot\|)'$. Then one can embed $B(\mathscr{B}a, E)$ into $C_b(X,E)''_{\beta_\sigma}$ by the mapping $\overline{\pi} : B(\mathscr{B}a,E) \to C_b(X,E)''_{\beta_\sigma}$, where, for $g \in B(\mathscr{B}a, E)$,

$$\overline{\pi}(g)\left(\Phi_\mu\right) := \int_X g\, d\mu \quad \text{for } \mu \in M_\sigma\left(\mathscr{B}a, E'\right). \tag{37}$$

(Here $\Phi_\mu(f) = \int_X f\, d\mu$ for $f \in C_b(X,E)$).

Proposition 33. *Let $T : C_b(X,E) \to F$ be a $(\beta_\sigma, \|\cdot\|_F)$-continuous linear operator such that $T_x : C_b(X) \to F$ is weakly compact for each $x \in E$, and let $m \in M_\sigma(X, \mathscr{L}(E,F''))$ be its representing measure. Then the following statements hold.*

(i) *$(T'' \circ \overline{\pi})(g) = \int_X g\, d\overline{m}_{y'}$ for $g \in B(\mathscr{B}a, E)$.*

(ii) *$i_F(\overline{m}(A)(x)) = (T'' \circ \overline{\pi})(\mathbb{1}_A \otimes x)$ for $A \in \mathscr{B}a$, $x \in E$.*

Proof. (i) Let $T' : F' \to C_b(X,E)'_{\beta_\sigma}$ and $T'' : C_b(X,E)''_{\beta_\sigma} \to F''$ stand for the conjugate and the biconjugate operators of T, respectively. Then for each $y' \in F'$, $y'(T(f)) = \int_X f\, dm_{y'} = \int_X f\, d\overline{m}_{y'}$ for $f \in C_b(X,E)$, and hence, for $g \in B(\mathscr{B}a, E)$,

$$\begin{aligned}\left(\left(T'' \circ \overline{\pi}\right)(g)\right)\left(y'\right) &= \overline{\pi}(g)\left(T'\left(y'\right)\right) = \overline{\pi}(g)\left(y' \circ T\right) \\ &= \int_X g\, d\overline{m}_{y'}.\end{aligned} \tag{38}$$

(ii) Let $A \in \mathscr{B}a$. Then by (i) for each $x \in E$ and $y' \in F'$ we get

$$\begin{aligned}&\left(\left(T'' \circ \overline{\pi}\right)\left(\mathbb{1}_A \otimes x\right)\right)\left(y'\right)\\&= \int_X \left(\mathbb{1}_A \otimes x\right) d\overline{m}_{y'}\\&= \overline{m}_{y'}(A)(x) = y'(\overline{m}(A)(x)) = i_F(\overline{m}(A)(x))\left(y'\right).\end{aligned} \tag{39}$$

Hence $i_F(\overline{m}(A)(x)) = (T'' \circ \overline{\pi})(\mathbb{1}_A \otimes x)$ for $A \in \mathscr{B}a, x \in E$. □

For $\overline{m} : \mathscr{B}a \to \mathscr{L}(E, F)$ by $|\overline{m}|_{\text{nuc}}(A)$ we denote the variation of $\overline{m}$ on $A \in \mathscr{B}a$; that is,

$$|\overline{m}|_{\text{nuc}}(A) := \sup \sum_{i=1}^{n} \left\|\overline{m}(A_i)\right\|_{\text{nuc}}, \tag{40}$$

where the supremum is taken over all finite $\mathscr{B}a$-partitions $(A_i)_{i=1}^n$ of A.

Theorem 34. *Let $T : C_b(X, E) \to F$ be a $(\beta_\sigma, \|\cdot\|_F)$-continuous and nuclear operator and let $m \in M_\sigma(X, \mathscr{L}(E, F''))$ be its representing measure. Then the following statements hold.*

(i) *T is strongly bounded.*

(ii) *For each $A \in \mathscr{B}a$, $\overline{m}(A) \in \mathscr{N}(E, F)$.*

(iii) *$|\overline{m}|_{nuc}(X) < \infty$ and $\overline{m} : \mathscr{B}a \to \mathscr{N}(E, F)$ is $\|\cdot\|_{nuc}$-countably additive.*

Proof. (i) In view of [46, Ch. 3, §7, Corollary 1] T is $(\beta_\sigma, \|\cdot\|_F)$-compact. Hence by Theorem 29 T is strongly bounded.

(ii) Assume that T is of the form

$$T(f) = \sum_{n=1}^{\infty} \lambda_n \Phi_n(f) y_n \quad \text{for each } f \in C_b(X, E), \tag{41}$$

where (Φ_n) is a β_σ-equicontinuous sequence in $C_b(X, E)'_{\beta_\sigma}$, (y_n) is a bounded sequence in F, and (λ_n) is a sequence in $\mathbb{R}$ such that $\sum_{n=1}^{\infty} |\lambda_n| < \infty$. Then for $n \in \mathbb{N}$, $\Phi_n(f) = \Phi_{\mu_n}(f) = \int_X f\, d\mu_n$, where $\mu_n \in M_\sigma(\mathscr{B}a, E')$ and $|\mu_n|(X) = \|\Phi_{\mu_n}\|$ (see Remark 9). It follows that $\sup_n |\mu_n|(X) = \sup \|\Phi_{\mu_n}\| < \infty$. Assume that $A \in \mathscr{B}a$. Then for $x \in E$, $y' \in F'$, using Proposition 33 we get

$$\begin{aligned}y'(\overline{m}(A)(x)) &= \left(\left(T'' \circ \overline{\pi}\right)\left(\mathbb{1}_A \otimes x\right)\right)\left(y'\right)\\&= \overline{\pi}\left(\mathbb{1}_A \otimes x\right)\left(y' \circ T\right)\\&= \overline{\pi}\left(\mathbb{1}_A \otimes x\right)\left(\sum_{n=1}^{\infty} \lambda_n y'(y_n) \Phi_{\mu_n}\right)\\&= \sum_{n=1}^{\infty} \lambda_n y'(y_n) \overline{\pi}\left(\mathbb{1}_A \otimes x\right)\left(\Phi_{\mu_n}\right)\\&= \sum_{n=1}^{\infty} \lambda_n y'(y_n) \int_X \left(\mathbb{1}_A \otimes x\right) d\mu_n\\&= \sum_{n=1}^{\infty} \lambda_n y'(y_n) \mu_n(A)(x)\\&= y'\left(\sum_{n=1}^{\infty} \lambda_n \mu_n(A)(x)\, y_n\right).\end{aligned} \tag{42}$$

Hence

$$\overline{m}(A)(x) = \sum_{n=1}^{\infty} \lambda_n \mu_n(A)(x)\, y_n \quad \text{for } x \in E. \tag{43}$$

Note that $\|\mu_n(A)\|_{E'} \le |\mu_n|(A) \le |\mu_n|(X)$, and hence

$$\begin{aligned}\sum_{n=1}^{\infty} \|\mu_n(A)\|_{E'} \|\lambda_n y_n\|_F &\le \sum_{n=1}^{\infty} |\mu_n|(X) |\lambda_n| \|y_n\|_F\\&\le \sup_n |\mu_n|(X) \sup_n \|y_n\|_F \sum_{n=1}^{\infty} |\lambda_n| < \infty.\end{aligned} \tag{44}$$

This means that $\overline{m}(A) : E \to F$ is a nuclear operators, as desired.

(iii) To show that $|\overline{m}|_{\text{nuc}}(X) < \infty$, assume $(A_i)_{i=1}^k$ is a $\mathscr{B}a$-partition of X. Then using (43)

$$\begin{aligned}\sum_{i=1}^{k} \left\|\overline{m}(A_i)\right\|_{\text{nuc}} &\le \sum_{i=1}^{k}\sum_{n=1}^{\infty} |\lambda_n| \left\|\mu_n(A_i)\right\|_{E'} \|y_n\|_F\\&\le \sum_{i=1}^{k}\sum_{n=1}^{\infty} |\lambda_n| |\mu_n|(A_i) \|y_n\|_F\\&\le \sum_{i=1}^{k}\left(\sum_{n=1}^{\infty} |\lambda_n| |\mu_n|(A_i) \|y_n\|_F\right)\\&\le \sum_{n=1}^{\infty} |\lambda_n| |\mu_n|(X) \|y_n\|_F\\&\le \sup_n |\mu_n|(X) \sup_n \|y_n\|_F \sum_{n=1}^{\infty} |\lambda_n| < \infty.\end{aligned} \tag{45}$$

Hence $|\overline{m}|_{\text{nuc}}(X) < \infty$, as desired. Now we will show that $\overline{m} : \mathscr{B}a \to \mathscr{N}(E, F)$ is $\|\cdot\|_{\text{nuc}}$-countably additive. Let $\varepsilon > 0$ be given. Since $\sum_{n=1}^{\infty} |\lambda_n||\mu_n|(X) \le \sup_n |\mu_n|(X) \sum_{n=1}^{\infty} |\lambda_n| < \infty$, one can choose $n_\varepsilon \in \mathbb{N}$ such that

$$\sum_{n=n_\varepsilon+1}^{\infty} |\lambda_n| |\mu_n|(X) \le \frac{\varepsilon}{2a}, \quad \text{where } a = \sup_n \|y_n\|_F. \tag{46}$$

Since $\mu_n \in M_\sigma(\mathscr{B}a, E')$, for $n \in \mathbb{N}$, there exists $k \in \mathbb{N}$ such that

$$\left|\lambda_j\right| \left|\mu_j\right|\left(\bigcup_{i=k}^{\infty} A_i\right) \le \frac{\varepsilon}{2n_\varepsilon a} \quad \text{for } j = 1, \dots, n_\varepsilon. \tag{47}$$

Hence

$$\begin{aligned}
&\left\|\overline{m}\left(\bigcup_{i=k}^{\infty}A_i\right)-\sum_{i=1}^{k-1}\overline{m}\left(A_i\right)\right\|_{\mathrm{nuc}} \\
&\quad=\left\|\overline{m}\left(\bigcup_{i=k}^{\infty}A_i\right)\right\|_{\mathrm{nuc}} \\
&\quad\le\sum_{n=1}^{\infty}\left|\lambda_n\right|\left\|\mu_n\left(\bigcup_{i=k}^{\infty}A_i\right)\right\|_{E'}\left\|y_n\right\|_F \\
&\quad\le a\sum_{n=1}^{n_\varepsilon}\left|\lambda_n\right|\left|\mu_n\right|\left(\bigcup_{i=k}^{\infty}A_i\right)+a\sum_{n=n_\varepsilon+1}^{\infty}\left|\lambda_n\right|\left|\mu_n\right|\left(\bigcup_{i=k}^{\infty}A_i\right) \\
&\quad\le a\frac{\varepsilon}{2a}+a\frac{\varepsilon}{2a}=\varepsilon.
\end{aligned}\tag{48}$$

This means that $\overline{m}:\mathscr{B}a\to\mathscr{N}(E,F)$ is $\|\cdot\|_{\mathrm{nuc}}$-countably additive. □

9. Strictly Singular Operators on $C_b(X,E)$

Definition 35. A bounded linear operator $T:C_b(X,E)\to F$ is said to be *strictly singular* if it does not have a bounded inverse on any infinite-dimensional subspace contained in $C_b(X,E)$.

Bilyeu and Lewis [21, Theorem 4.1] showed that if X is compact, then every strictly singular operator $T:C(X,E)\to F$ is strongly bounded and, for each Borel set in X, $m(A):E\to F$ is strictly singular. Strictly singular operators $T:C(X,E)\to F$ have been studied by Bessaga and Pełczyński [44] and Abbott et al. [18].

Now we show an analogue of Theorem 4.1 of [21] for $(\beta_\sigma,\|\cdot\|)$-continuous and strictly singular operators $T:C_b(X,E)\to F$, where X is a completely regular Hausdorff space.

Theorem 36. *Let $T:C_b(X,E)\to F$ be a $(\beta_\sigma,\|\cdot\|_F)$-continuous and strictly singular linear operator and let $m\in M_\sigma(X,\mathscr{L}(E,F''))$ be its representing measure. Then the following statements hold.*

(i) *T is strongly bounded.*

(ii) *For each $A\in\mathscr{B}a$, $\overline{m}(A):\to F$ is strictly singular.*

Proof. Since T is strictly singular, T is unconditionally converging (see [47, Proposition 1.5]) and hence by Theorem 17 T is strongly bounded. Suppose that there is $A\in\mathscr{B}a$ such that $\overline{m}(A):E\to F$ is not strictly singular. Then there is an infinite-dimensional subspace M of E so that $\overline{m}(A)|_M$ has a bounded inverse. Therefore, there is $a>0$ so that $\|\overline{m}(A)(x)\|_F\ge a\|x\|_E$ for each $x\in M$.

Let $\varepsilon>0$ be given such that $2\varepsilon<a$. Hence by Corollary 12, there exist $Z\in\mathscr{Z}$, $Z\subset A$ and $U\in\mathscr{P}$, $U\supset A$ such that $\widetilde{m}(U\backslash Z)\le\varepsilon$. Choose a function $u_o\in C_b(X)$ with $0\le u_o\le\mathbb{1}_X$ such that $u_o|_Z\equiv 1$ and $u_o|_{X\backslash U}\equiv 0$. For $x\in E$ let $h_x=u_o\otimes x$. Then, by Theorem 8,

$$\begin{aligned}
\left\|T(h_x)\right\|_F&=\left\|\int_U h_x d\overline{m}\right\|_F \\
&\ge\left\|\int_Z h_x d\overline{m}\right\|_F-\left\|\int_{U\backslash Z} h_x d\overline{m}\right\|_F \\
&\ge\left\|\int_Z h_x d\overline{m}\right\|_F-\left\|\int_{U\backslash Z} h_x d\overline{m}\right\|_F \\
&\ge\|\overline{m}(Z)(x)\|_F-\widetilde{m}\left(U\setminus Z\right)\|x\|_E \\
&\ge\|\overline{m}(A)(x)\|_F-\|\overline{m}(A\setminus Z)(x)\|_F-\varepsilon\|x\|_E \\
&\ge\|\overline{m}(A)(x)\|_F-\widetilde{\overline{m}}\left(A\setminus Z\right)\|x\|_E-\varepsilon\|x\|_E \\
&\ge a\|x\|_E-\varepsilon\|x\|_E-\varepsilon\|x\|_E\ge(a-2\varepsilon)\|x\|_E.
\end{aligned}\tag{49}$$

Let $E_M=\{u_o\otimes x:x\in M\}$. Then E_M is an infinite-dimensional subspace of $C_b(X,E)$, and this means that T is not strictly singular, a contradiction. □

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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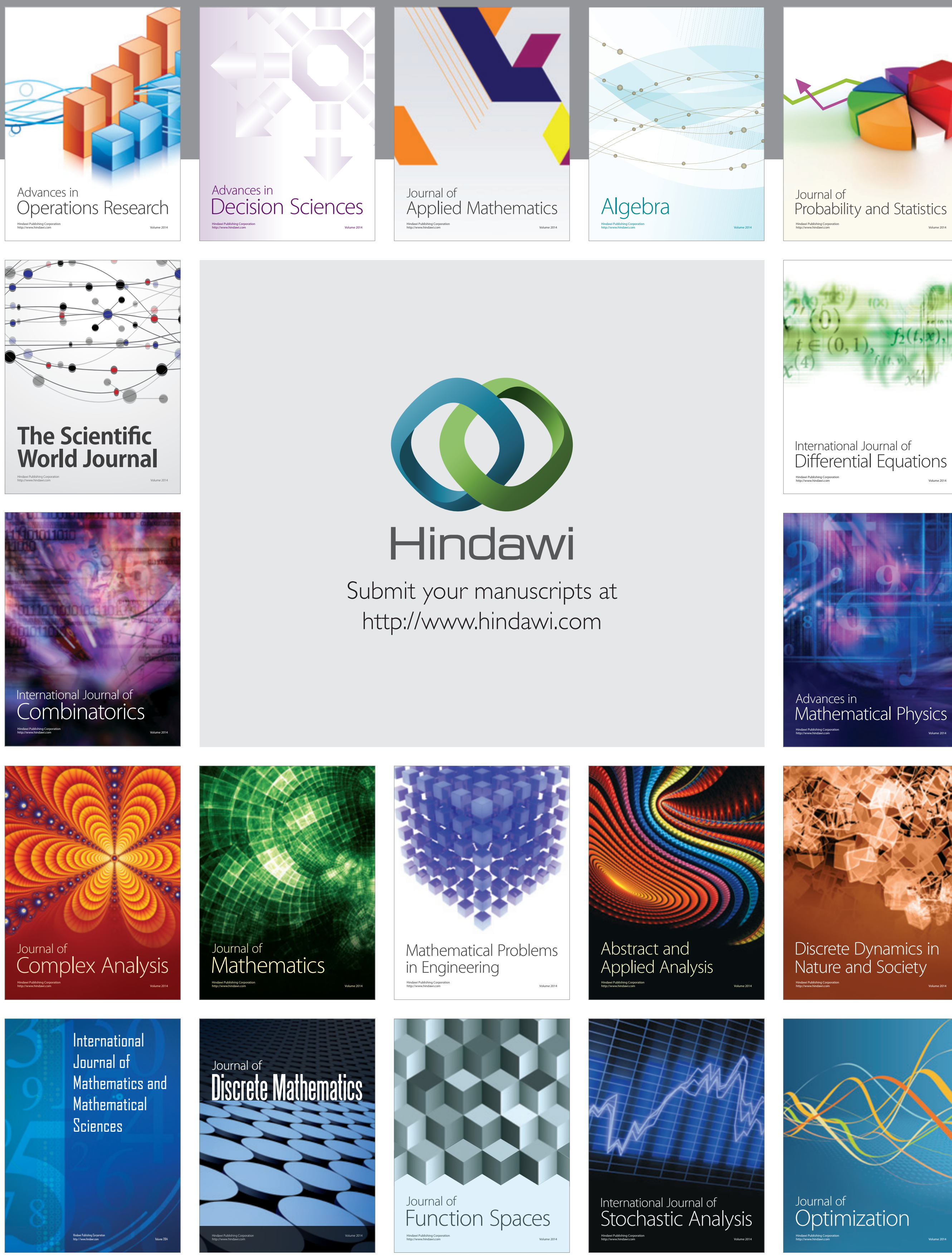
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