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Research Article

A New Characterization of k-Uniformly Rotund Banach Space

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A new characterization of k-uniformly rotund Banach space with $1 < P < +\infty$ is given. Moreover, a corresponding result in the locally k-uniformly rotund Banach space with $1 < P < +\infty$ is given.

1. Introduction

In the geometric theory of Banach spaces the concept of uniform convexity plays a very significant role and is frequently used in functional analysis. The concept of a uniformly rotund (or uniformly convex) Banach space was first introduced by Clarkson [1] in 1936 and this class of Banach space is very interesting and has numerous applications (cf. [2–8]). In 1979, Sullivan [9] introduced the k-uniformly rotund spaces as a generalization of uniformly rotund Banach spaces. Indeed, the 1-uniformly rotund Banach spaces coincide with usually uniformly rotund Banach spaces.

The purpose of this paper is to give a character inequality of k-uniformly rotund Banach space (or locally k-uniformly rotund Banach space) with $1 < p < +\infty$. Throughout the sequel, the symbol X denotes a real Banach space and X^* denotes its dual space. $B(X) = \{x : x \in X, \|x\| \leq 1\}$ and $S(X) = \{x : x \in X, \|x\| = 1\}$ denote, respectively, the unit ball and the unit sphere in X. For $x_1, x_2, \dots, x_{k+1} \in S(X)$, the k-dimensional volume enclosed by $x_1, x_2, \dots, x_{k+1}$ is given by

$$A(x_1, x_2, \dots, x_{k+1}) = \sup\left\{\begin{vmatrix} 1 & 1 & \cdots & 1 \\ f_1(x_1) & f_1(x_2) & \cdots & f_1(x_{k+1}) \\ \cdots & \cdots & \cdots & \cdots \\ f_k(x_1) & f_k(x_2) & \cdots & f_k(x_{k+1}) \end{vmatrix}; \; f_i \in S(X^*),\ i = 1, \dots, k\right\}. \quad (1)$$

A Banach space X is said to be k-uniformly rotund [9] if, for any $\epsilon > 0$, there is a $\delta(\epsilon) > 0$, such that, for $x_1, x_2, \dots, x_{k+1} \in S(X)$, if $\|x_1 + x_2 + \cdots + x_{k+1}\| > (k+1) - \delta$ then $A(x_1, x_2, \dots, x_{k+1}) < \epsilon$.

A Banach space X is said to be locally k-uniformly rotund [9] if, for $\forall \epsilon > 0$, $x \in S(X)$, there is a $\delta(\epsilon, x) > 0$, such that, for $x_1, \dots, x_k \in S(X)$, if $\|x + x_1 + \cdots + x_k\| > (k+1) - \delta$ then $A(x, x_1, \dots, x_k) < \epsilon$.

2. A New Characterization of k-Uniformly Rotund Banach Spaces

Theorem 1. *Let $1 < p < +\infty$, let X be a Banach space, and let M be an arbitrary bounded subset of X. Then, X is k-uniformly rotund space if and only if, for any $\epsilon > 0$, there exists $0 < \delta(\epsilon, p) < 1$, such that the inequality*

$$\left\|\frac{x + x_1 + \cdots + x_k}{k+1}\right\|^p \leq (1 - \delta(\epsilon, p)) \frac{\|x\|^p + \|x_1\|^p + \cdots + \|x_k\|^p}{k+1} \quad (2)$$

holds for all $x \in M$ *and* $x_1, \ldots, x_k \in X$ *with* $A(x, x_1, \ldots, x_k) \geq \epsilon$.

In order to prove Theorem 1, we give three lemmas.

Lemma 2 (Yu [10]). *X is k-uniformly rotund space if and only if for any* $k+1$ *sequences*

$$\{x_1^{(n)}\}_{n=1}^{\infty}, \{x_2^{(n)}\}_{n=1}^{\infty}, \ldots, \{x_{k+1}^{(n)}\}_{n=1}^{\infty} \subset X, \quad (3)$$

if $\|x_i^{(n)}\| \to a$ $(i = 1, 2, \ldots, k+1)$, $\|x_1^{(n)} + x_2^{(n)} + \cdots + x_{k+1}^{(n)}\| \to (k+1)a$, $(n \to \infty)$, *then*

$$A\left(x_1^{(n)}, x_2^{(n)}, \ldots, x_{k+1}^{(n)}\right) \longrightarrow 0, \quad (n \longrightarrow \infty). \quad (4)$$

For the sake of completeness of this paper, here we present the proof of Lemma 2.

The sufficiency of Lemma 2 is clear.

The Proof of Necessity. Without loss of generality, we may assume that $a = 1$.

Suppose that $\{x_i^{(n)}\}_{n=1}^{\infty} \subset X$ $(i = 1, 2, \ldots, k+1)$, satisfying the conditions given in Lemma 2. Then, for each $\delta > 0$, by the assumption that $\|x_1^{(n)} + x_2^{(n)} + \cdots + x_{k+1}^{(n)}\| \to k+1$ $(n \to \infty)$, there exists $N_0 > 0$, such that the inequality

$$\left\|x_1^{(n)} + x_2^{(n)} + \cdots + x_{k+1}^{(n)}\right\| > k + 1 - \frac{\delta}{2} \quad (5)$$

holds for all $n > N_0$.

On the other hand, since $\|x_i^{(n)}\| \to 1 (i = 1, 2, \ldots, k+1, n \to \infty)$, so there exists $N_1 \geq N_0$, such that the inequality

$$\left|\left\|x_i^{(n)}\right\| - 1\right| < \frac{\delta}{2(k+1)}, \quad (i = 1, 2, \ldots, k+1) \quad (6)$$

holds for all $n > N_1$.

Therefore, by letting $y_i^{(n)} = x_i^{(n)}/\|x_i^{(n)}\|$, $i = 1, 2, \ldots, k+1$, $n = 1, 2, \ldots$, we can deduce that

$$\begin{aligned}
&\left\|y_1^{(n)} + y_2^{(n)} + \cdots + y_{k+1}^{(n)}\right\| \\
&\quad = \left\|x_1^{(n)} + \cdots + x_{k+1}^{(n)} + y_1^{(n)} - x_1^{(n)} + \cdots + y_{k+1}^{(n)} - x_{k+1}^{(n)}\right\| \\
&\quad \geq \left\|x_1^{(n)} + \cdots + x_{k+1}^{(n)}\right\| - \sum_{i=1}^{k+1} \left\| \frac{x_i^{(n)}}{\left\|x_i^{(n)}\right\|} - x_i^{(n)} \right\| \\
&\quad > k + 1 - \frac{\delta}{2} - \frac{\delta(k+1)}{\delta(k+2)} \\
&\quad = k + 1 - \delta.
\end{aligned} \quad (7)$$

By the assumption that X is k-uniformly rotund space, we may take $\delta = \delta(\epsilon/2)$ for any $\epsilon > 0$. Therefore, by the above proof, there exists an N_1 corresponding to $\delta = \delta(\epsilon/2)$ such that the inequality

$$A\left(y_1^{(n)}, y_2^{(n)}, \ldots, y_{k+1}^{(n)}\right) < \frac{\epsilon}{2} \quad (8)$$

holds for all $n > N_1$.

Furthermore, by using inequality (8), we easily obtain the desired result that

$$A\left(x_1^{(n)}, x_2^{(n)}, \ldots, x_{k+1}^{(n)}\right) \longrightarrow 0, \quad (n \longrightarrow \infty). \quad (9)$$

Lemma 3. *Let* $1 < p < +\infty$, *then one has*

$$\left(\frac{1 + t_1 + \cdots + t_k}{k+1}\right)^p \leq \frac{1 + t_1^p + \cdots + t_k^p}{k+1}, \quad (10)$$

where $t_1, t_2, \ldots, t_k \geq 0$, *and the sign of equality holds if and only if* $t_1 = t_2 = \cdots = t_k = 1$.

Proof. (1°) When $k = 1$, we construct a function $f(t_1) = ((1 + t_1)/2)^p - (1 + t_1^p)/2$; then

$$f'(t_1) = \frac{p}{2}\left[\left(\frac{1+t_1}{2}\right)^{p-1} - t_1^{p-1}\right]. \quad (11)$$

Obviously,

$$\begin{aligned}
&f'(t_1) = 0 \quad \text{if } t_1 = 1 \\
&f'(t_1) > 0 \quad \text{if } t_1 < 1 \\
&f'(t_1) < 0 \quad \text{if } t_1 > 1.
\end{aligned} \quad (12)$$

It is easy to see that the function $f(t_1)$ attains its maximum value at point $t_1 = 1$ and $f(1) = 0$. Hence $f(t_1) \leq f(1) = 0$; that is,

$$\left(\frac{1+t_1}{2}\right)^p \leq \frac{1+t_1^p}{2}. \quad (13)$$

And the sign of equality holds if and only if $t_1 = 1$.

(2°) Suppose the conclusion of Lemma 3 is true when $k = n - 1$; that is, the inequality

$$\left(\frac{1 + t_1 + \cdots + t_{n-1}}{n}\right)^p \leq \frac{1 + t_1^p + \cdots + t_{n-1}^p}{n} \quad (14)$$

holds and the sign of equality holds if and only if $t_1 = t_2 = \cdots = t_{n-1} = 1$.

(3°) When $k = n$, we construct a multivariate function

$$f(t_1, t_2, \ldots, t_n) = \left(\frac{1 + t_1 + \cdots + t_n}{n+1}\right)^p - \frac{1 + t_1^p + \cdots + t_n^p}{n+1}; \tag{15}$$

then

$$\frac{\partial f}{\partial t_n} = \frac{p}{n+1}\left\{\left(\frac{1 + t_1 + \cdots + t_n}{n+1}\right)^{p-1} - t_n^{p-1}\right\}. \tag{16}$$

Now, let us fix variables $t_1, t_2, \ldots, t_{n-1}$. Then the function $f(t_1, t_2, \ldots, t_n)$ attains its maximum value at point $t_n = (1 + t_1 + \cdots + t_{n-1})/n$. Hence

$$\begin{aligned} f(t_1, t_2, \ldots, t_n) &\le \left(\frac{1 + t_1 + \cdots + t_{n-1} + (1 + t_1 + \cdots + t_{n-1})/n}{n+1}\right)^p \\ &\quad - \frac{1 + t_1^p + \cdots + t_{n-1}^p + ((1 + t_1 + \cdots + t_{n-1})/n)^p}{n+1} \\ &= \frac{n}{n+1}\left\{\left(\frac{1 + t_1 + \cdots + t_{n-1}}{n}\right)^p - \frac{1 + t_1^p + \cdots + t_{n-1}^p}{n}\right\} \\ &\le 0. \end{aligned} \tag{17}$$

This shows that the inequality

$$\left(\frac{1 + t_1 + \cdots + t_n}{n+1}\right)^p \le \frac{1 + t_1^p + \cdots + t_n^p}{n+1} \tag{18}$$

holds and the sign of equality holds if and only if $t_1 = t_2 = \cdots = t_n = 1$.

Combining (1°), (2°), and (3°), we have

$$\left(\frac{1 + t_1 + \cdots + t_k}{k+1}\right)^p \le \frac{1 + t_1^p + \cdots + t_k^p}{k+1}, \tag{19}$$

and the sign of equality holds if and only if $t_1 = t_2 = \cdots = t_k = 1$. □

Lemma 4. *Let $1 < p < +\infty$, then one has*

$$\begin{aligned} &\frac{((1 + t_1 + \cdots + t_k)/(k+1))^p}{(1 + t_1^p + \cdots + t_k^p)/(k+1)} \\ &\quad \le \frac{1}{(k+1)^{p-1}}\left(k - 1 + \left(\frac{(1+t_1)^p}{1+t_1^p}\right)^{1/(p-1)}\right)^{p-1}, \end{aligned} \tag{20}$$

where $t_1, t_2, \ldots, t_k \ge 0$.

Proof. (1°) When $k = 1$, the conclusion of Lemma 4 is obvious. When $k = 2$, we construct a function

$$f(t_1, t_2) = \frac{((1 + t_1 + t_2)/3)^p}{(1 + t_1^p + t_2^p)/3} = \frac{1}{3^{p-1}}\frac{(1 + t_1 + t_2)^p}{1 + t_1^p + t_2^p}; \tag{21}$$

then

$$\frac{\partial f}{\partial t_2} = \frac{1}{3^{p-1}}\frac{p(1 + t_1 + t_2)^{p-1}\left\{1 + t_1^p - t_2^{p-1} - t_2^{p-1}t_1\right\}}{(1 + t_1^p + t_2^p)^2}. \tag{22}$$

It is easy to see that the function $f(t_1, t_2)$ attains its maximum value at point $t_2 = ((1 + t_1^p)/(1 + t_1))^{1/(p-1)}$. Hence

$$\begin{aligned} f(t_1, t_2) &\le \frac{1}{3^{p-1}}\frac{\left(1 + t_1 + \left((1 + t_1^p)/(1 + t_1)\right)^{1/(p-1)}\right)^p}{1 + t_1^p + \left((1 + t_1^p)/(1 + t_1)\right)^{p/(p-1)}} \\ &= \frac{1}{3^{p-1}}\left(1 + \left(\frac{(1+t_1)^p}{1+t_1^p}\right)^{1/(p-1)}\right)^{p-1}. \end{aligned} \tag{23}$$

(2°) Suppose the conclusion of Lemma 4 is true when $k = n - 1$; that is, we have

$$\begin{aligned} &\frac{((1 + t_1 + \cdots + t_{n-1})/n)^p}{(1 + t_1^p + \cdots + t_{n-1}^p)/n} \\ &\quad \le \frac{1}{n^{p+1}}\left(n - 2 + \left(\frac{(1+t_1)^p}{1+t_1^p}\right)^{1/(p-1)}\right)^{p-1}. \end{aligned} \tag{24}$$

(3°) When $k = n$, we construct a multivariate function

$$\begin{aligned} f(t_1, t_2, \ldots, t_n) &= \frac{((1 + t_1 + \cdots + t_n)/(n+1))^p}{(1 + t_1^p + \cdots + t_n^p)/(n+1)} \\ &= \frac{1}{(n+1)^{p-1}}\frac{(1 + t_1 + \cdots + t_n)^p}{1 + t_1^p + \cdots + t_n^p}; \end{aligned} \tag{25}$$

then

$$\frac{\partial f}{\partial t_n} = \frac{1}{(n+1)^{p-1}}\frac{p(1 + t_1 + \cdots + t_n)^{p-1}\left\{1 + t_1^p + \cdots + t_{n-1}^p - t_n^p(1 + t_1 + \cdots + t_{n-1})\right\}}{(1 + t_1^p + \cdots + t_n^p)^2}. \tag{26}$$

Now, let us fix variables $t_1, t_2, \ldots, t_{n-1}$. Then the function $f(t_1, t_2, \ldots, t_n)$ attains its maximum value at point $t_n = ((1 + t_1^p + \cdots + t_{n-1}^p)/(1 + t_1 + \cdots + t_{n-1}))^{1/(p-1)}$. Hence,

$$f(t_1, t_2, \ldots, t_n) \le \frac{1}{(n+1)^{p-1}} \frac{\left(1 + t_1 + \cdots + t_{n-1} + \left(\left(1 + t_1^p + \cdots + t_{n-1}^p\right) / \left(1 + t_1 + \cdots + t_{n-1}\right)\right)^{1/(p-1)}\right)^p}{1 + t_1^p + \cdots + t_{n-1}^p + \left(\left(1 + t_1^p + \cdots + t_{n-1}^p\right) / \left(1 + t_1 + \cdots + t_{n-1}\right)\right)^{p/(p-1)}}$$
$$= \frac{1}{(n+1)^{p-1}} \left(1 + \left(\frac{(1 + t_1 + \cdots + t_{n-1})^p}{1 + t_1^p + \cdots + t_{n-1}^p}\right)^{1/(p-1)}\right)^{p-1}. \quad (27)$$

By using inequality (24) we have

$$\frac{((1 + t_1 + \cdots + t_n) / (n+1))^p}{\left(1 + t_1^p + \cdots + t_n^p\right) / (n+1)}$$
$$\le \frac{1}{(n+1)^{p-1}} \left(1 + \left(\frac{(1 + t_1 + \cdots + t_{n-1})^p}{1 + t_1^p + \cdots + t_{n-1}^p}\right)^{1/(p-1)}\right)^{p-1}$$
$$\le \frac{1}{(n+1)^{p-1}} \left(n - 1 + \left(\frac{(1 + t_1)^p}{1 + t_1^p}\right)^{1/(p-1)}\right)^{p-1}. \quad (28)$$

Combining (1°), (2°), and (3°), we have

$$\frac{((1 + t_1 + \cdots + t_k) / (k+1))^p}{\left(1 + t_1^p + \cdots + t_k^p\right) / (k+1)}$$
$$\le \frac{1}{(k+1)^{p-1}} \left(k - 1 + \left(\frac{(1 + t_1)^p}{1 + t_1^p}\right)^{1/(p-1)}\right)^{p-1}. \quad (29)$$

□

Proof of Theorem 1.

Proof of Sufficiency. Suppose that, for $\forall \epsilon > 0$, there is a $0 < \delta_1 = \delta_1(\epsilon, p) < 1$, such that the inequality

$$\left\| \frac{x_1 + x_2 + \cdots + x_{k+1}}{k+1} \right\|^p \le 1 - \delta_1 \quad (30)$$

holds for all $x_1, x_2, \ldots, x_{k+1} \in S(X)$ with $A(x_1, x_2, \ldots, x_{k+1}) \ge \epsilon$.

Let $\delta_2 = 1 - (1 - \delta_1)^{1/p} > 0$ and $\delta = (k+1)\delta_2$; then $\|x_1 + x_2 + \cdots + x_{k+1}\| \le 1 - \delta$.

By the definition of k-uniformly rotund space, we know that X is k-uniformly rotund space.

Proof of Necessity. Suppose inequality (2) is not true. Then there exist $b \in R^+$, $\epsilon_0 > 0$, such that, for $\forall 1/n > 0$, there exist $\{x_1^{(n)}\}_{n=1}^{\infty}, \{x_2^{(n)}\}_{n=1}^{\infty}, \ldots, \{x_{k+1}^{(n)}\}_{n=1}^{\infty} \subset X$, satisfying $\|x_1^{(n)}\| \le b$. When $A(x_1^{(n)}, x_2^{(n)}, \ldots, x_{k+1}^{(n)}) \ge \epsilon_0$, we have

$$\left\| \frac{x_1^{(n)} + x_2^{(n)} + \cdots + x_{k+1}^{(n)}}{k+1} \right\|^p$$
$$> \left(1 - \frac{1}{n}\right) \frac{\left\|x_1^{(n)}\right\|^p + \left\|x_2^{(n)}\right\|^p + \cdots + \left\|x_{k+1}^{(n)}\right\|^p}{k+1}. \quad (31)$$

Take $u_1^{(n)} = x_1^{(n)}/\|x_1^{(n)}\|$, $u_2^{(n)} = x_2^{(n)}/\|x_1^{(n)}\|, \ldots, u_{k+1}^{(n)} = x_{k+1}^{(n)}/\|x_1^{(n)}\|$. Then

$$\left\|u_1^{(n)}\right\| = 1,$$
$$A\left(u_1^{(n)}, u_2^{(n)}, \ldots, u_{k+1}^{(n)}\right) = \frac{1}{\left\|x_1^{(n)}\right\|^k} A\left(x_1^{(n)}, x_2^{(n)}, \ldots, x_{k+1}^{(n)}\right)$$
$$\ge \frac{\epsilon_0}{b^k}. \quad (32)$$

By Lemma 3 we know that

$$1 - \frac{1}{n} < \frac{\left\|\left(u_1^{(n)} + u_2^{(n)} + \cdots + u_{k+1}^{(n)}\right) / (k+1)\right\|^p}{\left(\left\|u_1^{(n)}\right\|^p + \left\|u_2^{(n)}\right\|^p + \cdots + \left\|u_{k+1}^{(n)}\right\|^p\right) / (k+1)}$$
$$\le \frac{\left(\left(1 + \left\|u_2^{(n)}\right\| + \cdots + \left\|u_{k+1}^{(n)}\right\|\right) / (k+1)\right)^p}{\left(1 + \left\|u_2^{(n)}\right\|^p + \cdots + \left\|u_{k+1}^{(n)}\right\|^p\right) / (k+1)} \le 1. \quad (33)$$

It follows that

$$\lim_{n \to \infty} \frac{\left\|\left(u_1^{(n)} + u_2^{(n)} + \cdots + u_{k+1}^{(n)}\right) / (k+1)\right\|^p}{\left(\left\|u_1^{(n)}\right\|^p + \left\|u_2^{(n)}\right\|^p + \cdots + \left\|u_{k+1}^{(n)}\right\|^p\right) / (k+1)} = 1. \quad (34)$$

Now we prove that

$$\lim_{n \to \infty} \left\|u_i^{(n)}\right\| = 1, \quad i = 2, 3, \ldots, k+1. \quad (35)$$

If (35) does not hold, then there exists subsequence $\{u_j^{(n_i)}\}$ of $\{u_j^{(n)}\}$ satisfying

$$\left\|u_j^{(n_i)}\right\| \le \alpha_{j-1} < 1 \quad \text{or} \quad \left\|u_j^{(n_i)}\right\| \ge \alpha_{j-1} > 1, \quad j = 2, \dots, k+1. \tag{36}$$

Hence, by Lemma 4 we have

$$\begin{aligned} &\frac{\left\|\left(u_1^{(n_i)} + u_2^{(n_i)} + \cdots + u_{k+1}^{(n_i)}\right)/(k+1)\right\|^p}{\left(\left\|u_1^{(n_i)}\right\|^p + \left\|u_2^{(n_i)}\right\|^p + \cdots + \left\|u_{k+1}^{(n_i)}\right\|^p\right)/(k+1)} \\ &\le \frac{1}{(k+1)^{p-1}} \frac{\left(1 + \left\|u_2^{(n_i)}\right\| + \cdots + \left\|u_{k+1}^{(n_i)}\right\|\right)^p}{1 + \left\|u_2^{(n_i)}\right\|^p + \cdots + \left\|u_{k+1}^{(n_i)}\right\|^p} \\ &\le \frac{1}{(k+1)^{p-1}} \left(k - 1 + \left(\frac{\left(1 + \left\|u_2^{(n_i)}\right\|\right)^p}{1 + \left\|u_2^{(n_i)}\right\|^p}\right)^{1/(p-1)}\right)^{p-1}. \end{aligned} \tag{37}$$

Considering a function $(1+t)^p/(1+t^p)$ and noticing that

$$\begin{aligned} \left(\frac{(1+t)^p}{1+t^p}\right)'(t) &= \frac{p(1+t)^{p-1}(1+t^p) - pt^{p-1}(1+t)^p}{(1+t^p)^2} \\ &= \frac{p(1+t)^{p-1}\left(1 - t^{p-1}\right)}{(1+t^p)^2}, \end{aligned} \tag{38}$$

we know that $(1+t)^p/(1+t^p)$ is strictly increasing (decreasing) function when $t < 1$ ($t > 1$) and attains its maximum value at point $t = 1$.

Hence

$$\begin{aligned} &\frac{\left\|\left(u_1^{(n_i)} + u_2^{(n_i)} + \cdots + u_{k+1}^{(n_i)}\right)/(k+1)\right\|^p}{\left(\left\|u_1^{(n_i)}\right\|^p + \left\|u_2^{(n_i)}\right\|^p + \cdots + \left\|u_{k+1}^{(n_i)}\right\|^p\right)/(k+1)} \\ &\le \frac{1}{(k+1)^{p-1}} \left(k - 1 + \left(\frac{\left(1 + \left\|u_2^{(n_i)}\right\|\right)^p}{1 + \left\|u_2^{(n_i)}\right\|^p}\right)^{1/(p-1)}\right)^{p-1} \\ &\le \frac{1}{(k+1)^{p-1}} \left(k - 1 + \left(\frac{(1+\alpha_1)^p}{1+\alpha_1{}^p}\right)^{1/(p-1)}\right)^{p-1} \\ &< 1; \end{aligned} \tag{39}$$

this contradicts (34), so $\lim_{n\to\infty}\|u_2^{(n)}\| = 1$.

Similarly, we can deduce that $\lim_{n\to\infty}\|u_3^{(n)}\| = 1, \dots, \lim_{n\to\infty}\|u_{k+1}^{(n)}\| = 1$. It follows that

$$\lim_{n\to\infty} \left\|\frac{u_1^{(n)} + u_2^{(n)} + \cdots + u_{k+1}^{(n)}}{k+1}\right\| = 1. \tag{40}$$

Let $z_n = u_{k+1}^{(n)}/\|u_{k+1}^{(n)}\|$; then

$$\left\|z_n - u_{k+1}^{(n)}\right\| = \left\|\frac{u_{k+1}^{(n)}}{\left\|u_{k+1}^{(n)}\right\|} - u_{k+1}^{(n)}\right\| \longrightarrow 0, \quad (n \longrightarrow \infty),$$

$$\begin{aligned} 1 &= \lim_{n\to\infty} \left\|\frac{u_1^{(n)} + u_2^{(n)} + \cdots + u_{k+1}^{(n)}}{k+1}\right\| \\ &\le \lim_{n\to\infty} \left(\left\|\frac{u_1^{(n)} + u_2^{(n)} + \cdots + z_n}{k+1}\right\| + \left\|\frac{u_{k+1}^{(n)} - z_n}{k+1}\right\|\right) \\ &= \lim_{n\to\infty} \left\|\frac{u_1^{(n)} + u_2^{(n)} + \cdots + z_n}{k+1}\right\| \\ &\le 1. \end{aligned} \tag{41}$$

This means that

$$\lim_{n\to\infty} \left\|\frac{u_1^{(n)} + u_2^{(n)} + \cdots + z_n}{k+1}\right\| = 1. \tag{42}$$

But

$$\begin{aligned} \lim_{n\to\infty} A\left(u_1^{(n)}, u_2^{(n)}, \dots, z_n\right) &= \lim_{n\to\infty} A\left(u_1^{(n)}, u_2^{(n)}, \dots, u_{k+1}^{(n)}\right) \\ &\ge \frac{\epsilon_o}{b^k}. \end{aligned} \tag{43}$$

This contradicts that X is k-uniformly rotund space from Lemma 2. □

Theorem 5. *Let $1 < p < +\infty$, let X be a Banach space, and let M be an arbitrary bounded subset of X. Then, X is locally k-uniformly rotund space if and only if, for any $\epsilon > 0$ and $x_1 \in M$, there exists $0 < \delta(\epsilon, p, x_1) < 1$, such that the inequality*

$$\begin{aligned} &\left\|\frac{x_1 + x_2 + \cdots + x_{k+1}}{k+1}\right\|^p \\ &\le (1 - \delta(\epsilon, p, x_1)) \frac{\|x_1\|^p + \|x_2\|^p + \cdots + \|x_{k+1}\|^p}{k+1} \end{aligned} \tag{44}$$

holds for all $x_2, x_3, \dots, x_{k+1} \in X$ with $A(x_1, x_2, \dots, x_{k+1}) \ge \epsilon$.

The proof of Theorem 5 is greatly similar to the proof of Theorem 1.

In particular, considering the special cases of Theorems 1 and 5 when $k = 1$, we give a new characterization of uniformly rotund (resp., locally uniformly rotund) Banach space; that is, we have the following two corollaries.

Corollary 6. *Let $1 < p < +\infty$, let X be a Banach space, and let M be an arbitrary bounded subset of X. Then, X is uniformly rotund space if and only if, for any $\epsilon > 0$, there exists $0 < \delta(\epsilon, p) < 1$, such that the inequality*

$$\left\| \frac{x+y}{2} \right\|^p \leq \left(1 - \delta\left(\epsilon, p\right)\right) \frac{\|x\|^p + \|y\|^p}{2} \tag{45}$$

holds for all $x \in M$ and $y \in X$ with $\|x - y\| \geq \epsilon$.

Corollary 7. *Let $1 < p < +\infty$, let X be a Banach space, and let M be an arbitrary bounded subset of X. Then, X is locally uniformly rotund space if and only if, for any $\epsilon > 0$ and $x \in M$, there exists $0 < \delta(\epsilon, p, x) < 1$, such that the inequality*

$$\left\| \frac{x+y}{2} \right\|^p \leq \left(1 - \delta\left(\epsilon, p, x\right)\right) \frac{\|x\|^p + \|y\|^p}{2} \tag{46}$$

holds for all $y \in X$ with $\|x - y\| \geq \epsilon$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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