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Research Article

Topological and Functional Properties of Some F-Algebras of Holomorphic Functions

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Let N^p $(1 < p < \infty)$ be the Privalov class of holomorphic functions on the open unit disk $\mathbb{D}$ in the complex plane. The space N^p equipped with the topology given by the metric d_p defined by $d_p(f,g) = (\int_0^{2\pi}(\log(1+|f^*(e^{i\theta}) - g^*(e^{i\theta})|))^p(d\theta/2\pi))^{1/p}$, $f, g \in N^p$, becomes an F-algebra. For each $p > 1$, we also consider the countably normed Fréchet algebra F^p of holomorphic functions on $\mathbb{D}$ which is the Fréchet envelope of the space N^p. Notice that the spaces F^p and N^p have the same topological duals. In this paper, we give a characterization of bounded subsets of the spaces F^p and weakly bounded subsets of the spaces N^p with $p > 1$. If $(F^p)^*$ denotes the strong dual space of F^p and $(N^p)^*_w$ denotes the space S_p of complex sequences $\gamma = \{\gamma_n\}_n$ satisfying the condition $\gamma_n = O(\exp(-cn^{1/(p+1)}))$, equipped with the topology of uniform convergence on weakly bounded subsets of N^p, then we prove that $(F^p)^* = (N^p)^*_w$ both set theoretically and topologically. We prove that for each $p > 1$ F^p is a Montel space and that both spaces F^p and $(F^p)^*$ are reflexive.

1. Introduction and Preliminaries

Let $\mathbb{D}$ denote the open unit disk in the complex plane and let $\mathbb{T}$ denote the boundary of $\mathbb{D}$.

The *Privalov class* N^p $(1 < p < \infty)$ consists of all holomorphic functions f on $\mathbb{D}$ for which

$$\sup_{0<r<1} \int_0^{2\pi} \left(\log^+ \left|f\left(re^{i\theta}\right)\right|\right)^p \frac{d\theta}{2\pi} < +\infty. \tag{1}$$

These classes were firstly considered by Privalov in [1, page 93], where N^p is denoted as A_q.

Notice that, for $p = 1$, condition (1) defines the *Nevanlinna class* N of holomorphic functions in $\mathbb{D}$. Recall that the *Smirnov class* N^+ is the set of all functions f holomorphic on $\mathbb{D}$ such that

$$\lim_{r\to 1} \int_0^{2\pi} \log^+ \left|f\left(re^{i\theta}\right)\right| \frac{d\theta}{2\pi} = \int_0^{2\pi} \log^+ \left|f^*\left(e^{i\theta}\right)\right| \frac{d\theta}{2\pi} < +\infty, \tag{2}$$

where f^* is the boundary function of f on $\mathbb{T}$; that is,

$$f^*\left(e^{i\theta}\right) = \lim_{r\to 1-} f\left(re^{i\theta}\right) \tag{3}$$

is the *radial limit* of f which exists for almost every $e^{i\theta}$. We denote by H^q $(0 < q \le \infty)$ the classical *Hardy space* on $\mathbb{D}$. It is known (see [2, 3]) that

$$\begin{gathered} N^r \subset N^p \quad (r > p), \\ \bigcup_{q>0} H^q \subset \bigcap_{p>1} N^p, \\ \bigcup_{p>1} N^p \subset N^+ \subset N, \end{gathered} \tag{4}$$

where the above containment relations are proper.

Notice that Privalov in [1, page 98] established the inner-outer factorization theorem for the spaces N^p $(1 < p < \infty)$ (for another proof see [4]). The study of the spaces N^p $(1 < p < \infty)$ on the unit disk was continued in 1977 by Stoll [5] (with the notation $(\log^+ H)^\alpha$ in [5]). The topological and functional properties of these spaces were extensively studied in [6]. Different topologies on the spaces N^p were considered and compared in [7] with related applications. Complex-linear isometries of N^p are investigated in [8]. Motivated by some results of Matsugu [9], in [10] the structure of closed weakly dense ideals in Privalov spaces N^p $(1 < p < \infty)$ was studied. Notice that the structure of maximal ideals of the algebras N^p and their Fréchet envelopes F^p $(1 < p < \infty)$ was investigated in [11]. The interpolation problems for the spaces N^p are treated in [12].

Stoll [5, Theorem 4.2] showed that the space N^p (with the notation $(\log^+ H)^\alpha$ in [5]) with the topology given by the metric d_p defined by

$$d_p(f,g) = \left(\int_0^{2\pi} \left(\log\left(1 + \left|f^*\left(e^{i\theta}\right) - g^*\left(e^{i\theta}\right)\right|\right)\right)^p \frac{d\theta}{2\pi}\right)^{1/p}, \quad f, g \in N^p, \tag{5}$$

becomes an F-algebra.

Recall that the function $d_1 = d$ defined on the Smirnov class N^+ by (5) with $p = 1$ induces the metric topology on N^+. Yanagihara [13] showed that, under this topology, N^+ is an F-space.

Recently, in [14] the author of this note characterized some topological properties of the spaces N^p $(1 < p < \infty)$. For these purposes, the fact that the metric ρ_p defined on N^p as

$$\rho_p(f,g) = \left(\int_0^{2\pi} \log^p\left(1 + M(f-g)(\theta)\right) \frac{d\theta}{2\pi}\right)^{1/p}, \tag{6}$$

with $f, g \in N^p$ and

$$Mf(\theta) = \sup_{0 \le r < 1} \left|f\left(re^{i\theta}\right)\right|, \tag{7}$$

induces on the space N^p the same topological structure as the initial metric d_p given by (5) (see [14, Theorem 16]) is used.

Furthermore, in connection with the spaces N^p $(1 < p < \infty)$, Stoll [5] (also see [15] and [10, Section 3]) also studied the spaces F^q $(0 < q < \infty)$ (with the notation $F_{1/q}$ in [5]), consisting of those functions f holomorphic on $\mathbb{D}$ for which

$$\lim_{r \to 1} (1-r)^{1/q} \log^+ M_\infty(r, f) = 0, \tag{8}$$

where

$$M_\infty(r, f) = \max_{|z| \le r} |f(z)|. \tag{9}$$

Stoll [5, Theorem 3.2] proved that the space F^q with the topology given by the family of seminorms $\{\|\cdot\|_{q,c}\}_{c>0}$ defined for $f \in F^q$ as

$$\|f\|_{q,c} := \sum_{n=0}^{\infty} |a_n| \exp\left(-cn^{1/(q+1)}\right) < \infty \tag{10}$$

is a countably normed *Fréchet algebra*.

Here, as always in the sequel, we will need some Stoll's results concerning the spaces F^q only with $1 < q < \infty$, and, hence, we will assume that $q = p > 1$ be any fixed number.

Theorem 1 (see [5, Theorem 2.2]). *Suppose that $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is a holomorphic function on $\mathbb{D}$. Then the following statements are equivalent:*

(a) $f \in F^p$.

(b) *There exists a sequence $\{c_n\}_n$ of positive real numbers with $c_n \to 0$ such that*

$$|a_n| \le \exp\left(c_n n^{1/(p+1)}\right), \quad n = 0, 1, 2, \ldots. \tag{11}$$

(c) *For any $c > 0$,*

$$\|f\|_{p,c} := \sum_{n=0}^{\infty} |a_n| \exp\left(-cn^{1/(p+1)}\right) < \infty. \tag{12}$$

Remark 2. Notice that, in view of Theorem 1 ((a)⇔(c)), by (10) the family of seminorms $\{\|\cdot\|_{p,c}\}_{c>0}$ on F^p is well defined.

Recall that a locally convex F-space is called a *Fréchet space*, and a *Fréchet algebra* is a Fréchet space that is an algebra in which multiplication is continuous.

Notice that the space N^p is not locally convex (see [15, Theorem 4.2] and [16, Corollary]), and, hence, N^p is properly contained in F^p. Moreover, N^p is not locally bounded (see [17, Theorem 1.1]). The most important connection between spaces N^p and F^p is given by the following result.

Theorem 3 (see [5, Theorem 4.3]). *For any fixed $p > 1$ the following assertions hold:*

(a) *N^p is a dense subspace of F^p.*

(b) *The topology on F^p defined by the family of seminorms (10) is weaker than the topology on N^p given by the metric d_p defined by (5).*

Remark 4. For $p = 1$, the space F_1 has been denoted by F^+ and has been studied by Yanagihara in [13, 18]. It was shown in [13, 18] that F^+ is actually the containing Fréchet space for N^+; that is, N^+ with the initial topology embeds densely into F^+, under the natural inclusion, and F^+ and the Smirnov class N^+ have the same topological duals.

Observe that the space F^p topologised by the family of seminorms $\{\|\cdot\|_{p,c}\}_{c>0}$ given by (10) is metrizable by the metric λ_p defined as

$$\lambda_p(f,g) = \sum_{n=1}^{\infty} 2^{-n} \frac{\|f-g\|_{p,1/n^{p/(p+1)}}}{1 + \|f-g\|_{p,1/n^{p/(p+1)}}}, \quad f, g \in F^p. \tag{13}$$

The following Stoll's result describes the topological dual of the space F^p.

Theorem 5 (see [5, Theorem 3.3]). *If γ is a continuous linear functional on F^p, then there exists a sequence $\{\gamma_n\}_n$ of complex numbers with $\gamma_n = O(\exp(-cn^{1/(p+1)}))$, for some $c > 0$, such that*

$$\gamma(f) = \sum_{n=0}^{\infty} a_n \gamma_n, \tag{14}$$

where $f(z) = \sum_{n=0}^{\infty} a_n z^n \in F^p$, *with convergence being absolute. Conversely, if* $\{\gamma_n\}_n$ *is a sequence of complex numbers for which*

$$\gamma_n = O\left(\exp\left(-cn^{1/(p+1)}\right)\right), \tag{15}$$

then (14) defines a continuous linear functional on F^p.

Let us recall that if $X = (X, \tau)$ is an F-space whose topological dual (the set of all continuous linear functionals on X) X^* separates the points of X, then its *Fréchet envelope* $\widehat{X}$ is defined to be the completion of the space (X, τ^c), where τ^c is the strongest locally convex (necessarily metrizable) topology on X that is weaker than τ. In fact, it is known that τ^c is equal to the *Mackey topology* of the dual pair (X, X^*), that is, to the unique maximal locally convex topology on X for which X still has dual space X^* (see [19, Theorem 1]). For each metrizable locally convex topology τ on X, (X, τ) is a *Mackey space*; that is, τ coincides with the Mackey topology of the dual pair (X, X^*) (see [20, Corollary 22.3, page 210]).

Eoff [15, the proof of Theorem 4.2] showed that the topology of F^p, $p > 1$ (resp., $F^1 = F^+$), is stronger than that of the Fréchet envelope of N^p (resp., N^+). As an immediate consequence of this result, we obtain the following statements.

Theorem 6 (see [15, Theorem 4.2, the case $p > 1$]). *For each* $p > 1$, F^p *is the Fréchet envelope of* N^p.

Theorem 7 (see [21, Theorem 2]; also see [14, Theorem 17]). *The spaces* N^p *and* F^p *have the same dual spaces in the sense that every continuous linear functional on* F^p *(given by (14)) is restricted to one on* N^p*, and every continuous linear functional on* N^p *extends continuously to one on* F^p.

Hence, the dual spaces of N^p *and* F^p *can be identified with the collection* S_p *of complex sequences* $\{\gamma_n\}$ *satisfying the growth condition (15).*

Remark 8. Theorem 7 is proved in [21, Theorem 1] directly, by using the characterization of multipliers from N^q to H^∞. Notice also that the dual space of the Smirnov class N^+ is described by Yanagihara in [13] and applying another method by McCarthy in [22].

Remark 9. Recall that we may introduce *the weak topology* on N^p in the usual way. The basic weak neighborhoods of zero are defined by $V(\varphi_1, \dots, \varphi_n; \varepsilon) = \{f \in N^p : |\varphi_i(f)| < \varepsilon,\ i = 1, \dots, n\}$, where $\varepsilon > 0$ and $n \in \mathbb{N}$ are arbitrary, and $\varphi_1, \dots, \varphi_n$ are arbitrary continuous linear functionals on N^p. The weak topology of N^p is locally convex, and, hence, by [20, Corollary 17.3, page 154], a linear functional on N^p is weakly continuous if and only if it is continuous with respect to the initial metric topology d_p.

In Section 2, we give a characterization of bounded subsets of the spaces F^p $(1 < p < \infty)$ (Theorem 10). As an application, we obtain a characterization of weakly bounded subsets of the spaces N^p (Theorem 11). In Section 3, we prove that $(F^p)^* = (N^p)_w^*$ both set theoretically and topologically (Theorem 12). Here $(F^p)^*$ denotes the strong dual space of F^p, and $(N^p)_w^*$ is the space S_p of complex sequences $\gamma = \{\gamma_n\}_n$ satisfying the growth condition (15), equipped with the topology of uniform convergence on weakly bounded subsets of N^p. Finally, we prove that F^p is a Montel space (Theorem 13) and that both spaces F^p and $(F^p)^*$ are reflexive (Theorem 14).

2. Bounded Subsets of the Spaces F^p

The following result characterizes bounded sets of the space F^p.

Theorem 10. *Let* $p > 1$ *and let* E *be a subset of* F^p. *Then the following assertions are equivalent:*

(i) *E is a bounded subset of* F^p.

(ii) *E is a relatively compact subset of* F^p.

(iii) *There exists a constant $A > 0$ depending on E and a sequence $\{c_n\}$ of positive real numbers such that $c_n \downarrow 0$ and*

$$|a_n| \le A \exp\left(c_n n^{1/(p+1)}\right) \quad \text{for each function } f(z) = \sum_{n=0}^{\infty} a_n z^n \in E. \tag{16}$$

Proof. (ii)⇒(i): it follows immediately from the fact that every relatively compact set in a topological vector space is bounded.

(iii)⇒(i): for given $c > 0$ choose a positive integer m such that $c_n < c/2$ for each $n > m$. Then by condition (iii) for every function $f(z) = \sum_{n=0}^{\infty} a_n z^n \in E$ we have

$$\begin{aligned}\|f\|_{p,c} &= \sum_{n=0}^{\infty} |a_n| \exp\left(-cn^{1/(p+1)}\right) \\ &\le A \sum_{n=0}^{\infty} \exp\left((c_n - c)\, n^{1/(p+1)}\right) \\ &\le A \left(\sum_{n=0}^{m} \exp\left((c_n - c)\, n^{1/(p+1)}\right) \right. \\ &\quad \left. + \sum_{n=m+1}^{\infty} \exp\left(-\frac{c}{2} n^{1/(p+1)}\right) \right) = K(c) = K,\end{aligned} \tag{17}$$

where K is a constant depending only on c. Hence, E is a bounded set in every normed space $(F^p, \|\cdot\|_{p,c})$, $c > 0$, and, thus, E is a bounded subset of F^p.

(i)⇒(iii): suppose that E is a bounded set in F^p. For arbitrary $c > 0$ and $\eta > 0$ the set V defined as

$$V = \left\{g \in F^p : \|g\|_{p,c} < \eta\right\} \tag{18}$$

is a neighbourhood of 0 in the space F^p. Since, by the assumption, E is a bounded set in F^p, there exists $\alpha > 0$ such that $\alpha E \subset V$. This yields

$$\sum_{n=0}^{\infty} \left|\alpha a_n\right| \exp\left(-cn^{1/(p+1)}\right) < \eta \qquad \text{for each } f(z) = \sum_{n=0}^{\infty} a_n z^n \in E. \tag{19}$$

From (19) we find that

$$\left|a_n\right| \le \left|\frac{\eta}{\alpha}\right| \exp\left(cn^{1/(p+1)}\right) \quad \text{for each } n \in \mathbb{N}. \tag{20}$$

Then putting

$$a_n^* = \left\{\sup\left|a_n(f)\right| : f(z) = \sum_{n=0}^{\infty} a_n(f) z^n \in E\right\} \tag{21}$$

into (20), we find that

$$a_n^* \le \left|\frac{\eta}{\alpha}\right| \exp\left(cn^{1/(p+1)}\right) \quad \text{for each } n \in \mathbb{N}. \tag{22}$$

Since $c > 0$ is arbitrary, from (22) we immediately have

$$a_n^* = O\left(\exp\left(o\left(n^{1/(p+1)}\right)\right)\right), \tag{23}$$

which immediately yields (16).

(iii)⇒(ii): since in a metric space compactness and sequential compactness are equivalent, it is necessary to show that a set E satisfying condition (16) is a relatively compact subset of F^p; that is, for every sequence $\{f_n\} \subset E$ there exists a subsequence of $\{f_n\}$ which is convergent in F^p. We will inductively construct such a subsequence of a sequence $\{f_n\}$ in the following way. Take

$$f_n(z) = \sum_{k=0}^{\infty} a_{kn} z^k \quad \text{for each } n \in \mathbb{N}. \tag{24}$$

Since, by assumption (16),

$$\left|a_{0n}\right| \le A \exp\left(c_0\right) \quad \text{for each } n \in \mathbb{N}, \tag{25}$$

it follows that there is a subsequence $\{f_n^{(0)}\}$ of $\{f_n\}$ ($f_n^{(0)}$ denotes the nth term of this subsequence) such that the appropriate subsequence $\{a_{0n}^{(0)}\}$ of a sequence $\{a_{0n}\}$ is convergent; assume that

$$\lim_{n\to\infty} a_{0n}^{(0)} = a_0. \tag{26}$$

Take $f_{i_0} = f_0^{(0)}$; that is, denote by f_{i_0} the first term of the obtained subsequence $\{f_n^{(0)}\}$.

Since, by assumption (16), we have

$$\left|a_{1n}\right| \le A \exp\left(c_1\right) \quad \text{for each } n \in \mathbb{N}, \tag{27}$$

it follows that there exists a subsequence $\{f_n^{(1)}\}$ of $\{f_n^{(0)}\}$ ($f_n^{(1)}$ denotes the nth term of this subsequence) such that the corresponding subsequence $\{a_{1n}^{(1)}\}$ of a sequence $\{a_{0n}^{(0)}\}$ is convergent; assume that

$$\lim_{n\to\infty} a_{1n}^{(1)} = a_1. \tag{28}$$

Put $f_{i_1} = f_1^{(1)}$; that is, denote by f_{i_1} the first term of the obtained subsequence $\{f_n^{(1)}\}$ which is different from f_{i_0}.

By continuing the above diagonal procedure and taking into account that in view of (16) and (24)

$$\left|a_{kn}\right| \le A \exp\left(c_k k^{1/(p+1)}\right) \quad \text{for each } n \in \mathbb{N}, \tag{29}$$

after $s+1$ steps we obtain a subsequence $\{f_n^{(s)}\}$ of $\{f_n^{(s-1)}\}$ ($f_n^{(s)}$ denotes the nth term of this subsequence) such that the corresponding subsequence $\{a_{sn}^{(s)}\}$ of $\{a_{(s-1)n}^{(s-1)}\}$ converges; assume that

$$\lim_{n\to\infty} a_{sn}^{(s)} = a_s. \tag{30}$$

Take $f_{i_s} = f_s^{(s)}$; that is, denote by f_{i_s} the first term of the obtained subsequence $\{f_n^{(s)}\}$ which is different from $\{f_n^{(j)}\}$ for all $j = 0, 1, \ldots, s-1$.

In this way we have obviously constructed a subsequence $\{f_{i_n}\}$ of a sequence $\{f_n\}$ such that

$$f_{i_n}(z) = \sum_{k=0}^{\infty} a_{kn}^{(n)} z^k \quad \text{for each } n \in \mathbb{N}. \tag{31}$$

Furthermore, by the above construction, $\{a_{kn}^{(k)}\}_n$ is a convergent sequence for any fixed $k \in \mathbb{N}$; assume that

$$\lim_{n\to\infty} a_{kn}^{(k)} = a_k \quad \text{for each fixed } k \in \mathbb{N}. \tag{32}$$

Let f be a function defined as

$$f(z) = \sum_{k=0}^{\infty} a_k z^k, \tag{33}$$

where the coefficients a_k are defined by (26) and (32). Since by (29) and (30) we have

$$\left|a_k\right| \le A \exp\left(c_k k^{1/(p+1)}\right) \quad \text{for each } k \in \mathbb{N}, \tag{34}$$

it follows from Theorem 1 that f belongs to the space F^p.

It remains to show that $f_{i_n} \to f$ in F^p as $n \to \infty$. Let $c > 0$ be any fixed positive real number. Choose a nonnegative integer $k_1 = k_1(c)$ such that

$$c_k < \frac{c}{2} \quad \text{for each } k > k_1, \tag{35}$$

and also choose a nonnegative integer $k_2 = k_2(c)$ for which

$$\sum_{k=k_2+1}^{\infty} \exp\left(-\frac{c}{2} k^{1/(p+1)}\right) < \frac{\varepsilon}{4A}. \tag{36}$$

Take $k_0 = \max\{k_1, k_2\}$. For every $k \in \{0, 1, \ldots, k_0\}$ choose $n_k \in \mathbb{N}$ such that

$$\left|a_{kn}^{(n)} - a_k\right| < \frac{\varepsilon}{2(k_0+1)} \exp\left(ck^{1/(p+1)}\right) \quad \text{for each } n > n_k. \tag{37}$$

Take $m = \max\{n_k : k = 0, 1, \ldots, k_0\}$. Then from (32), (33), (35), (36), and (37) it follows that for each $n > m$ there holds

$$\begin{aligned}
\left\|f_{i_n} - f\right\|_{p,c} &= \sum_{k=0}^{\infty} \left|a_{kn}^{(n)} - a_k\right| \exp\left(-ck^{1/(p+1)}\right) \\
&= \sum_{k=0}^{k_0} \left|a_{kn}^{(n)} - a_k\right| \exp\left(-ck^{1/(p+1)}\right) \\
&\quad + \sum_{k=k_0+1}^{\infty} \left|a_{kn}^{(n)} - a_k\right| \exp\left(-ck^{1/(p+1)}\right) \\
&\le \frac{\varepsilon}{2(k_0+1)}(k_0+1) \\
&\quad + 2A\left(\sum_{k=k_0+1}^{\infty} \exp\left((c_k - c)k^{1/(p+1)}\right)\right) \\
&\le \frac{\varepsilon}{2} + 2A\left(\sum_{k=k_2+1}^{\infty} \exp\left(-\frac{c}{2}k^{1/(p+1)}\right)\right) \\
&\le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned} \tag{38}$$

Therefore, $f_{i_n} \to f$ in the space $(F^p, \|\cdot\|_{p,c})$. From this and the fact that $c > 0$ is arbitrary we conclude that $f_{i_n} \to f$ in F^p. Hence, E is a relatively compact subset of F^p. This completes the proof. □

Theorem 11. *Let E be a subset of N^p. E is a weakly bounded subset of N^p if and only if there is a constant $A > 0$ depending on E and a sequence $\{c_n\}$ of positive real numbers with $c_n \downarrow 0$ such that*

$$|a_n| \le A \exp\left(c_n n^{1/(p+1)}\right) \quad \text{for each } f(z) = \sum_{n=0}^{\infty} a_n z^n \in E. \tag{39}$$

Proof. The proof follows immediately from Theorem 7, the equivalence (i)⇔(iii) of Theorem 10, and the well known fact that in every locally convex topological vector space a set is bounded if and only if it is weakly bounded (see, e.g., [23, page 68]). □

3. The Dual Spaces of the Spaces F^p and N^p

Let X be a topological vector space over the field $\mathbb{C}$. Consider its *strong dual space* X^* which consists of all continuous linear functionals $f : X \to \mathbb{C}$ and is equipped with the usual strong topology $\beta(X^*, X)$. Recall that in the case when X is a locally convex space, the strong topology $\beta(X^*, X)$ on X^* coincides with the topology of uniform convergence on bounded subsets in X. In this case the topology $\beta(X^*, X)$ coincides with the topology of uniform convergence on bounded sets in X, that is, with the topology on X^* generated by the seminorms of the form

$$|||\varphi|||_B = \sup_{x\in B} |\varphi(x)|, \quad \varphi \in X^*, \tag{40}$$

where B runs over the family of all bounded sets in X (see, e.g., [24, § 21, page 21]).

The space X^* is a locally convex topological vector space, so one can consider its strong dual space $X^{**} := (X^*)^*$ which is called *strong bidual space* for X. More precisely, the strong bidual space X^{**} for X consists of all continuous linear functionals $h : X^* \to \mathbb{C}$ and is equipped with the strong topology $\beta(X^{**}, X^*)$. Furthermore, X is called *reflexive* if the evaluation map $J : X \to (X)^{**}$ defined as $J(x)(f) = f(x)$ ($x \in X$, $f \in X^*$) is surjective and continuous (in this case J is an isomorphism of topological vector spaces) (see [20, page 189]). Accordingly, X is a reflexive space if and only if X coincides with the continuous dual of its continuous dual space X^*, both as linear space and as topological space.

In particular, in the case of locally convex space F^p, the strong topology $\beta((F^p)^*, F^p)$ coincides with the topology on $(F^p)^* = S_p$ (see Theorem 7) generated by the family of seminorms of the form

$$|||\gamma|||_B = \sup_{f\in B} |\gamma(f)|, \quad \gamma \in (F^p)^* = S_p, \tag{41}$$

where B is an arbitrary bounded subset of F^p.

Furthermore, denote by $(N^p)_w^*$ the space S_p of complex sequences $\gamma = \{\gamma_n\}_n$ satisfying the growth condition (15), equipped with the topology of uniform convergence on weakly bounded subsets of N^p. (Let us recall that a subset $E \subset N^p$ is *weakly bounded* if it is bounded with respect to the weak topology on N^p described in Remark 9.) More precisely, this topology on the space S_p is defined by the family of seminorms of the form

$$|||\gamma|||'_E = \sup_{f\in E} |\gamma(f)|, \quad \gamma \in (N^p)^* = S_p, \tag{42}$$

where E is an arbitrary weakly bounded subset of N^p.

Notice that (see Theorems 5 and 7) for $\gamma = \{\gamma_n\}_{n=0}^{\infty} \in S_p$ and $f(z) = \sum_{n=0}^{\infty} a_n z^n \in F^p$ (or $f \in N^p$)

$$\gamma(f) = \sum_{n=0}^{\infty} a_n \gamma_n. \tag{43}$$

Then we have the following result which is analogous to Yanagihara's result [25, Theorem 3] concerning the spaces F^+ and N^+.

Theorem 12. *For each $p > 1$ $(F^p)^* = (N^p)_w^*$ both set theoretically and topologically.*

Proof. Proof follows immediately from Theorem 7, the equivalence (i)⇔(iii) of Theorems 10 and 11 in view of (41)–(43). □

A locally convex topological vector space is called barrelled if every closed, absolutely convex, absorbing set is a neighborhood of zero. It is known that every F-space is barreled (see [24, page 263 § 21, the assertion (3)]). The most important fact about barrelled spaces is that every pointwise bounded family of continuous linear functionals is equicontinuous. Furthermore, each barrelled space is a Mackey space (see, e.g., [20, pages 171–173]). Recall that a *Montel space* is a barrelled topological vector space in which every closed bounded set is compact. It is known that every Montel space is reflexive (e.g., see [24, page 369]).

Theorem 13. *F^p is a Montel space for each $p > 1$.*

Proof. As noticed above, every F-space is barreled (see [24, page 263 § 21, 5.(3)]). Furthermore, by Theorem 10, every closed bounded subset of F^p is compact. Therefore, F^p is a Montel space. □

Theorem 14. *The space F^p is reflexive. Hence, $(F^p)^*$ is reflexive.*

Proof. The first assertion follows from Theorem 13 and the fact that every Montel space is reflexive (see [24, page 369, the assertion (1)]). The second assertion follows from the first assertion and the fact that the strong dual of a reflexive F-space is also reflexive (see [20, page 303, the assertion (5)]). □

Theorem 15. *For each $p > 1$ $(F^p)^{**} = F^p$.*

Proof. The assertion follows immediately from Theorem 14. □

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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