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Research Article

Boundedness of θ-Type Calderón–Zygmund Operators and Commutators in the Generalized Weighted Morrey Spaces

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We first introduce some new Morrey type spaces containing generalized Morrey space and weighted Morrey space as special cases. Then, we discuss the strong-type and weak-type estimates for a class of Calderón–Zygmund type operators T_θ in these new Morrey type spaces. Furthermore, the strong-type estimate and endpoint estimate of commutators $[b, T_\theta]$ formed by b and T_θ are established. Also, we study related problems about two-weight, weak-type inequalities for T_θ and $[b, T_\theta]$ in the Morrey type spaces and give partial results.

1. Introduction

Calderón–Zygmund singular integral operators and their generalizations on the Euclidean space $\mathbb{R}^n$ have been extensively studied (see [1–5], for instance). In particular, Yabuta [5] introduced certain θ-type Calderón–Zygmund operators to facilitate his study of certain classes of pseudodifferential operators. Following the terminology of Yabuta [5], we introduce the so-called θ-type Calderón–Zygmund operators.

Definition 1. Let θ be a nonnegative, nondecreasing function on $\mathbb{R}^+ = (0, +\infty)$ with

$$\int_0^1 \frac{\theta(t)}{t} dt < \infty. \tag{1}$$

A measurable function $K(\cdot, \cdot)$ on $\mathbb{R}^n \times \mathbb{R}^n \setminus \{(x, x) : x \in \mathbb{R}^n\}$ is said to be a θ-type kernel if it satisfies

$$\text{(i)}\ |K(x, y)| \le \frac{C}{|x - y|^n}, \quad \text{for any } x \ne y; \tag{2}$$

$$\begin{aligned}\text{(ii)}\ & |K(x, y) - K(z, y)| + |K(y, x) - K(y, z)| \\ & \le \frac{C}{|x - y|^n} \cdot \theta\left(\frac{|x - z|}{|x - y|}\right), \quad \text{for } |x - z| < \frac{|x - y|}{2}.\end{aligned} \tag{3}$$

Definition 2. Let T_θ be a linear operator from $\mathcal{S}(\mathbb{R}^n)$ into its dual $\mathcal{S}'(\mathbb{R}^n)$. One can say that T_θ is a θ-type Calderón–Zygmund operator if

(1) T_θ can be extended to be a bounded linear operator on $L^2(\mathbb{R}^n)$;

(2) there is a θ-type kernel $K(x, y)$ such that

$$T_\theta f(x) := \int_{\mathbb{R}^n} K(x, y) f(y)\, dy \tag{4}$$

for all $f \in C_0^\infty(\mathbb{R}^n)$ and for all $x \notin \operatorname{supp} f$, where $C_0^\infty(\mathbb{R}^n)$ is the space consisting of all infinitely differentiable functions on $\mathbb{R}^n$ with compact supports.

Note that the classical Calderón–Zygmund operator with standard kernel (see [1, 2]) is a special case of θ-type operator T_θ when $\theta(t) = t^\delta$ with $0 < \delta \le 1$.

Definition 3. Given a locally integrable function b defined on $\mathbb{R}^n$ and given a θ-type Calderón–Zygmund operator T_θ, the linear commutator $[b, T_\theta]$ is defined for smooth, compactly supported functions f as

$$\begin{aligned}[b, T_\theta] f(x) &:= b(x) \cdot T_\theta f(x) - T_\theta(b \cdot f)(x) \\ &= \int_{\mathbb{R}^n} [b(x) - b(y)] K(x, y) f(y)\, dy.\end{aligned} \tag{5}$$

Throughout the paper, let us suppose that θ is a nonnegative, nondecreasing function on $\mathbb{R}^+ = (0, +\infty)$ satisfying condition (1). Let us give the following weighted result of T_θ obtained by Quek and Yang in [6].

Theorem 4 (see [6]). *Let $1 \le p < \infty$ and $w \in A_p$. Then, the θ-type Calderón–Zygmund operator T_θ is bounded on $L^p_w(\mathbb{R}^n)$ for $p > 1$ and bounded from $L^1_w(\mathbb{R}^n)$ into $WL^1_w(\mathbb{R}^n)$ for $p = 1$.*

Since linear commutator has a greater degree of singularity than the corresponding θ-type Calderón–Zygmund operator, we need a slightly stronger version of condition (8) given below. The following weighted endpoint estimate for commutator $[b, T_\theta]$ of the θ-type Calderón–Zygmund operator was established in [7] under a stronger version of condition (8) assumed on θ, if $b \in \mathrm{BMO}(\mathbb{R}^n)$ (for the unweighted case, see [8]).

Let us now recall the definition of the space of $\mathrm{BMO}(\mathbb{R}^n)$ (see [9]). $\mathrm{BMO}(\mathbb{R}^n)$ is the Banach function space modulo constants with the norm $\|\cdot\|_*$ defined by

$$\|b\|_* := \sup_B \frac{1}{|B|} \int_B |b(x) - b_B|\, dx < \infty, \tag{6}$$

where the supremum is taken over all balls B in $\mathbb{R}^n$ and b_B stands for the mean value of b over B; that is,

$$b_B := \frac{1}{|B|} \int_B b(y)\, dy. \tag{7}$$

Theorem 5 (see [7]). *Let θ be a nonnegative, nondecreasing function on $\mathbb{R}^+ = (0, +\infty)$ with*

$$\int_0^1 \frac{\theta(t) \cdot |\log t|}{t} dt < \infty, \tag{8}$$

and let $w \in A_1$ and $b \in BMO(\mathbb{R}^n)$. Then, for all $\sigma > 0$, there is a constant $C > 0$ independent of f and $\sigma > 0$ such that

$$\begin{aligned} & w(\{x \in \mathbb{R}^n : |[b, T_\theta](f)(x)| > \sigma\}) \\ & \quad \le C \int_{\mathbb{R}^n} \Phi\left(\frac{|f(x)|}{\sigma}\right) \cdot w(x)\, dx, \end{aligned} \tag{9}$$

where $\Phi(t) = t \cdot (1 + \log^+ t)$ and $\log^+ t = \max\{\log t, 0\}$.

On the other hand, the classical Morrey space was originally introduced by Morrey in [10] to study the local behavior of solutions to second-order elliptic partial differential equations. Since then, this space played an important role in studying the regularity of solutions to partial differential equations. In [11], Mizuhara introduced the generalized Morrey space $\mathscr{L}^{p,\Psi}$ which was later extended and studied by many authors. In [12], Komori and Shirai defined a version of the weighted Morrey space $\mathscr{L}^{p,\kappa}(w)$ which is a natural generalization of the weighted Lebesgue space. Let T_θ be the θ-type Calderón–Zygmund operator, and let $[b, T_\theta]$ be its linear commutator. The main purpose of this paper is twofold. We first define a new kind of Morrey type spaces $\mathscr{M}^{p,\Psi}(w)$ containing generalized Morrey space $\mathscr{L}^{p,\Psi}$ and weighted Morrey space $\mathscr{L}^{p,\kappa}(w)$ as special cases, and then we will establish the weighted strong type and endpoint estimates for T_θ and $[b, T_\theta]$ in these Morrey type spaces $\mathscr{M}^{p,\Psi}(w)$ for all $1 \le p < \infty$ and $w \in A_p$. In addition, we will discuss two-weight, weak-type norm inequalities for T_θ and $[b, T_\theta]$ in $\mathscr{M}^{p,\Psi}(w)$ and give some partial results.

Throughout this paper, C will denote a positive constant whose value may change at each appearance. We also use $A \approx B$ to denote the equivalence of A and B; that is, there exist two positive constants C_1 and C_2 independent of A and B such that $C_1 A \le B \le C_2 A$.

2. Statements of the Main Results

2.1. Notation and Preliminaries. Let $\mathbb{R}^n$ be the n-dimensional Euclidean space of points $x = (x_1, x_2, \dots, x_n)$ with norm $|x| = (\sum_{i=1}^n x_i^2)^{1/2}$. For $x_0 \in \mathbb{R}^n$ and $r > 0$, let $B(x_0, r) = \{x \in \mathbb{R}^n : |x - x_0| < r\}$ denote the open ball centered at x_0 of radius r, $B(x_0, r)^c$ denote its complement, and $|B(x_0, r)|$ be the Lebesgue measure of the ball $B(x_0, r)$. A weight w is a nonnegative locally integrable function on $\mathbb{R}^n$ that takes values in $(0, +\infty)$ almost everywhere. A weight w is said to belong to Muckenhoupt's class A_p for $1 < p < \infty$, if there exists a constant $C > 0$ such that

$$\left(\frac{1}{|B|} \int_B w(x)\, dx\right)^{1/p} \left(\frac{1}{|B|} \int_B w(x)^{-p'/p}\, dx\right)^{1/p'} \le C \tag{10}$$

for every ball $B \subset \mathbb{R}^n$, where p' is the dual of p such that $1/p + 1/p' = 1$. The class A_1 is defined replacing the inequality above by

$$\frac{1}{|B|} \int_B w(x)\, dx \le C \cdot \operatorname*{ess\,inf}_{x \in B} w(x) \tag{11}$$

for every ball $B \subset \mathbb{R}^n$. We also define $A_\infty = \bigcup_{1 \le p < \infty} A_p$. Given a ball B and $\lambda > 0$, λB will denote the ball with the same center as B whose radius is λ times that of B. Given a Lebesgue measurable set E and a weight function w, we denote the characteristic function of E by χ_E, the Lebesgue measure of E by $|E|$, and the weighted measure of E by $w(E)$, where $w(E) = \int_E w(x) dx$. It is well known that if $w \in A_p$ with $1 \le p < \infty$ (or $w \in A_\infty$), then w satisfies the doubling condition; that is, for any ball B, there exists an absolute constant $C > 0$ such that (see [2])

$$w(2B) \le C w(B). \tag{12}$$

Moreover, if $w \in A_\infty$, then for any ball B and any measurable subset E of a ball B, there exists a number $\delta > 0$ independent of E and B such that (see [2])

$$\frac{w(E)}{w(B)} \le C \left(\frac{|E|}{|B|}\right)^\delta. \tag{13}$$

Given a weight function w on $\mathbb{R}^n$, as usual, the weighted Lebesgue space $L^p_w(\mathbb{R}^n)$ for $1 \le p < \infty$ is defined as the set of all functions f such that

$$\|f\|_{L^p_w} := \left(\int_{\mathbb{R}^n} |f(x)|^p w(x)\, dx\right)^{1/p} < \infty. \tag{14}$$

We also denote by $WL^p_w(\mathbb{R}^n)$ $(1 \le p < \infty)$ the weighted weak Lebesgue space consisting of all measurable functions f such that

$$\|f\|_{WL^p_w} := \sup_{\lambda>0} \lambda \cdot [w(\{x \in \mathbb{R}^n : |f(x)| > \lambda\})]^{1/p} < \infty. \tag{15}$$

We next recall some basic definitions and facts about Orlicz spaces needed for the proof of the main results. For further information on the subject, one can see [13]. A function $\mathscr{A}$ is called a Young function if it is continuous, nonnegative, convex, and strictly increasing on $[0, +\infty)$ with $\mathscr{A}(0) = 0$ and $\mathscr{A}(t) \to +\infty$ as $t \to +\infty$. An important example of Young function is $\mathscr{A}(t) = t^p(1+\log^+ t)^p$ with some $1 \le p < \infty$. Given a Young function $\mathscr{A}$, we define the $\mathscr{A}$-average of a function f over a ball B by means of the following Luxemburg norm:

$$\|f\|_{\mathscr{A},B} := \inf\left\{\lambda > 0 : \frac{1}{|B|}\int_B \mathscr{A}\left(\frac{|f(x)|}{\lambda}\right) dx \le 1\right\}. \tag{16}$$

When $\mathscr{A}(t) = t^p$, $1 \le p < \infty$, it is easy to see that

$$\|f\|_{\mathscr{A},B} = \left(\frac{1}{|B|}\int_B |f(x)|^p\, dx\right)^{1/p}; \tag{17}$$

that is, the Luxemburg norm coincides with the normalized L^p norm. Given a Young function $\mathscr{A}$, we use $\overline{\mathscr{A}}$ to denote the complementary Young function associated with $\mathscr{A}$. Then, the following generalized Hölder's inequality holds for any given ball B:

$$\frac{1}{|B|}\int_B |f(x)\cdot g(x)|\, dx \le 2\|f\|_{\mathscr{A},B}\|g\|_{\overline{\mathscr{A}},B}. \tag{18}$$

In particular, when $\mathscr{A}(t) = t\cdot(1+\log^+ t)$, we know that its complementary Young function is $\overline{\mathscr{A}}(t) \approx \exp(t) - 1$. In this situation, we denote

$$\|f\|_{L\log L,B} = \|f\|_{\mathscr{A},B}, \qquad \|g\|_{\exp L,B} = \|g\|_{\overline{\mathscr{A}},B}. \tag{19}$$

So we have

$$\frac{1}{|B|}\int_B |f(x)\cdot g(x)|\, dx \le 2\|f\|_{L\log L,B}\|g\|_{\exp L,B}. \tag{20}$$

2.2. Morrey Type Spaces. Let us begin with the definitions of the weighted Morrey space and generalized Morrey space.

Definition 6 (see [12]). Let $1 \le p < \infty$, $0 < \kappa < 1$, and w be a weight function on $\mathbb{R}^n$. Then, the weighted Morrey space $\mathscr{L}^{p,\kappa}(w)$ is defined by

$$\mathscr{L}^{p,\kappa}(w) := \left\{f \in L^p_{\text{loc}}(w) : \|f\|_{\mathscr{L}^{p,\kappa}(w)} = \sup_B \left(\frac{1}{w(B)^\kappa}\int_B |f(x)|^p w(x)\, dx\right)^{1/p} < \infty\right\}, \tag{21}$$

where the supremum is taken over all balls B in $\mathbb{R}^n$. We also denote by $W\mathscr{L}^{1,\kappa}(w)$ the weighted weak Morrey space of all measurable functions f such that

$$\sup_B \sup_{\lambda>0} \frac{1}{w(B)^\kappa}\lambda\cdot w(\{x \in B : |f(x)| > \lambda\}) \le C < \infty. \tag{22}$$

Let $\Psi = \Psi(r)$, $r > 0$, be a growth function, that is, a positive increasing function in $(0, +\infty)$, and satisfy the following doubling condition:

$$\Psi(2r) \le D\cdot\Psi(r), \quad \forall r > 0, \tag{23}$$

where $D = D(\Psi) \ge 1$ is a doubling constant independent of r.

Definition 7 (see [11]). Let $1 \le p < \infty$ and Ψ be a growth function in $(0, +\infty)$. Then, the generalized Morrey space $\mathscr{L}^{p,\Psi}(\mathbb{R}^n)$ is defined by

$$\mathscr{L}^{p,\Psi}(\mathbb{R}^n) := \left\{f \in L^p_{\text{loc}}(\mathbb{R}^n) : \|f\|_{\mathscr{L}^{p,\Psi}} = \sup_{r>0;B(x_0,r)} \left(\frac{1}{\Psi(r)}\int_{B(x_0,r)} |f(x)|^p\, dx\right)^{1/p} < \infty\right\}, \tag{24}$$

where the supremum is taken over all balls $B(x_0, r)$ in $\mathbb{R}^n$ with $x_0 \in \mathbb{R}^n$. One can also denote by $W\mathscr{L}^{1,\Psi}(\mathbb{R}^n)$ the generalized weak Morrey space of all measurable functions f for which

$$\sup_{B(x_0,r)} \sup_{\lambda>0} \frac{1}{\Psi(r)}\lambda\cdot|\{x \in B(x_0,r) : |f(x)| > \lambda\}| \le C < \infty. \tag{25}$$

In order to unify these two definitions, we now introduce Morrey type spaces associated with ψ as follows. Let $0 \le \kappa < 1$. Assume that $\psi(\cdot)$ is a positive increasing function defined in $(0, +\infty)$ and satisfies the following $\mathscr{D}_\kappa$ condition:

$$\frac{\psi(\xi)}{\xi^\kappa} \le C\cdot\frac{\psi(\xi')}{(\xi')^\kappa}, \quad \text{for any } 0 < \xi' < \xi < +\infty, \tag{26}$$

where $C > 0$ is a constant independent of ξ and ξ'.

Definition 8. Let $1 \le p < \infty$, $0 \le \kappa < 1$, ψ satisfy the $\mathscr{D}_\kappa$ condition (26), and w be a weight function on $\mathbb{R}^n$. We denote by $\mathscr{M}^{p,\psi}(w)$ the generalized weighted Morrey space, the space of all locally integrable functions f with finite norm:

$$\|f\|_{\mathscr{M}^{p,\psi}(w)} := \sup_B \left(\frac{1}{\psi(w(B))}\int_B |f(x)|^p w(x)\, dx\right)^{1/p} \le C < \infty. \tag{27}$$

Then, we know that $\mathscr{M}^{p,\psi}(w)$ becomes a Banach function space with respect to the norm $\|\cdot\|_{\mathscr{M}^{p,\psi}(w)}$. Furthermore, we denote by $W\mathcal{M}^{p,\psi}(w)$ the generalized weighted weak Morrey space of all measurable functions f for which

$$\|f\|_{W\mathcal{M}^{p,\psi}(w)} := \sup_B \sup_{\sigma>0} \frac{1}{\psi(w(B))^{1/p}}\sigma \cdot [w(\{x \in B : |f(x)| > \sigma\})]^{1/p} \le C < \infty. \tag{28}$$

Definition 9. In the unweighted case (when w equals a constant function), one can denote the generalized unweighted Morrey space by $\mathcal{M}^{p,\psi}(\mathbb{R}^n)$ and weak Morrey space by $W\mathcal{M}^{p,\psi}(\mathbb{R}^n)$. That is, let $1 \le p < \infty$ and ψ satisfy the $\mathcal{D}_\kappa$ condition (26) with $0 \le \kappa < 1$; one can define

$$\begin{aligned} \mathcal{M}^{p,\psi}(\mathbb{R}^n) &:= \left\{ f \in L^p_{\mathrm{loc}}(\mathbb{R}^n) : \|f\|_{\mathcal{M}^{p,\psi}} = \sup_B \left(\frac{1}{\psi(|B|)} \int_B |f(x)|^p\,dx \right)^{1/p} < \infty \right\}, \\ W\mathcal{M}^{p,\psi}(\mathbb{R}^n) &:= \left\{ f : \|f\|_{W\mathcal{M}^{p,\psi}} = \sup_B \sup_{\sigma>0} \frac{1}{\psi(|B|)^{1/p}} \sigma \cdot \left|\{x \in B : |f(x)| > \sigma\}\right|^{1/p} < \infty \right\}. \end{aligned} \tag{29}$$

Note the following:

(i) If $\psi(x) \equiv 1$, then $\mathcal{M}^{p,\psi}(w) = L^p_w(\mathbb{R}^n)$ and $W\mathcal{M}^{p,\psi}(w) = WL^p_w(\mathbb{R}^n)$. Thus, our (weak) Morrey type space is an extension of the weighted (weak) Lebesgue space.

(ii) If $\psi(x) = x^\kappa$ with $0 < \kappa < 1$, then $\mathcal{M}^{p,\psi}(w)$ is just the weighted Morrey space $\mathcal{L}^{p,\kappa}(w)$, and $W\mathcal{M}^{1,\psi}(w)$ is just the weighted weak Morrey space $W\mathcal{L}^{1,\kappa}(w)$.

(iii) If $w(x) \equiv 1$, below we will show that $\mathcal{M}^{p,\psi}(\mathbb{R}^n)$ reduces to the generalized Morrey space $\mathcal{L}^{p,\Psi}(\mathbb{R}^n)$, and $W\mathcal{M}^{1,\psi}(\mathbb{R}^n)$ reduces to the generalized weak Morrey space $W\mathcal{L}^{1,\Psi}(\mathbb{R}^n)$.

Our main results on the boundedness of T_θ in the Morrey type spaces $\mathcal{M}^{p,\psi}(w)$ can be formulated as follows.

Theorem 10. *Let $1 < p < \infty$ and $w \in A_p$. Assume that ψ satisfies the $\mathcal{D}_\kappa$ condition (26) with $0 \le \kappa < 1$; then, the θ-type Calderón–Zygmund operator T_θ is bounded on $\mathcal{M}^{p,\psi}(w)$.*

Theorem 11. *Let $p = 1$ and $w \in A_1$. Assume that ψ satisfies the $\mathcal{D}_\kappa$ condition (26) with $0 \le \kappa < 1$; then, the θ-type Calderón–Zygmund operator T_θ is bounded from $\mathcal{M}^{1,\psi}(w)$ into $W\mathcal{M}^{1,\psi}(w)$.*

Let θ be a nonnegative, nondecreasing function on $\mathbb{R}^+ = (0, +\infty)$ satisfying condition (8), and let $[b, T_\theta]$ be the commutator formed by T_θ and BMO function b. For the strong-type estimate of the linear commutator $[b, T_\theta]$ in $\mathcal{M}^{p,\psi}(w)$ with $1 < p < \infty$, we will prove the following.

Theorem 12. *Let $1 < p < \infty$, $w \in A_p$, and $b \in BMO(\mathbb{R}^n)$. Assume that θ satisfies (8) and ψ satisfies the $\mathcal{D}_\kappa$ condition (26) with $0 \le \kappa < 1$; then, the commutator operator $[b, T_\theta]$ is bounded on $\mathcal{M}^{p,\psi}(w)$.*

To obtain endpoint estimate for the linear commutator $[b, T_\theta]$ in $\mathcal{M}^{1,\psi}(w)$, we first need to define the weighted $\mathcal{A}$-average of a function f over a ball B by means of the weighted Luxemburg norm; that is, given a Young function $\mathcal{A}$ and $w \in A_\infty$, we define (see [13, 14])

$$\|f\|_{\mathcal{A}(w),B} := \inf\left\{ \sigma > 0 : \frac{1}{w(B)} \int_B \mathcal{A}\left(\frac{|f(x)|}{\sigma}\right) \cdot w(x)\,dx \le 1 \right\}. \tag{30}$$

When $\mathcal{A}(t) = t$, this norm is denoted by $\|\cdot\|_{L(w),B}$; when $\mathcal{A}(t) = t\cdot(1+\log^+ t)$, this norm is also denoted by $\|\cdot\|_{L\log L(w),B}$. The complementary Young function of $t\cdot(1+\log^+ t)$ is $\exp t - 1$ with mean Luxemburg norm denoted by $\|\cdot\|_{\exp L(w),B}$. For $w \in A_\infty$ and for every ball B in $\mathbb{R}^n$, we can also show the weighted version of (20). Namely, the following generalized Hölder's inequality in the weighted setting

$$\frac{1}{w(B)} \int_B |f(x)\cdot g(x)|\, w(x)\,dx \le C\|f\|_{L\log L(w),B} \|g\|_{\exp L(w),B} \tag{31}$$

is valid (see [14], for instance). Now we introduce new Morrey type spaces of $L\log L$ type associated with ψ as follows.

Definition 13. Let $p = 1$, $0 \le \kappa < 1$, ψ satisfy the $\mathcal{D}_\kappa$ condition (26), and w be a weight function on $\mathbb{R}^n$. One can denote by $\mathcal{M}^{1,\psi}_{L\log L}(w)$ the generalized weighted Morrey space of $L\log L$ type, the space of all locally integrable functions f defined on $\mathbb{R}^n$ with finite norm $\|f\|_{\mathcal{M}^{1,\psi}_{L\log L}(w)}$.

$$\mathcal{M}^{1,\psi}_{L\log L}(w) := \left\{ f \in L^1_{\mathrm{loc}}(w) : \|f\|_{\mathcal{M}^{1,\psi}_{L\log L}(w)} < \infty \right\}, \tag{32}$$

where

$$\|f\|_{\mathcal{M}^{1,\psi}_{L\log L}(w)} := \sup_B \left\{ \frac{w(B)}{\psi(w(B))} \cdot \|f\|_{L\log L(w),B} \right\}. \tag{33}$$

Note that $t \le t\cdot(1+\log^+ t)$ for all $t > 0$; then, for any ball $B \subset \mathbb{R}^n$ and $w \in A_\infty$, we have $\|f\|_{L(w),B} \le \|f\|_{L\log L(w),B}$ by definition; that is, the inequality

$$\|f\|_{L(w),B} = \frac{1}{w(B)} \int_B |f(x)| \cdot w(x)\,dx \le \|f\|_{L\log L(w),B} \tag{34}$$

holds for any ball $B \subset \mathbb{R}^n$. From this, we can further see that when ψ satisfies the $\mathcal{D}_\kappa$ condition (26) with $0 \le \kappa < 1$,

$$\begin{aligned} \frac{1}{\psi(w(B))} \int_B |f(x)| \cdot w(x)\,dx &= \frac{w(B)}{\psi(w(B))} \cdot \frac{1}{w(B)} \int_B |f(x)| \cdot w(x)\,dx \\ &= \frac{w(B)}{\psi(w(B))} \cdot \|f\|_{L(w),B} \\ &\le \frac{w(B)}{\psi(w(B))} \cdot \|f\|_{L\log L(w),B}. \end{aligned} \tag{35}$$

Hence, we have $\mathcal{M}^{1,\psi}_{L\log L}(w) \subset \mathcal{M}^{1,\psi}(w)$ by definition.

Definition 14. In the unweighted case (when w equals a constant function), one can denote by $\mathcal{M}^{1,\psi}_{L\log L}(\mathbb{R}^n)$ the generalized unweighted Morrey space of $L\log L$ type. That is, let $p = 1$ and ψ satisfy the $\mathcal{D}_\kappa$ condition (26) with $0 \le \kappa < 1$; one can define

$$\mathcal{M}^{1,\psi}_{L\log L}(\mathbb{R}^n) := \left\{ f \in L^1_{\mathrm{loc}}(\mathbb{R}^n) : \|f\|_{\mathcal{M}^{1,\psi}_{L\log L}(\mathbb{R}^n)} < \infty \right\}, \tag{36}$$

where

$$\|f\|_{\mathcal{M}^{1,\psi}_{L\log L}(\mathbb{R}^n)} := \sup_B \left\{ \frac{|B|}{\psi(|B|)} \cdot \|f\|_{L\log L,B} \right\}. \tag{37}$$

We also consider the special case when ψ is taken to be $\psi(x) = x^\kappa$ with $0 < \kappa < 1$ and denote the corresponding space by $\mathcal{L}^{1,\kappa}_{L\log L}(w)$.

Definition 15. Let $p = 1$, $0 < \kappa < 1$, and w be a weight function on $\mathbb{R}^n$. One can denote by $\mathcal{L}^{1,\kappa}_{L\log L}(w)$ the weighted Morrey space of $L\log L$ type, the space of all locally integrable functions f defined on $\mathbb{R}^n$ with finite norm $\|f\|_{\mathcal{L}^{1,\kappa}_{L\log L}(w)}$.

$$\mathcal{L}^{1,\kappa}_{L\log L}(w) := \left\{ f \in L^1_{\mathrm{loc}}(w) : \|f\|_{\mathcal{L}^{1,\kappa}_{L\log L}(w)} < \infty \right\}, \tag{38}$$

where

$$\|f\|_{\mathcal{L}^{1,\kappa}_{L\log L}(w)} := \sup_B \left\{ w(B)^{1-\kappa} \cdot \|f\|_{L\log L(w),B} \right\}. \tag{39}$$

In this situation, we have $\mathcal{L}^{1,\kappa}_{L\log L}(w) \subset \mathcal{L}^{1,\kappa}(w)$.

For the endpoint case, we will also prove the following weak-type $L\log L$ estimate of the linear commutator $[b, T_\theta]$ in the Morrey type space associated with ψ.

Theorem 16. *Let $p = 1$, $w \in A_1$, and $b \in BMO(\mathbb{R}^n)$. Assume that θ satisfies (8) and ψ satisfies the $\mathcal{D}_\kappa$ condition (26) with $0 \le \kappa < 1$; then, for any given $\sigma > 0$ and any ball $B \subset \mathbb{R}^n$, there exists a constant $C > 0$ independent of f, B, and $\sigma > 0$ such that*

$$\begin{aligned} &\frac{1}{\psi(w(B))} \cdot w(\{x \in B : |[b, T_\theta](f)(x)| > \sigma\}) \\ &\quad \le C \cdot \left\| \Phi\left(\frac{|f|}{\sigma}\right) \right\|_{\mathcal{M}^{1,\psi}_{L\log L}(w)}, \end{aligned} \tag{40}$$

where $\Phi(t) = t \cdot (1 + \log^+ t)$. From the definitions, we can roughly say that the commutator operator $[b, T_\theta]$ is bounded from $\mathcal{M}^{1,\psi}_{L\log L}(w)$ into $W\mathcal{M}^{1,\psi}(w)$.

In particular, if we take $\psi(x) = x^\kappa$ with $0 < \kappa < 1$, then we immediately get the following strong-type estimate and endpoint estimate of T_θ and $[b, T_\theta]$ in the weighted Morrey spaces $\mathcal{L}^{p,\kappa}(w)$ for all $0 < \kappa < 1$ and $1 \le p < \infty$.

Corollary 17. *Let $1 < p < \infty$, $0 < \kappa < 1$, and $w \in A_p$. Then, the θ-type Calderón–Zygmund operator T_θ is bounded on $\mathcal{L}^{p,\kappa}(w)$.*

Corollary 18. *Let $p = 1$, $0 < \kappa < 1$, and $w \in A_1$. Then, the θ-type Calderón–Zygmund operator T_θ is bounded from $\mathcal{L}^{1,\kappa}(w)$ into $W\mathcal{L}^{1,\kappa}(w)$.*

Corollary 19. *Let $1 < p < \infty$, $0 < \kappa < 1$, $w \in A_p$, and $b \in BMO(\mathbb{R}^n)$. Assume that θ satisfies (8); then, the commutator operator $[b, T_\theta]$ is bounded on $\mathcal{L}^{p,\kappa}(w)$.*

Corollary 20. *Let $p = 1$, $0 < \kappa < 1$, $w \in A_1$, and $b \in BMO(\mathbb{R}^n)$. Assume that θ satisfies (8); then, for any given $\sigma > 0$ and any ball $B \subset \mathbb{R}^n$, there exists a constant $C > 0$ independent of f, B, and $\sigma > 0$ such that*

$$\begin{aligned} &\frac{1}{w(B)^\kappa} \cdot w(\{x \in B : |[b, T_\theta](f)(x)| > \sigma\}) \\ &\quad \le C \cdot \left\| \Phi\left(\frac{|f|}{\sigma}\right) \right\|_{\mathcal{L}^{1,\kappa}_{L\log L}(w)}, \end{aligned} \tag{41}$$

where $\Phi(t) = t \cdot (1 + \log^+ t)$.

Naturally, when $w(x) \equiv 1$, we have the following unweighted results.

Corollary 21. *Let $1 < p < \infty$. Assume that ψ satisfies the $\mathcal{D}_\kappa$ condition (26) with $0 \le \kappa < 1$; then, the θ-type Calderón–Zygmund operator T_θ is bounded on $\mathcal{M}^{p,\psi}(\mathbb{R}^n)$.*

Corollary 22. *Let $p = 1$. Assume that ψ satisfies the $\mathcal{D}_\kappa$ condition (26) with $0 \le \kappa < 1$; then, the θ-type Calderón–Zygmund operator T_θ is bounded from $\mathcal{M}^{1,\psi}(\mathbb{R}^n)$ into $W\mathcal{M}^{1,\psi}(\mathbb{R}^n)$.*

Corollary 23. *Let $1 < p < \infty$ and $b \in BMO(\mathbb{R}^n)$. Assume that θ satisfies (8) and ψ satisfies the $\mathcal{D}_\kappa$ condition (26) with $0 \le \kappa < 1$; then, the commutator operator $[b, T_\theta]$ is bounded on $\mathcal{M}^{p,\psi}(\mathbb{R}^n)$.*

Corollary 24. *Let $p = 1$ and $b \in BMO(\mathbb{R}^n)$. Assume that θ satisfies (8) and ψ satisfies the $\mathscr{D}_\kappa$ condition (26) with $0 \le \kappa < 1$; then, for any given $\sigma > 0$ and any ball $B \subset \mathbb{R}^n$, there exists a constant $C > 0$ independent of f, B, and $\sigma > 0$ such that*

$$\frac{1}{\psi(|B|)} \cdot \left|\left\{x \in B : \left|[b, T_\theta](f)(x)\right| > \sigma\right\}\right| \le C \cdot \left\|\Phi\left(\frac{|f|}{\sigma}\right)\right\|_{\mathscr{M}^{1,\psi}_{L\log L}(\mathbb{R}^n)}, \tag{42}$$

where $\Phi(t) = t \cdot (1 + \log^+ t)$.

Let $\Psi = \Psi(r)$, $r > 0$, be a growth function with doubling constant $D(\Psi) : 1 \le D(\Psi) < 2^n$. If, for any fixed $x_0 \in \mathbb{R}^n$ and $r > 0$, we set $\psi(|B(x_0, r)|) = \Psi(r)$, then

$$\psi\left(2^n \left|B(x_0, r)\right|\right) = \psi\left(\left|B(x_0, 2r)\right|\right) = \Psi(2r). \tag{43}$$

For the doubling constant $D(\Psi)$ satisfying $1 \le D(\Psi) < 2^n$, which means that $D(\Psi) = 2^{\kappa \cdot n}$ for some $0 \le \kappa < 1$, then we are able to verify that ψ is an increasing function and satisfies the $\mathscr{D}_\kappa$ condition (26) with some $0 \le \kappa < 1$.

Definition 25. Let $p = 1$ and Ψ be a growth function in $(0, +\infty)$. One can denote by $\mathscr{L}^{1,\Psi}_{L\log L}(\mathbb{R}^n)$ the generalized Morrey space of $L\log L$ type, which is defined by

$$\mathscr{L}^{1,\Psi}_{L\log L}(\mathbb{R}^n) := \left\{f \in L^1_{\mathrm{loc}}(\mathbb{R}^n) : \|f\|_{\mathscr{L}^{1,\Psi}_{L\log L}(\mathbb{R}^n)} < \infty\right\}, \tag{44}$$

where

$$\|f\|_{\mathscr{L}^{1,\Psi}_{L\log L}(\mathbb{R}^n)} := \sup_{r>0; B(x_0,r)} \left\{\frac{\left|B(x_0, r)\right|}{\Psi(r)} \cdot \|f\|_{L\log L, B(x_0,r)}\right\}. \tag{45}$$

In this situation, we also have $\mathscr{L}^{1,\Psi}_{L\log L}(\mathbb{R}^n) \subset \mathscr{L}^{1,\Psi}(\mathbb{R}^n)$.

From the definitions given above, we get $\mathscr{M}^{p,\psi}(\mathbb{R}^n) = \mathscr{L}^{p,\Psi}(\mathbb{R}^n)$, $W\mathscr{M}^{1,\psi}(\mathbb{R}^n) = W\mathscr{L}^{1,\Psi}(\mathbb{R}^n)$, and $\mathscr{M}^{1,\psi}_{L\log L}(\mathbb{R}^n) = \mathscr{L}^{1,\Psi}_{L\log L}(\mathbb{R}^n)$ by the choice of Ψ. Thus, by the above unweighted results (Corollaries 21–24), we can also obtain strong-type estimate and endpoint estimate of T_θ and $[b, T_\theta]$ in the generalized Morrey spaces $\mathscr{L}^{p,\Psi}(\mathbb{R}^n)$ when $1 \le p < \infty$ and Ψ satisfies the doubling condition (23).

Corollary 26. *Let $1 < p < \infty$. Suppose that Ψ satisfies the doubling condition (23) and $1 \le D(\Psi) < 2^n$; then, the θ-type Calderón–Zygmund operator T_θ is bounded on $\mathscr{L}^{p,\Psi}(\mathbb{R}^n)$.*

Corollary 27. *Let $p = 1$. Suppose that Ψ satisfies the doubling condition (23) and $1 \le D(\Psi) < 2^n$; then, the θ-type Calderón–Zygmund operator T_θ is bounded from $\mathscr{L}^{1,\Psi}(\mathbb{R}^n)$ into $W\mathscr{L}^{1,\Psi}(\mathbb{R}^n)$.*

Corollary 28. *Let $1 < p < \infty$ and $b \in BMO(\mathbb{R}^n)$. Suppose that θ satisfies (8) and Ψ satisfies the doubling condition (23) with $1 \le D(\Psi) < 2^n$; then, the commutator operator $[b, T_\theta]$ is bounded on $\mathscr{L}^{p,\Psi}(\mathbb{R}^n)$.*

Corollary 29. *Let $p = 1$ and $b \in BMO(\mathbb{R}^n)$. Suppose that θ satisfies (8) and Ψ satisfies the doubling condition (23) with $1 \le D(\Psi) < 2^n$; then, for any given $\sigma > 0$ and any ball $B(x_0, r) \subset \mathbb{R}^n$, there exists a constant $C > 0$ independent of f, $B(x_0, r)$, and $\sigma > 0$ such that*

$$\frac{1}{\Psi(r)} \cdot \left|\left\{x \in B(x_0, r) : \left|[b, T_\theta](f)(x)\right| > \sigma\right\}\right| \le C \cdot \left\|\Phi\left(\frac{|f|}{\sigma}\right)\right\|_{\mathscr{L}^{1,\Psi}_{L\log L}(\mathbb{R}^n)}, \tag{46}$$

where $\Phi(t) = t \cdot (1 + \log^+ t)$.

3. Proof of Theorems 10 and 11

Proof of Theorem 10. Let $f \in \mathscr{M}^{p,\psi}(w)$ with $1 < p < \infty$ and $w \in A_p$. For an arbitrary point $x_0 \in \mathbb{R}^n$, set $B = B(x_0, r_B)$ for the ball centered at x_0 and of radius r_B, $2B = B(x_0, 2r_B)$. We represent f as

$$f = f \cdot \chi_{2B} + f \cdot \chi_{(2B)^c} := f_1 + f_2; \tag{47}$$

by the linearity of the θ-type Calderón–Zygmund operator T_θ, we write

$$\begin{aligned} &\frac{1}{\psi(w(B))^{1/p}} \left(\int_B \left|T_\theta(f)(x)\right|^p w(x)\, dx\right)^{1/p} \\ &\quad\le \frac{1}{\psi(w(B))^{1/p}} \left(\int_B \left|T_\theta(f_1)(x)\right|^p w(x)\, dx\right)^{1/p} \\ &\quad\quad + \frac{1}{\psi(w(B))^{1/p}} \left(\int_B \left|T_\theta(f_2)(x)\right|^p w(x)\, dx\right)^{1/p} \\ &\quad:= I_1 + I_2. \end{aligned} \tag{48}$$

Below, we will give the estimates of I_1 and I_2, respectively. By the weighted L^p boundedness of T_θ (see Theorem 4), we have

$$\begin{aligned} I_1 &\le \frac{1}{\psi(w(B))^{1/p}} \left\|T_\theta(f_1)\right\|_{L^p_w} \\ &\le C \cdot \frac{1}{\psi(w(B))^{1/p}} \left(\int_{2B} \left|f(x)\right|^p w(x)\, dx\right)^{1/p} \\ &\le C \|f\|_{\mathscr{M}^{p,\psi}(w)} \cdot \frac{\psi(w(2B))^{1/p}}{\psi(w(B))^{1/p}}. \end{aligned} \tag{49}$$

Moreover, since $0 < w(B) < w(2B) < +\infty$ when $w \in A_p$ with $1 < p < \infty$, then by the $\mathscr{D}_\kappa$ condition (26) of ψ and inequality (12), we obtain

$$I_1 \le C \|f\|_{\mathscr{M}^{p,\psi}(w)} \cdot \frac{w(2B)^{\kappa/p}}{w(B)^{\kappa/p}} \le C \|f\|_{\mathscr{M}^{p,\psi}(w)}. \tag{50}$$

As for the term I_2, it is clear that when $x \in B$ and $y \in (2B)^c$, we get $|x-y| \approx |x_0-y|$. We then decompose $\mathbb{R}^n$ into a geometrically increasing sequence of concentric balls and obtain the following pointwise estimate:

$$\begin{aligned}|T_\theta(f_2)(x)| &\le \int_{\mathbb{R}^n}\frac{|f_2(y)|}{|x-y|^n}dy \le C\int_{(2B)^c}\frac{|f(y)|}{|x_0-y|^n}dy \\ &\le C\sum_{j=1}^{\infty}\frac{1}{|2^{j+1}B|}\int_{2^{j+1}B}|f(y)|\,dy.\end{aligned} \tag{51}$$

From this, it follows that

$$I_2 \le C\cdot\frac{w(B)^{1/p}}{\psi(w(B))^{1/p}}\sum_{j=1}^{\infty}\frac{1}{|2^{j+1}B|}\int_{2^{j+1}B}|f(y)|\,dy. \tag{52}$$

By using Hölder's inequality and A_p condition on w, we get

$$\begin{aligned}&\frac{1}{|2^{j+1}B|}\int_{2^{j+1}B}|f(y)|\,dy \\ &\quad\le \frac{1}{|2^{j+1}B|}\left(\int_{2^{j+1}B}|f(y)|^p w(y)\,dy\right)^{1/p} \\ &\quad\cdot\left(\int_{2^{j+1}B}w(y)^{-p'/p}\,dy\right)^{1/p'} \le C\|f\|_{\mathscr{M}^{p,\psi}(w)} \\ &\quad\cdot\frac{\psi\left(w\left(2^{j+1}B\right)\right)^{1/p}}{w\left(2^{j+1}B\right)^{1/p}}.\end{aligned} \tag{53}$$

Hence,

$$\begin{aligned}I_2 &\le C\|f\|_{\mathscr{M}^{p,\psi}(w)}\sum_{j=1}^{\infty}\frac{\psi\left(w\left(2^{j+1}B\right)\right)^{1/p}}{\psi(w(B))^{1/p}} \\ &\quad\cdot\frac{w(B)^{1/p}}{w\left(2^{j+1}B\right)^{1/p}}.\end{aligned} \tag{54}$$

Notice that $w \in A_p \subset A_\infty$ for $1<p<\infty$; then, by using the $\mathscr{D}_\kappa$ condition (26) of ψ again, inequality (13), and the fact that $0 \le \kappa < 1$, we find that

$$\begin{aligned}&\sum_{j=1}^{\infty}\frac{\psi\left(w\left(2^{j+1}B\right)\right)^{1/p}}{\psi(w(B))^{1/p}}\cdot\frac{w(B)^{1/p}}{w\left(2^{j+1}B\right)^{1/p}} \\ &\quad\le C\sum_{j=1}^{\infty}\frac{w(B)^{(1-\kappa)/p}}{w\left(2^{j+1}B\right)^{(1-\kappa)/p}} \le C\sum_{j=1}^{\infty}\left(\frac{|B|}{|2^{j+1}B|}\right)^{\delta(1-\kappa)/p} \\ &\quad\le C\sum_{j=1}^{\infty}\left(\frac{1}{2^{(j+1)n}}\right)^{\delta(1-\kappa)/p} \le C,\end{aligned} \tag{55}$$

which gives our desired estimate $I_2 \le C\|f\|_{\mathscr{M}^{p,\psi}(w)}$. Combining the estimates above for I_1 and I_2 and then taking the supremum over all balls $B \subset \mathbb{R}^n$, we complete the proof of Theorem 10. □

Proof of Theorem 11. Let $f \in \mathscr{M}^{1,\psi}(w)$ with $w \in A_1$. For an arbitrary ball $B = B(x_0, r_B) \subset \mathbb{R}^n$, we represent f as

$$f = f\cdot\chi_{2B} + f\cdot\chi_{(2B)^c} := f_1 + f_2; \tag{56}$$

then, for any given $\sigma > 0$, by the linearity of the θ-type Calderón–Zygmund operator T_θ, one can write

$$\begin{aligned}&\frac{1}{\psi(w(B))}\sigma\cdot w\left(\{x\in B : |T_\theta(f)(x)| > \sigma\}\right) \\ &\quad\le \frac{1}{\psi(w(B))}\sigma\cdot w\left(\left\{x\in B : |T_\theta(f_1)(x)| > \frac{\sigma}{2}\right\}\right) \\ &\quad+\frac{1}{\psi(w(B))}\sigma \\ &\quad\cdot w\left(\left\{x\in B : |T_\theta(f_2)(x)| > \frac{\sigma}{2}\right\}\right) := I_1' + I_2'.\end{aligned} \tag{57}$$

We first consider the term I_1'. By the weighted weak (1, 1) boundedness of T_θ (see Theorem 4), we have

$$\begin{aligned}I_1' &\le C\cdot\frac{1}{\psi(w(B))}\|f_1\|_{L^1_w} \\ &= C\cdot\frac{1}{\psi(w(B))}\left(\int_{2B}|f(x)|\,w(x)\,dx\right) \\ &\le C\|f\|_{\mathscr{M}^{1,\psi}(w)}\cdot\frac{\psi(w(2B))}{\psi(w(B))}.\end{aligned} \tag{58}$$

Moreover, since $0 < w(B) < w(2B) < +\infty$ when $w \in A_1$, then we apply the $\mathscr{D}_\kappa$ condition (26) of ψ and inequality (12) to obtain that

$$I_1' \le C\|f\|_{\mathscr{M}^{1,\psi}(w)}\cdot\frac{w(2B)^\kappa}{w(B)^\kappa} \le C\|f\|_{\mathscr{M}^{1,\psi}(w)}. \tag{59}$$

As for the term I_2', it follows directly from Chebyshev's inequality and the pointwise estimate (51) that

$$\begin{aligned}I_2' &\le \frac{1}{\psi(w(B))}\sigma\cdot\frac{2}{\sigma}\int_B|T_\theta(f_2)(x)|\,w(x)\,dx \\ &\le C\cdot\frac{w(B)}{\psi(w(B))}\sum_{j=1}^{\infty}\frac{1}{|2^{j+1}B|}\int_{2^{j+1}B}|f(y)|\,dy.\end{aligned} \tag{60}$$

Another application of A_1 condition on w gives that

$$\begin{aligned}&\frac{1}{|2^{j+1}B|}\int_{2^{j+1}B}|f(y)|\,dy \\ &\quad\le C\frac{1}{w\left(2^{j+1}B\right)}\cdot\operatorname*{ess\,inf}_{y\in 2^{j+1}B}w(y)\int_{2^{j+1}B}|f(y)|\,dy \\ &\quad\le C\frac{1}{w\left(2^{j+1}B\right)}\left(\int_{2^{j+1}B}|f(y)|\,w(y)\,dy\right) \\ &\quad\le C\|f\|_{\mathscr{M}^{1,\psi}(w)}\cdot\frac{\psi\left(w\left(2^{j+1}B\right)\right)}{w\left(2^{j+1}B\right)}.\end{aligned} \tag{61}$$

Consequently,

$$I_2' \le C \|f\|_{\mathcal{M}^{1,\psi}(w)} \sum_{j=1}^{\infty} \frac{\psi\left(w\left(2^{j+1}B\right)\right)}{\psi(w(B))} \cdot \frac{w(B)}{w(2^{j+1}B)}. \quad (62)$$

Recall that $w \in A_1 \subset A_\infty$; therefore, by using the $\mathcal{D}_\kappa$ condition (26) of ψ again, inequality (13), and the fact that $0 \le \kappa < 1$, we get

$$\begin{aligned}\sum_{j=1}^{\infty} \frac{\psi\left(w\left(2^{j+1}B\right)\right)}{\psi(w(B))} \cdot \frac{w(B)}{w(2^{j+1}B)} &\le C \sum_{j=1}^{\infty} \frac{w(B)^{1-\kappa}}{w\left(2^{j+1}B\right)^{1-\kappa}} \\ &\le C \sum_{j=1}^{\infty} \left(\frac{|B|}{|2^{j+1}B|}\right)^{\delta^*(1-\kappa)} \le C \sum_{j=1}^{\infty} \left(\frac{1}{2^{(j+1)n}}\right)^{\delta^*(1-\kappa)} \\ &\le C,\end{aligned} \quad (63)$$

which implies our desired estimate $I_2' \le C\|f\|_{\mathcal{M}^{1,\psi}(w)}$. Summing up the estimates above for I_1' and I_2' and then taking the supremum over all balls $B \subset \mathbb{R}^n$ and all $\sigma > 0$, we finish the proof of Theorem 11. □

4. Proof of Theorems 12 and 16

To prove our main theorems in this section, we need the following lemma about BMO functions.

Lemma 30. *Let b be a function in BMO($\mathbb{R}^n$). Then,*

(i) *for every ball B in $\mathbb{R}^n$ and for all $j \in \mathbb{Z}^+$,*

$$\left|b_{2^{j+1}B} - b_B\right| \le C \cdot (j+1) \|b\|_*; \quad (64)$$

(ii) *for every ball B in $\mathbb{R}^n$ and for all $w \in A_p$ with $1 \le p < \infty$,*

$$\left(\int_B \left|b(x) - b_B\right|^p w(x)\, dx\right)^{1/p} \le C \|b\|_* \cdot w(B)^{1/p}. \quad (65)$$

Proof. For the proof of (i), we refer the reader to [3]. For the proof of (ii), we refer the reader to [15]. □

Proof of Theorem 12. Let $f \in \mathcal{M}^{p,\psi}(w)$ with $1 < p < \infty$ and $w \in A_p$. For each fixed ball $B = B(x_0, r_B) \subset \mathbb{R}^n$, as before, we represent f as $f = f_1 + f_2$, where $f_1 = f \cdot \chi_{2B}$ and $2B = B(x_0, 2r_B) \subset \mathbb{R}^n$. By the linearity of the commutator operator $[b, T_\theta]$, we write

$$\begin{aligned}&\frac{1}{\psi(w(B))^{1/p}} \left(\int_B \left|[b, T_\theta](f)(x)\right|^p w(x)\, dx\right)^{1/p} \\ &\quad \le \frac{1}{\psi(w(B))^{1/p}} \left(\int_B \left|[b, T_\theta](f_1)(x)\right|^p w(x)\, dx\right)^{1/p} \\ &\quad + \frac{1}{\psi(w(B))^{1/p}} \left(\int_B \left|[b, T_\theta](f_2)(x)\right|^p w(x)\, dx\right)^{1/p} \\ &\quad := J_1 + J_2.\end{aligned} \quad (66)$$

Since T_θ is bounded on $L^p_w(\mathbb{R}^n)$ for $1 < p < \infty$ and $w \in A_p$, then by the well-known boundedness criterion for the commutators of linear operators, which was obtained by Álvarez et al. in [16], we know that $[b, T_\theta]$ is also bounded on $L^p_w(\mathbb{R}^n)$ for all $1 < p < \infty$ and $w \in A_p$, whenever $b \in \text{BMO}(\mathbb{R}^n)$. This fact together with the $\mathcal{D}_\kappa$ condition (26) of ψ and inequality (12) implies

$$\begin{aligned}J_1 &\le \frac{1}{\psi(w(B))^{1/p}} \left\|[b, T_\theta](f_1)\right\|_{L^p_w} \\ &\le C \cdot \frac{1}{\psi(w(B))^{1/p}} \left(\int_{2B} |f(x)|^p w(x)\, dx\right)^{1/p} \\ &\le C \|f\|_{\mathcal{M}^{p,\psi}(w)} \cdot \frac{\psi(w(2B))^{1/p}}{\psi(w(B))^{1/p}} \\ &\le C \|f\|_{\mathcal{M}^{p,\psi}(w)} \cdot \frac{w(2B)^{\kappa/p}}{w(B)^{\kappa/p}} \le C \|f\|_{\mathcal{M}^{p,\psi}(w)}.\end{aligned} \quad (67)$$

Let us now turn to the estimate of J_2. By definition, for any $x \in B$, we have

$$\begin{aligned}\left|[b, T_\theta](f_2)(x)\right| &\le \left|b(x) - b_B\right| \cdot \left|T_\theta(f_2)(x)\right| \\ &\quad + \left|T_\theta\left(\left[b_B - b\right] f_2\right)(x)\right|.\end{aligned} \quad (68)$$

In the proof of Theorem 10, we have already shown that (see (51))

$$\left|T_\theta(f_2)(x)\right| \le C \sum_{j=1}^{\infty} \frac{1}{|2^{j+1}B|} \int_{2^{j+1}B} |f(y)|\, dy. \quad (69)$$

Following the same arguments as in (51), we can also prove that

$$\begin{aligned}\left|T_\theta\left(\left[b_B - b\right] f_2\right)(x)\right| &\le \int_{\mathbb{R}^n} \frac{\left|\left[b_B - b(y)\right] f_2(y)\right|}{|x - y|^n} dy \\ &\le C \int_{(2B)^c} \frac{\left|\left[b_B - b(y)\right] f(y)\right|}{|x_0 - y|^n} dy \\ &\le C \sum_{j=1}^{\infty} \frac{1}{|2^{j+1}B|} \int_{2^{j+1}B} \left|b(y) - b_B\right| \cdot |f(y)|\, dy.\end{aligned} \quad (70)$$

Hence, from the pointwise estimates above for $|T_\theta(f_2)(x)|$ and $|T_\theta([b_B - b] f_2)(x)|$, it follows that

$$\begin{aligned}J_2 &\le \frac{C}{\psi(w(B))^{1/p}} \left(\int_B \left|b(x) - b_B\right|^p w(x)\, dx\right)^{1/p} \\ &\quad \cdot \left(\sum_{j=1}^{\infty} \frac{1}{|2^{j+1}B|} \int_{2^{j+1}B} |f(y)|\, dy\right) + C\end{aligned}$$

$$\cdot \frac{w(B)^{1/p}}{\psi(w(B))^{1/p}} \sum_{j=1}^{\infty} \frac{1}{|2^{j+1}B|} \int_{2^{j+1}B} |b_{2^{j+1}B} - b_B| \cdot |f(y)|\, dy + C \cdot \frac{w(B)^{1/p}}{\psi(w(B))^{1/p}} \sum_{j=1}^{\infty} \frac{1}{|2^{j+1}B|} \cdot \int_{2^{j+1}B} |b(y) - b_{2^{j+1}B}| \cdot |f(y)|\, dy := J_3 + J_4 + J_5. \quad (71)$$

Below, we will give the estimates of J_3, J_4, and J_5, respectively. Using (ii) of Lemma 30, Hölder's inequality, and the A_p condition, we obtain

$$\begin{aligned} J_3 &\le C\|b\|_* \cdot \frac{w(B)^{1/p}}{\psi(w(B))^{1/p}} \left(\sum_{j=1}^{\infty} \frac{1}{|2^{j+1}B|} \int_{2^{j+1}B} |f(y)|\, dy \right) \\ &\le C\|b\|_* \cdot \frac{w(B)^{1/p}}{\psi(w(B))^{1/p}} \sum_{j=1}^{\infty} \frac{1}{|2^{j+1}B|} \cdot \left(\int_{2^{j+1}B} |f(y)|^p w(y)\, dy \right)^{1/p} \cdot \left(\int_{2^{j+1}B} w(y)^{-p'/p}\, dy \right)^{1/p'} \le C\|f\|_{\mathscr{M}^{p,\psi}(w)} \\ &\quad \cdot \sum_{j=1}^{\infty} \frac{\psi(w(2^{j+1}B))^{1/p}}{\psi(w(B))^{1/p}} \cdot \frac{w(B)^{1/p}}{w(2^{j+1}B)^{1/p}} \le C\|f\|_{\mathscr{M}^{p,\psi}(w)}, \end{aligned} \quad (72)$$

where in the last inequality we have used the estimate (55). Applying (i) of Lemma 30, Hölder's inequality, and the A_p condition, we can deduce that

$$\begin{aligned} J_4 &\le C\|b\|_* \cdot \frac{w(B)^{1/p}}{\psi(w(B))^{1/p}} \sum_{j=1}^{\infty} \frac{(j+1)}{|2^{j+1}B|} \int_{2^{j+1}B} |f(y)|\, dy \\ &\le C\|b\|_* \cdot \frac{w(B)^{1/p}}{\psi(w(B))^{1/p}} \sum_{j=1}^{\infty} \frac{(j+1)}{|2^{j+1}B|} \cdot \left(\int_{2^{j+1}B} |f(y)|^p w(y)\, dy \right)^{1/p} \cdot \left(\int_{2^{j+1}B} w(y)^{-p'/p}\, dy \right)^{1/p'} \le C\|f\|_{\mathscr{M}^{p,\psi}(w)} \\ &\quad \cdot \sum_{j=1}^{\infty} (j+1) \cdot \frac{\psi(w(2^{j+1}B))^{1/p}}{\psi(w(B))^{1/p}} \cdot \frac{w(B)^{1/p}}{w(2^{j+1}B)^{1/p}}. \end{aligned} \quad (73)$$

For any $j \in \mathbb{Z}^+$, since $0 < w(B) < w(2^{j+1}B) < +\infty$ when $w \in A_p$ with $1 < p < \infty$, then by using the $\mathscr{D}_\kappa$ condition (26) of ψ and inequality (13) together with the fact that $0 \le \kappa < 1$, we thus obtain

$$\begin{aligned} &\sum_{j=1}^{\infty} (j+1) \cdot \frac{\psi(w(2^{j+1}B))^{1/p}}{\psi(w(B))^{1/p}} \cdot \frac{w(B)^{1/p}}{w(2^{j+1}B)^{1/p}} \\ &\quad \le C \sum_{j=1}^{\infty} (j+1) \cdot \frac{w(B)^{(1-\kappa)/p}}{w(2^{j+1}B)^{(1-\kappa)/p}} \\ &\quad \le C \sum_{j=1}^{\infty} (j+1) \cdot \left(\frac{|B|}{|2^{j+1}B|} \right)^{\delta(1-\kappa)/p} \\ &\quad \le C \sum_{j=1}^{\infty} (j+1) \cdot \left(\frac{1}{2^{(j+1)n}} \right)^{\delta(1-\kappa)/p} \le C, \end{aligned} \quad (74)$$

where the last series is convergent since the exponent $\delta(1-\kappa)/p$ is positive. This implies our desired estimate $J_4 \le C\|f\|_{\mathscr{M}^{p,\psi}(w)}$. It remains to estimate the last term J_5. An application of Hölder's inequality gives us that

$$\begin{aligned} J_5 &\le C \cdot \frac{w(B)^{1/p}}{\psi(w(B))^{1/p}} \sum_{j=1}^{\infty} \frac{1}{|2^{j+1}B|} \cdot \left(\int_{2^{j+1}B} |f(y)|^p w(y)\, dy \right)^{1/p} \\ &\quad \cdot \left(\int_{2^{j+1}B} |b(y) - b_{2^{j+1}B}|^{p'} w(y)^{-p'/p}\, dy \right)^{1/p'}. \end{aligned} \quad (75)$$

If we set $\mu(y) = w(y)^{-p'/p}$, then we have $\mu \in A_{p'}$ because $w \in A_p$ (see [1, 2]). Thus, it follows from (ii) of Lemma 30 and the A_p condition that

$$\begin{aligned} &\left(\int_{2^{j+1}B} |b(y) - b_{2^{j+1}B}|^{p'} \mu(y)\, dy \right)^{1/p'} \\ &\quad \le C\|b\|_* \cdot \mu(2^{j+1}B)^{1/p'} \\ &\quad = C\|b\|_* \cdot \left(\int_{2^{j+1}B} w(y)^{-p'/p}\, dy \right)^{1/p'} \\ &\quad \le C\|b\|_* \cdot \frac{|2^{j+1}B|}{w(2^{j+1}B)^{1/p}}. \end{aligned} \quad (76)$$

Therefore, in view of estimate (55), we conclude that

$$\begin{aligned} J_5 &\le C\|b\|_* \cdot \frac{w(B)^{1/p}}{\psi(w(B))^{1/p}} \sum_{j=1}^{\infty} \frac{1}{w(2^{j+1}B)^{1/p}} \cdot \left(\int_{2^{j+1}B} |f(y)|^p w(y)\, dy \right)^{1/p} \le C\|f\|_{\mathscr{M}^{p,\psi}(w)} \\ &\quad \cdot \sum_{j=1}^{\infty} \frac{\psi(w(2^{j+1}B))^{1/p}}{\psi(w(B))^{1/p}} \cdot \frac{w(B)^{1/p}}{w(2^{j+1}B)^{1/p}} \le C\|f\|_{\mathscr{M}^{p,\psi}(w)}. \end{aligned} \quad (77)$$

Summarizing the estimates derived above and then taking the supremum over all balls $B \subset \mathbb{R}^n$, we complete the proof of Theorem 12. □

Proof of Theorem 16. For any fixed ball $B = B(x_0, r_B)$ in $\mathbb{R}^n$, as before, we represent f as $f = f_1 + f_2$, where $f_1 = f \cdot \chi_{2B}$ and $2B = B(x_0, 2r_B) \subset \mathbb{R}^n$. Then, for any given $\sigma > 0$, by the linearity of the commutator operator $[b, T_\theta]$, one can write

$$\begin{aligned} &\frac{1}{\psi(w(B))} \cdot w(\{x \in B : |[b, T_\theta](f)(x)| > \sigma\}) \\ &\le \frac{1}{\psi(w(B))} \\ &\quad \cdot w\left(\left\{x \in B : |[b, T_\theta](f_1)(x)| > \frac{\sigma}{2}\right\}\right) \\ &\quad + \frac{1}{\psi(w(B))} \\ &\quad \cdot w\left(\left\{x \in B : |[b, T_\theta](f_2)(x)| > \frac{\sigma}{2}\right\}\right) \\ &\coloneqq J_1' + J_2'. \end{aligned} \tag{78}$$

By using Theorem 5 and the previous estimate (35), we get

$$\begin{aligned} J_1' &\le C \cdot \frac{1}{\psi(w(B))} \int_{\mathbb{R}^n} \Phi\left(\frac{|f_1(x)|}{\sigma}\right) \cdot w(x)\, dx \\ &= C \cdot \frac{1}{\psi(w(B))} \int_{2B} \Phi\left(\frac{|f(x)|}{\sigma}\right) \cdot w(x)\, dx \\ &= C \cdot \frac{\psi(w(2B))}{\psi(w(B))} \cdot \frac{1}{\psi(w(2B))} \int_{2B} \Phi\left(\frac{|f(x)|}{\sigma}\right) \\ &\quad \cdot w(x)\, dx \\ &\le C \cdot \frac{\psi(w(2B))}{\psi(w(B))} \cdot \frac{w(2B)}{\psi(w(2B))} \\ &\quad \cdot \left\| \Phi\left(\frac{|f|}{\sigma}\right) \right\|_{L\log L(w), 2B}. \end{aligned} \tag{79}$$

Moreover, since $0 < w(B) < w(2B) < +\infty$ when $w \in A_1$, then by the $\mathscr{D}_\kappa$ condition (26) of ψ and inequality (12), we have

$$\begin{aligned} J_1' &\le C \cdot \frac{w(2B)^\kappa}{w(B)^\kappa} \\ &\quad \cdot \left\{ \frac{w(2B)}{\psi(w(2B))} \cdot \left\| \Phi\left(\frac{|f|}{\sigma}\right) \right\|_{L\log L(w), 2B} \right\} \\ &\le C \cdot \left\| \Phi\left(\frac{|f|}{\sigma}\right) \right\|_{\mathscr{M}^{1,\psi}_{L\log L}(w)}, \end{aligned} \tag{80}$$

which is our desired estimate. We now turn to deal with the term J_2'. Recall that the following inequality

$$\begin{aligned} |[b, T_\theta](f_2)(x)| &\le |b(x) - b_B| \cdot |T_\theta(f_2)(x)| \\ &\quad + |T_\theta([b_B - b] f_2)(x)| \end{aligned} \tag{81}$$

is valid. So we can further decompose J_2' as

$$\begin{aligned} J_2' &\le \frac{1}{\psi(w(B))} \\ &\quad \cdot w\left(\left\{x \in B : |b(x) - b_B| \cdot |T_\theta(f_2)(x)| > \frac{\sigma}{4}\right\}\right) \\ &\quad + \frac{1}{\psi(w(B))} \\ &\quad \cdot w\left(\left\{x \in B : |T_\theta([b_B - b] f_2)(x)| > \frac{\sigma}{4}\right\}\right) \\ &\coloneqq J_3' + J_4'. \end{aligned} \tag{82}$$

By using the previous pointwise estimate (51) and Chebyshev's inequality together with (ii) of Lemma 30, we deduce that

$$\begin{aligned} J_3' &\le \frac{1}{\psi(w(B))} \\ &\quad \cdot \frac{4}{\sigma} \int_B |b(x) - b_B| \cdot |T_\theta(f_2)(x)|\, w(x)\, dx \\ &\le C \sum_{j=1}^\infty \frac{1}{|2^{j+1}B|} \int_{2^{j+1}B} \frac{|f(y)|}{\sigma}\, dy \frac{1}{\psi(w(B))} \\ &\quad \cdot \int_B |b(x) - b_B|\, w(x)\, dx \\ &\le C \|b\|_* \sum_{j=1}^\infty \frac{1}{|2^{j+1}B|} \int_{2^{j+1}B} \frac{|f(y)|}{\sigma}\, dy \frac{w(B)}{\psi(w(B))}. \end{aligned} \tag{83}$$

Furthermore, note that $t \le \Phi(t) = t \cdot (1 + \log^+ t)$ for any $t > 0$. It then follows from the A_1 condition and the previous estimate (34) that

$$\begin{aligned} J_3' &\le C \sum_{j=1}^\infty \frac{1}{w(2^{j+1}B)} \int_{2^{j+1}B} \frac{|f(y)|}{\sigma} \\ &\quad \cdot w(y)\, dy \frac{w(B)}{\psi(w(B))} \le C \sum_{j=1}^\infty \frac{1}{w(2^{j+1}B)} \\ &\quad \cdot \int_{2^{j+1}B} \Phi\left(\frac{|f(y)|}{\sigma}\right) \cdot w(y)\, dy \frac{w(B)}{\psi(w(B))} \\ &\le C \sum_{j=1}^\infty \left\| \Phi\left(\frac{|f|}{\sigma}\right) \right\|_{L\log L(w), 2^{j+1}B} \frac{w(B)}{\psi(w(B))} \\ &= C \sum_{j=1}^\infty \left\{ \frac{w(2^{j+1}B)}{\psi(w(2^{j+1}B))} \cdot \left\| \Phi\left(\frac{|f|}{\sigma}\right) \right\|_{L\log L(w), 2^{j+1}B} \right\} \\ &\quad \cdot \frac{\psi(w(2^{j+1}B))}{\psi(w(B))} \cdot \frac{w(B)}{w(2^{j+1}B)} \le C \end{aligned}$$

$$\cdot \left\| \Phi\left(\frac{|f|}{\sigma}\right) \right\|_{\mathscr{M}^{1,\psi}_{L\log L}(w)} \sum_{j=1}^{\infty} \frac{\psi\left(w\left(2^{j+1}B\right)\right)}{\psi\left(w\left(B\right)\right)} \cdot \frac{w\left(B\right)}{w\left(2^{j+1}B\right)} \leq C \cdot \left\| \Phi\left(\frac{|f|}{\sigma}\right) \right\|_{\mathscr{M}^{1,\psi}_{L\log L}(w)}, \tag{84}$$

where in the last inequality we have used estimate (63). On the other hand, applying the pointwise estimate (70) and Chebyshev's inequality, we have

$$\begin{aligned} J_4' &\leq \frac{1}{\psi\left(w\left(B\right)\right)} \cdot \frac{4}{\sigma} \int_B \left| T_\theta\left(\left[b_B - b\right] f_2\right)(x) \right| w(x)\,dx \\ &\leq \frac{w(B)}{\psi\left(w\left(B\right)\right)} \cdot \frac{C}{\sigma} \sum_{j=1}^{\infty} \frac{1}{\left|2^{j+1}B\right|} \int_{2^{j+1}B} \left|b(y) - b_B\right| \cdot \left|f(y)\right| dy \\ &\leq \frac{w(B)}{\psi\left(w\left(B\right)\right)} \cdot \frac{C}{\sigma} \sum_{j=1}^{\infty} \frac{1}{\left|2^{j+1}B\right|} \int_{2^{j+1}B} \left|b(y) - b_{2^{j+1}B}\right| \cdot \left|f(y)\right| dy + \frac{w(B)}{\psi\left(w\left(B\right)\right)} \\ &\quad \cdot \frac{C}{\sigma} \sum_{j=1}^{\infty} \frac{1}{\left|2^{j+1}B\right|} \int_{2^{j+1}B} \left|b_{2^{j+1}B} - b_B\right| \cdot \left|f(y)\right| dy \\ &:= J_5' + J_6'. \end{aligned} \tag{85}$$

For the term J_5', since $w \in A_1$, by the A_1 condition and the fact that $t \leq \Phi(t)$,

$$\begin{aligned} J_5' &\leq \frac{C}{\sigma} \cdot \frac{w(B)}{\psi\left(w\left(B\right)\right)} \sum_{j=1}^{\infty} \frac{1}{w\left(2^{j+1}B\right)} \int_{2^{j+1}B} \left|b(y) - b_{2^{j+1}B}\right| \cdot \left|f(y)\right| w(y)\,dy \\ &\leq C \cdot \frac{w(B)}{\psi\left(w\left(B\right)\right)} \sum_{j=1}^{\infty} \frac{1}{w\left(2^{j+1}B\right)} \int_{2^{j+1}B} \left|b(y) - b_{2^{j+1}B}\right| \cdot \Phi\left(\frac{\left|f(y)\right|}{\sigma}\right) w(y)\,dy. \end{aligned} \tag{86}$$

Furthermore, we use the generalized Hölder's inequality with weight (31) to obtain

$$\begin{aligned} J_5' &\leq C \cdot \frac{w(B)}{\psi\left(w\left(B\right)\right)} \cdot \sum_{j=1}^{\infty} \left\| b - b_{2^{j+1}B} \right\|_{\exp L(w), 2^{j+1}B} \left\| \Phi\left(\frac{|f|}{\sigma}\right) \right\|_{L\log L(w), 2^{j+1}B} \\ &\leq C \left\| b \right\|_* \cdot \frac{w(B)}{\psi\left(w\left(B\right)\right)} \sum_{j=1}^{\infty} \left\| \Phi\left(\frac{|f|}{\sigma}\right) \right\|_{L\log L(w), 2^{j+1}B}. \end{aligned} \tag{87}$$

In the last inequality, we have used the well-known fact that (see [14])

$$\left\| b - b_B \right\|_{\exp L(w), B} \leq C \left\| b \right\|_*, \quad \text{for any ball } B \subset \mathbb{R}^n. \tag{88}$$

It is equivalent to the inequality

$$\frac{1}{w(B)} \int_B \exp\left(\frac{\left|b(y) - b_B\right|}{c_0 \left\| b \right\|_*}\right) w(y)\,dy \leq C, \tag{89}$$

which is just a corollary of the well-known John–Nirenberg's inequality (see [9]) and the comparison property of A_1 weights. Hence, by estimate (63),

$$\begin{aligned} J_5' &\leq C \left\| b \right\|_* \cdot \sum_{j=1}^{\infty} \left\{ \frac{w\left(2^{j+1}B\right)}{\psi\left(w\left(2^{j+1}B\right)\right)} \cdot \left\| \Phi\left(\frac{|f|}{\sigma}\right) \right\|_{L\log L(w), 2^{j+1}B} \right\} \cdot \frac{\psi\left(w\left(2^{j+1}B\right)\right)}{\psi\left(w\left(B\right)\right)} \cdot \frac{w(B)}{w\left(2^{j+1}B\right)} \leq C \\ &\quad \cdot \left\| \Phi\left(\frac{|f|}{\sigma}\right) \right\|_{\mathscr{M}^{1,\psi}_{L\log L}(w)} \sum_{j=1}^{\infty} \frac{\psi\left(w\left(2^{j+1}B\right)\right)}{\psi\left(w\left(B\right)\right)} \cdot \frac{w(B)}{w\left(2^{j+1}B\right)} \leq C \cdot \left\| \Phi\left(\frac{|f|}{\sigma}\right) \right\|_{\mathscr{M}^{1,\psi}_{L\log L}(w)}. \end{aligned} \tag{90}$$

For the last term J_6', we proceed as follows. Using (i) of Lemma 30 together with the facts $w \in A_1$ and $t \leq \Phi(t) = t \cdot (1 + \log^+ t)$, we deduce that

$$\begin{aligned} J_6' &\leq C \cdot \frac{w(B)}{\psi\left(w\left(B\right)\right)} \sum_{j=1}^{\infty} (j+1) \left\| b \right\|_* \cdot \frac{1}{\left|2^{j+1}B\right|} \cdot \int_{2^{j+1}B} \frac{\left|f(y)\right|}{\sigma} dy \leq C \cdot \frac{w(B)}{\psi\left(w\left(B\right)\right)} \sum_{j=1}^{\infty} (j+1) \left\| b \right\|_* \\ &\quad \cdot \frac{1}{w\left(2^{j+1}B\right)} \int_{2^{j+1}B} \frac{\left|f(y)\right|}{\sigma} \cdot w(y)\,dy \leq C \left\| b \right\|_* \cdot \frac{w(B)}{\psi\left(w\left(B\right)\right)} \sum_{j=1}^{\infty} \frac{(j+1)}{w\left(2^{j+1}B\right)} \int_{2^{j+1}B} \Phi\left(\frac{\left|f(y)\right|}{\sigma}\right) \\ &\quad \cdot w(y)\,dy = C \left\| b \right\|_* \cdot \sum_{j=1}^{\infty} \left\{ \frac{w\left(2^{j+1}B\right)}{\psi\left(w\left(2^{j+1}B\right)\right)} \cdot \left\| \Phi\left(\frac{|f|}{\sigma}\right) \right\|_{L\log L(w), 2^{j+1}B} \right\} \\ &\quad \cdot (j+1) \cdot \frac{\psi\left(w\left(2^{j+1}B\right)\right)}{\psi\left(w\left(B\right)\right)} \cdot \frac{w(B)}{w\left(2^{j+1}B\right)} \leq C \\ &\quad \cdot \left\| \Phi\left(\frac{|f|}{\sigma}\right) \right\|_{\mathscr{M}^{1,\psi}_{L\log L}(w)} \sum_{j=1}^{\infty} (j+1) \cdot \frac{\psi\left(w\left(2^{j+1}B\right)\right)}{\psi\left(w\left(B\right)\right)} \cdot \frac{w(B)}{w\left(2^{j+1}B\right)}. \end{aligned} \tag{91}$$

Recall that $w \in A_1 \subset A_\infty$. We can now argue exactly as we did in the estimation of (74) to get

$$\begin{aligned}\sum_{j=1}^{\infty}(j+1)\cdot\frac{\psi\left(w\left(2^{j+1}B\right)\right)}{\psi(w(B))}\cdot\frac{w(B)}{w\left(2^{j+1}B\right)}&\\ \leq C\sum_{j=1}^{\infty}(j+1)\cdot\frac{w(B)^{1-\kappa}}{w\left(2^{j+1}B\right)^{1-\kappa}}&\\ \leq C\sum_{j=1}^{\infty}(j+1)\cdot\left(\frac{|B|}{\left|2^{j+1}B\right|}\right)^{\delta^*(1-\kappa)}&\\ \leq C\sum_{j=1}^{\infty}(j+1)\cdot\left(\frac{1}{2^{(j+1)n}}\right)^{\delta^*(1-\kappa)}\leq C.&\end{aligned} \tag{92}$$

Let us now substitute this estimate (92) into the term J_6'; we get the desired inequality

$$J_6' \leq C\cdot\left\|\Phi\left(\frac{|f|}{\sigma}\right)\right\|_{\mathcal{M}^{1,\psi}_{L\log L}(w)}. \tag{93}$$

This completes the proof of Theorem 16. □

5. Partial Results on Two-Weight Problems

In the last section, we consider related problems about two-weight, weak-type (p, p) inequalities with $1 < p < \infty$. Let $\mathcal{T}$ be the classical Calderón–Zygmund operator with standard kernel; that is, $\mathcal{T} = T_\theta$ when $\theta(t) = t^\delta$ with $0 < \delta \leq 1$. It is well known that $\mathcal{T}$ is a bounded operator on $L^p_w(\mathbb{R}^n)$ for all $1 < p < \infty$ and $w \in A_p$, and, of course, $\mathcal{T}$ is a bounded operator from $L^p_w(\mathbb{R}^n)$ into $WL^p_w(\mathbb{R}^n)$. In the two-weight context, however, the A_p condition is "not" sufficient for the weak-type (p, p) inequality for $\mathcal{T}$. More precisely, given a pair of weights (u, v) and p, $1 < p < \infty$, the weak-type inequality

$$\begin{aligned}&u\left(\left\{x\in\mathbb{R}^n : \left|\mathcal{T}f(x)\right| > \sigma\right\}\right)\\ &\quad\leq \frac{C}{\sigma^p}\int_{\mathbb{R}^n}\left|f(x)\right|^p v(x)\,dx\end{aligned} \tag{94}$$

does not hold if $(u, v) \in A_p$: there exists a positive constant C such that, for every cube $Q \subset \mathbb{R}^n$,

$$\left(\frac{1}{|Q|}\int_Q u(x)\,dx\right)^{1/p}\left(\frac{1}{|Q|}\int_Q v(x)^{-p'/p}\,dx\right)^{1/p'} \leq C < \infty; \tag{95}$$

one can see [17, 18] for some counterexamples. Here, all cubes are assumed to have their sides parallel to the coordinate axes; $Q(x_0, \ell)$ will denote the cube centered at x_0 and has side length ℓ. In [17, 19], Cruz-Uribe and Pérez considered the problem of finding sufficient conditions on a pair of weights (u, v) such that $\mathcal{T}$ satisfies the weak-type (p, p) inequality (94) $(1 < p < \infty)$. They showed in [19] that if we strengthened the A_p condition (95) by adding a "power bump" to the left-hand term, then inequality (94) holds for all $f \in L^p_v(\mathbb{R}^n)$. More specifically, if there exists a number $r > 1$ such that, for every cube Q in $\mathbb{R}^n$,

$$\left(\frac{1}{|Q|}\int_Q u(x)^r\,dx\right)^{1/(rp)}\left(\frac{1}{|Q|}\int_Q v(x)^{-p'/p}\,dx\right)^{1/p'} \leq C < \infty, \tag{96}$$

then the classical Calderón–Zygmund operator $\mathcal{T}$ is bounded from $L^p_v(\mathbb{R}^n)$ into $WL^p_u(\mathbb{R}^n)$. Moreover, in [17], the authors improved this result by replacing the "power bump" in (96) by a smaller "Orlicz bump." To be more precise, they introduced the following A_p-type condition in the scale of Orlicz spaces:

$$\|u\|^{1/p}_{L(\log L)^{p-1+\delta},Q}\left(\frac{1}{|Q|}\int_Q v(x)^{-p'/p}\,dx\right)^{1/p'} \leq C < \infty, \quad \delta > 0, \tag{97}$$

where $\|u\|_{L(\log L)^{p-1+\delta},Q}$ is the mean Luxemburg norm of u on cube Q with Young function $\mathcal{A}(t) = t\cdot(1+\log^+ t)^{p-1+\delta}$. It was shown that inequality (94) still holds under the A_p-type condition on (u, v), and this result is sharp since it does not hold in general when $\delta = 0$.

On the other hand, the following Sharp function estimate for T_θ was established in [8]: there exists some δ, $0 < \delta < 1$, and a positive constant $C = C_\delta$ such that, for any $f \in C_0^\infty(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,

$$\left[M^\sharp\left(\left|T_\theta f\right|^\delta\right)(x)\right]^{1/\delta} \leq CMf(x), \tag{98}$$

where M is the standard Hardy–Littlewood maximal operator and $M^\sharp$ is the well-known Sharp maximal operator defined as

$$M^\sharp f(x) := \sup_{x\in Q}\frac{1}{|Q|}\int_Q\left|f(y) - f_Q\right|dy. \tag{99}$$

Here, the supremum is taken over all the cubes containing x and f_Q denotes the mean value of f over Q; namely, $f_Q = (1/|Q|)\int_Q f(x)dx$. It was pointed out in [19] (Remark 1.3) that, by using this Sharp function estimate (98), we can also show inequality (94) is true for more general operator T_θ, under condition (96) on (u, v). Then, we obtain a sufficient condition for T_θ to be weak (p, p) with $1 < p < \infty$.

Theorem 31. *Let $1 < p < \infty$. Given a pair of weights (u, v), suppose that, for some $r > 1$ and for all cubes Q,*

$$\left(\frac{1}{|Q|}\int_Q u(x)^r\,dx\right)^{1/(rp)}\left(\frac{1}{|Q|}\int_Q v(x)^{-p'/p}\,dx\right)^{1/p'} \leq C < \infty. \tag{100}$$

Then, the θ-type Calderón–Zygmund operator T_θ satisfies the weak-type (p, p) inequality:

$$\begin{aligned}&u\left(\left\{x\in\mathbb{R}^n : \left|T_\theta f(x)\right| > \sigma\right\}\right)\\ &\quad\leq \frac{C}{\sigma^p}\int_{\mathbb{R}^n}\left|f(x)\right|^p v(x)\,dx,\end{aligned} \tag{101}$$

where C does not depend on f and $\sigma > 0$.

We want to extend Theorem 31 to the Morrey type spaces. In order to do so, we need to define Morrey type spaces associated with ψ with two weights.

Definition 32. Let $1 \le p < \infty$, $0 \le \kappa < 1$, and ψ satisfy the $\mathscr{D}_\kappa$ condition (26). For two weights u and v, one can denote by $\mathscr{M}^{p,\psi}(v,u)$ the generalized weighted Morrey space, the space of all locally integrable functions f with finite norm.

$$\mathscr{M}^{p,\psi}(v,u) := \left\{ f \in L^p_{\mathrm{loc}}(v) : \|f\|_{\mathscr{M}^{p,\psi}(v,u)} < \infty \right\}, \quad (102)$$

where the norm is given by

$$\|f\|_{\mathscr{M}^{p,\psi}(v,u)} := \sup_Q \left(\frac{1}{\psi(u(Q))} \int_Q |f(x)|^p v(x)\,dx \right)^{1/p}. \quad (103)$$

Note that

(i) if $u = v = w$, then $\mathscr{M}^{p,\psi}(v,u)$ is the space $\mathscr{M}^{p,\psi}(w)$ in Definition 8;

(ii) if $\psi(x) = x^\kappa$ with $0 < \kappa < 1$, then $\mathscr{M}^{p,\psi}(v,u)$ is just the weighted Morrey space with two weights $\mathscr{L}^{p,\kappa}(v,u)$, which was introduced by Komori and Shirai in [12].

We are now ready to prove the following result.

Theorem 33. *Let $1 < p < \infty$ and $u \in A_\infty$. Given a pair of weights (u,v), suppose that, for some $r > 1$ and for all cubes Q,*

$$\left(\frac{1}{|Q|} \int_Q u(x)^r\,dx \right)^{1/(rp)} \left(\frac{1}{|Q|} \int_Q v(x)^{-p'/p}\,dx \right)^{1/p'} \le C < \infty. \quad (104)$$

If ψ satisfies the $\mathscr{D}_\kappa$ condition (26) with $0 \le \kappa < 1$, then the θ-type Calderón–Zygmund operator T_θ is bounded from $\mathscr{M}^{p,\psi}(v,u)$ into $W\mathscr{M}^{p,\psi}(u)$.

Proof of Theorem 33. Let $f \in \mathscr{M}^{p,\psi}(v,u)$ with $1 < p < \infty$. For any cube $Q = Q(x_0,\ell) \subset \mathbb{R}^n$ and $\lambda > 0$, we will denote by λQ the cube concentric with Q whose each edge is λ times as long; that is, $\lambda Q = Q(x_0, \lambda\ell)$. Let

$$f = f \cdot \chi_{2Q} + f \cdot \chi_{(2Q)^c} := f_1 + f_2, \quad (105)$$

where χ_{2Q} denotes the characteristic function of $2Q = Q(x_0, 2\ell)$. Then, for any given $\sigma > 0$, we write

$$\begin{aligned} &\frac{1}{\psi(u(Q))^{1/p}} \sigma \cdot \left[u\left(\left\{ x \in Q : |T_\theta(f)(x)| > \sigma \right\}\right) \right]^{1/p} \\ &\le \frac{1}{\psi(u(Q))^{1/p}} \sigma \cdot \left[u\left(\left\{ x \in Q : |T_\theta(f_1)(x)| > \frac{\sigma}{2} \right\}\right) \right]^{1/p} \\ &+ \frac{1}{\psi(u(Q))^{1/p}} \sigma \cdot \left[u\left(\left\{ x \in Q : |T_\theta(f_2)(x)| > \frac{\sigma}{2} \right\}\right) \right]^{1/p} \\ &:= K_1 + K_2. \end{aligned} \quad (106)$$

Using Theorem 31, the $\mathscr{D}_\kappa$ condition (26) of ψ, and inequality (12) (consider cube Q instead of ball B), we get

$$\begin{aligned} K_1 &\le C \cdot \frac{1}{\psi(u(Q))^{1/p}} \left(\int_{\mathbb{R}^n} |f_1(x)|^p v(x)\,dx \right)^{1/p} \\ &= C \cdot \frac{1}{\psi(u(Q))^{1/p}} \left(\int_{2Q} |f(x)|^p v(x)\,dx \right)^{1/p} \\ &\le C \|f\|_{\mathscr{M}^{p,\psi}(v,u)} \cdot \frac{\psi(u(2Q))^{1/p}}{\psi(u(Q))^{1/p}} \\ &\le C \|f\|_{\mathscr{M}^{p,\psi}(v,u)} \cdot \frac{u(2Q)^{\kappa/p}}{u(Q)^{\kappa/p}} \le C \|f\|_{\mathscr{M}^{p,\psi}(v,u)}. \end{aligned} \quad (107)$$

As for the term K_2, using the same methods and steps as those we dealt with I_2 in Theorem 10, we can also obtain that, for any $x \in Q$,

$$|T_\theta(f_2)(x)| \le C \sum_{j=1}^\infty \frac{1}{|2^{j+1}Q|} \int_{2^{j+1}Q} |f(y)|\,dy. \quad (108)$$

This pointwise estimate together with Chebyshev's inequality implies

$$\begin{aligned} K_2 &\le \frac{2}{\psi(u(Q))^{1/p}} \cdot \left(\int_Q |T_\theta(f_2)(x)|^p u(x)\,dx \right)^{1/p} \\ &\le C \cdot \frac{u(Q)^{1/p}}{\psi(u(Q))^{1/p}} \sum_{j=1}^\infty \frac{1}{|2^{j+1}Q|} \int_{2^{j+1}Q} |f(y)|\,dy. \end{aligned} \quad (109)$$

Moreover, an application of Hölder's inequality gives that

$$\begin{aligned} K_2 &\le C \cdot \frac{u(Q)^{1/p}}{\psi(u(Q))^{1/p}} \sum_{j=1}^\infty \frac{1}{|2^{j+1}Q|} \cdot \left(\int_{2^{j+1}Q} |f(y)|^p v(y)\,dy \right)^{1/p} \\ &\quad \cdot \left(\int_{2^{j+1}Q} v(y)^{-p'/p}\,dy \right)^{1/p'} \le C \|f\|_{\mathscr{M}^{p,\psi}(v,u)} \\ &\quad \cdot \frac{u(Q)^{1/p}}{\psi(u(Q))^{1/p}} \sum_{j=1}^\infty \frac{\psi\left(u\left(2^{j+1}Q\right)\right)^{1/p}}{|2^{j+1}Q|} \cdot \left(\int_{2^{j+1}Q} v(y)^{-p'/p}\,dy \right)^{1/p'}. \end{aligned} \quad (110)$$

For any $j \in \mathbb{Z}^+$, since $0 < u(Q) < u(2^{j+1}Q) < +\infty$ when u is a weight function, then by the $\mathcal{D}_\kappa$ condition (26) of ψ with $0 \le \kappa < 1$, we can see that

$$\frac{\psi\left(u\left(2^{j+1}Q\right)\right)^{1/p}}{\psi\left(u\left(Q\right)\right)^{1/p}} \le \frac{u\left(2^{j+1}Q\right)^{\kappa/p}}{u\left(Q\right)^{\kappa/p}}. \tag{111}$$

In addition, we apply Hölder's inequality with exponent r to get

$$\begin{aligned} u\left(2^{j+1}Q\right) &= \int_{2^{j+1}Q} u\left(y\right) dy \\ &\le \left|2^{j+1}Q\right|^{1/r'} \left(\int_{2^{j+1}Q} u\left(y\right)^r dy\right)^{1/r}. \end{aligned} \tag{112}$$

Hence, in view of (111) and (112) derived above, we have

$$\begin{aligned} K_2 &\le C \left\|f\right\|_{\mathcal{M}^{p,\psi}(v,u)} \sum_{j=1}^{\infty} \frac{u\left(Q\right)^{(1-\kappa)/p}}{u\left(2^{j+1}Q\right)^{(1-\kappa)/p}} \\ &\quad \cdot \frac{u\left(2^{j+1}Q\right)^{1/p}}{\left|2^{j+1}Q\right|} \left(\int_{2^{j+1}Q} v\left(y\right)^{-p'/p} dy\right)^{1/p'} \\ &\le C \left\|f\right\|_{\mathcal{M}^{p,\psi}(v,u)} \sum_{j=1}^{\infty} \frac{u\left(Q\right)^{(1-\kappa)/p}}{u\left(2^{j+1}Q\right)^{(1-\kappa)/p}} \\ &\quad \cdot \frac{\left|2^{j+1}Q\right|^{1/(r'p)}}{\left|2^{j+1}Q\right|} \left(\int_{2^{j+1}Q} u\left(y\right)^r dy\right)^{1/(rp)} \\ &\quad \cdot \left(\int_{2^{j+1}Q} v\left(y\right)^{-p'/p} dy\right)^{1/p'} \le C \left\|f\right\|_{\mathcal{M}^{p,\psi}(v,u)} \\ &\quad \cdot \sum_{j=1}^{\infty} \frac{u\left(Q\right)^{(1-\kappa)/p}}{u\left(2^{j+1}Q\right)^{(1-\kappa)/p}}. \end{aligned} \tag{113}$$

The last inequality is obtained by condition (96) on (u, v). Furthermore, by our additional hypothesis on u : $u \in A_\infty$ and inequality (13) (consider cube Q instead of ball B), we get

$$\begin{aligned} \sum_{j=1}^{\infty} \frac{u\left(Q\right)^{(1-\kappa)/p}}{u\left(2^{j+1}Q\right)^{(1-\kappa)/p}} &\le C \sum_{j=1}^{\infty} \left(\frac{|Q|}{\left|2^{j+1}Q\right|}\right)^{\delta(1-\kappa)/p} \\ &\le C \sum_{j=1}^{\infty} \left(\frac{1}{2^{(j+1)n}}\right)^{\delta(1-\kappa)/p} \le C, \end{aligned} \tag{114}$$

which implies our desired estimate $K_2 \le C\|f\|_{\mathcal{M}^{p,\psi}(v,u)}$. Summing up the estimates above for K_1 and K_2 and then taking the supremum over all cubes $Q \subset \mathbb{R}^n$ and all $\sigma > 0$, we finish the proof of Theorem 33. □

Let M denote the Hardy–Littlewood maximal operator and $M^\sharp$ denote the Sharp maximal operator. For $\delta > 0$, we define

$$\begin{aligned} M_\delta\left(f\right) &:= \left[M\left(\left|f\right|^\delta\right)\right]^{1/\delta}, \\ M_\delta^\sharp\left(f\right) &:= \left[M^\sharp\left(\left|f\right|^\delta\right)\right]^{1/\delta}. \end{aligned} \tag{115}$$

The maximal function associated with $\mathcal{A}(t) = t(1 + \log^+ t)$ is defined as

$$M_{L\log L} f\left(x\right) := \sup_{x \in Q} \left\|f\right\|_{L\log L, Q}, \tag{116}$$

where the supremum is taken over all the cubes containing x. Let $b \in \mathrm{BMO}(\mathbb{R}^n)$ and $[b, T_\theta]$ be the commutator of the θ-type Calderón–Zygmund operator. In [8], it was proved that if θ satisfies condition (8), then, for $0 < \delta < \varepsilon < 1$, there exists a positive constant $C = C_{\delta,\varepsilon}$ such that, for any $f \in C_0^\infty(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,

$$\begin{aligned} &M_\delta^\sharp\left(\left[b, T_\theta\right] f\right)\left(x\right) \\ &\quad \le C \left\|b\right\|_* \left(M_\varepsilon\left(T_\theta f\right)\left(x\right) + M_{L\log L} f\left(x\right)\right). \end{aligned} \tag{117}$$

Using this Sharp function estimate (117) and following the idea of the proof in [19], we can also establish the two-weight, weak-type norm inequality for $[b, T_\theta]$.

Theorem 34. *Let $1 < p < \infty$ and $b \in BMO(\mathbb{R}^n)$. Given a pair of weights (u, v), suppose that, for some $r > 1$ and for all cubes Q,*

$$\left(\frac{1}{|Q|} \int_Q u\left(x\right)^r dx\right)^{1/(rp)} \left\|v^{-1/p}\right\|_{\mathcal{A},Q} \le C < \infty, \tag{118}$$

where $\mathcal{A}(t) = t^{p'}(1 + \log^+ t)^{p'}$ is a Young function. If θ satisfies (8), then the commutator operator $[b, T_\theta]$ satisfies the weak-type (p, p) inequality:

$$\begin{aligned} &u\left(\left\{x \in \mathbb{R}^n : \left|\left[b, T_\theta\right] f\left(x\right)\right| > \sigma\right\}\right) \\ &\quad \le \frac{C}{\sigma^p} \int_{\mathbb{R}^n} \left|f\left(x\right)\right|^p v\left(x\right) dx, \end{aligned} \tag{119}$$

where $C > 0$ does not depend on f and $\sigma > 0$.

We will extend Theorem 34 to the Morrey type spaces. In order to do so, we need the following key lemma.

Lemma 35. *Given three Young functions $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ such that, for all $t > 0$,*

$$\mathcal{A}^{-1}\left(t\right) \cdot \mathcal{B}^{-1}\left(t\right) \le \mathcal{C}^{-1}\left(t\right), \tag{120}$$

where $\mathcal{A}^{-1}(t)$ is the inverse function of $\mathcal{A}(t)$, then one has the following generalized Hölder's inequality due to O'Neil [20]: for any cube $Q \subset \mathbb{R}^n$ and all functions f and g,

$$\left\|f \cdot g\right\|_{\mathcal{C},Q} \le 2 \left\|f\right\|_{\mathcal{A},Q} \left\|g\right\|_{\mathcal{B},Q}. \tag{121}$$

Theorem 36. *Let $1 < p < \infty$, $u \in A_\infty$, and $b \in BMO(\mathbb{R}^n)$. Given a pair of weights (u, v), suppose that, for some $r > 1$ and for all cubes Q,*

$$\left(\frac{1}{|Q|} \int_Q u\left(x\right)^r dx\right)^{1/(rp)} \left\|v^{-1/p}\right\|_{\mathcal{A},Q} \le C < \infty, \tag{122}$$

where $\mathcal{A}(t) = t^{p'}(1 + \log^+ t)^{p'}$. If θ satisfies (8) and ψ satisfies the $\mathcal{D}_\kappa$ condition (26) with $0 \le \kappa < 1$, then the commutator operator $[b, T_\theta]$ is bounded from $\mathcal{M}^{p,\psi}(v, u)$ into $W\mathcal{M}^{p,\psi}(u)$.

Proof of Theorem 36. Let $f \in \mathscr{M}^{p,\psi}(v,u)$ with $1 < p < \infty$. For an arbitrary cube $Q = Q(x_0, \ell)$ in $\mathbb{R}^n$, as before, we set

$$f = f_1 + f_2, \quad f_1 = f \cdot \chi_{2Q}, \quad f_2 = f \cdot \chi_{(2Q)^c}. \tag{123}$$

Then, for any given $\sigma > 0$, we write

$$\begin{aligned}
&\frac{1}{\psi(u(Q))^{1/p}}\sigma \cdot \left[u\left(\left\{x \in Q : \left|[b,T_\theta](f)(x)\right| > \sigma\right\}\right)\right]^{1/p} \\
&\le \frac{1}{\psi(u(Q))^{1/p}}\sigma \cdot \left[u\left(\left\{x \in Q : \left|[b,T_\theta](f_1)(x)\right| > \frac{\sigma}{2}\right\}\right)\right]^{1/p} \\
&+ \frac{1}{\psi(u(Q))^{1/p}}\sigma \cdot \left[u\left(\left\{x \in Q : \left|[b,T_\theta](f_2)(x)\right| > \frac{\sigma}{2}\right\}\right)\right]^{1/p} := K_1' + K_2'.
\end{aligned} \tag{124}$$

Using Theorem 34, the $\mathscr{D}_\kappa$ condition (26) of ψ, and inequality (12) (consider cube Q instead of ball B), we get

$$\begin{aligned}
K_1' &\le C \cdot \frac{1}{\psi(u(Q))^{1/p}} \left(\int_{\mathbb{R}^n} \left|f_1(x)\right|^p v(x)\,dx\right)^{1/p} \\
&= C \cdot \frac{1}{\psi(u(Q))^{1/p}} \left(\int_{2Q} \left|f(x)\right|^p v(x)\,dx\right)^{1/p} \\
&\le C\|f\|_{\mathscr{M}^{p,\psi}(v,u)} \cdot \frac{\psi(u(2Q))^{1/p}}{\psi(u(Q))^{1/p}} \\
&\le C\|f\|_{\mathscr{M}^{p,\psi}(v,u)} \cdot \frac{u(2Q)^{\kappa/p}}{u(Q)^{\kappa/p}} \le C\|f\|_{\mathscr{M}^{p,\psi}(v,u)}.
\end{aligned} \tag{125}$$

Next we estimate K_2'. For any $x \in Q$, from the definition of $[b, T_\theta]$, we can see that

$$\begin{aligned}
\left|[b,T_\theta](f_2)(x)\right| &\le \left|b(x) - b_Q\right| \cdot \left|T_\theta(f_2)(x)\right| + \left|T_\theta\left(\left[b_Q - b\right] f_2\right)(x)\right| \\
&:= \xi(x) + \eta(x).
\end{aligned} \tag{126}$$

Thus, we have

$$\begin{aligned}
K_2' &\le \frac{1}{\psi(u(Q))^{1/p}}\sigma \cdot \left[u\left(\left\{x \in Q : \xi(x) > \frac{\sigma}{4}\right\}\right)\right]^{1/p} \\
&+ \frac{1}{\psi(u(Q))^{1/p}}\sigma \cdot \left[u\left(\left\{x \in Q : \eta(x) > \frac{\sigma}{4}\right\}\right)\right]^{1/p} := K_3' + K_4'.
\end{aligned} \tag{127}$$

For the term K_3', it follows from the pointwise estimate (108) mentioned above and Chebyshev's inequality that

$$\begin{aligned}
K_3' &\le \frac{4}{\psi(u(Q))^{1/p}} \cdot \left(\int_Q \left|\xi(x)\right|^p u(x)\,dx\right)^{1/p} \\
&\le \frac{C}{\psi(u(Q))^{1/p}} \cdot \left(\int_Q \left|b(x) - b_Q\right|^p u(x)\,dx\right)^{1/p} \cdot \left(\sum_{j=1}^{\infty} \frac{1}{\left|2^{j+1}Q\right|} \int_{2^{j+1}Q} \left|f(y)\right| dy\right) \le C \\
&\cdot \frac{u(Q)^{1/p}}{\psi(u(Q))^{1/p}} \sum_{j=1}^{\infty} \frac{1}{\left|2^{j+1}Q\right|} \int_{2^{j+1}Q} \left|f(y)\right| dy,
\end{aligned} \tag{128}$$

where in the last inequality we have used the fact that Lemma 30(ii) still holds when u is an A_∞ weight with B replaced by Q. Repeating the arguments in the proof of Theorem 33, we can show that $K_3' \le C\|f\|_{\mathscr{M}^{p,\psi}(v,u)}$. As for the term K_4', using the same methods and steps as those we dealt with J_2 in Theorem 12, we can show the following pointwise estimate as well:

$$\begin{aligned}
\eta(x) &= \left|T_\theta\left(\left[b_Q - b\right] f_2\right)(x)\right| \\
&\le C \sum_{j=1}^{\infty} \frac{1}{\left|2^{j+1}Q\right|} \int_{2^{j+1}Q} \left|b(y) - b_Q\right| \cdot \left|f(y)\right| dy.
\end{aligned} \tag{129}$$

This together with Chebyshev's inequality yields

$$\begin{aligned}
K_4' &\le \frac{4}{\psi(u(Q))^{1/p}} \cdot \left(\int_Q \left|\eta(x)\right|^p u(x)\,dx\right)^{1/p} \\
&\le C \cdot \frac{u(Q)^{1/p}}{\psi(u(Q))^{1/p}} \cdot \sum_{j=1}^{\infty} \frac{1}{\left|2^{j+1}Q\right|} \int_{2^{j+1}Q} \left|b(y) - b_Q\right| \cdot \left|f(y)\right| dy \\
&\le C \cdot \frac{u(Q)^{1/p}}{\psi(u(Q))^{1/p}} \cdot \sum_{j=1}^{\infty} \frac{1}{\left|2^{j+1}Q\right|} \int_{2^{j+1}Q} \left|b(y) - b_{2^{j+1}Q}\right| \cdot \left|f(y)\right| dy \\
&+ C \cdot \frac{u(Q)^{1/p}}{\psi(u(Q))^{1/p}} \cdot \sum_{j=1}^{\infty} \frac{1}{\left|2^{j+1}Q\right|} \int_{2^{j+1}Q} \left|b_{2^{j+1}Q} - b_Q\right| \cdot \left|f(y)\right| dy \\
&:= K_5' + K_6'.
\end{aligned} \tag{130}$$

An application of Hölder's inequality yields that

$$\begin{aligned} K_5' &\le C \cdot \frac{u(Q)^{1/p}}{\psi(u(Q))^{1/p}} \cdot \sum_{j=1}^{\infty} \frac{1}{|2^{j+1}Q|} \\ &\cdot \left(\int_{2^{j+1}Q} |f(y)|^p v(y)\, dy \right)^{1/p} \\ &\cdot \left(\int_{2^{j+1}Q} |b(y) - b_{2^{j+1}Q}|^{p'} v(y)^{-p'/p}\, dy \right)^{1/p'} \\ &\le C \|f\|_{\mathscr{M}^{p,\psi}(v,u)} \cdot \frac{u(Q)^{1/p}}{\psi(u(Q))^{1/p}} \\ &\cdot \sum_{j=1}^{\infty} \frac{\psi\left(u\left(2^{j+1}Q\right)\right)^{1/p}}{|2^{j+1}Q|} \left|2^{j+1}Q\right|^{1/p'} \\ &\cdot \left\|\left(b - b_{2^{j+1}Q}\right) \cdot v^{-1/p}\right\|_{\mathscr{C},2^{j+1}Q}, \end{aligned} \tag{131}$$

where $\mathscr{C}(t) = t^{p'}$ is a Young function. For $1 < p < \infty$, we know the inverse function of $\mathscr{C}(t)$ is $\mathscr{C}^{-1}(t) = t^{1/p'}$. Observe that

$$\begin{aligned} \mathscr{C}^{-1}(t) = t^{1/p'} &= \frac{t^{1/p'}}{1 + \log^+ t} \left(1 + \log^+ t\right) \\ &= \mathscr{A}^{-1}(t) \cdot \mathscr{B}^{-1}(t), \end{aligned} \tag{132}$$

where

$$\begin{aligned} \mathscr{A}(t) &\approx t^{p'} \left(1 + \log^+ t\right)^{p'}, \\ \mathscr{B}(t) &\approx \exp(t) - 1. \end{aligned} \tag{133}$$

Thus, by Lemma 35 and estimate (88) (when $w \equiv 1$), we have

$$\begin{aligned} &\left\|\left(b - b_{2^{j+1}Q}\right) \cdot v^{-1/p}\right\|_{\mathscr{C},2^{j+1}Q} \\ &\quad \le C \left\|b - b_{2^{j+1}Q}\right\|_{\mathscr{B},2^{j+1}Q} \cdot \left\|v^{-1/p}\right\|_{\mathscr{A},2^{j+1}Q} \\ &\quad \le C \|b\|_* \cdot \left\|v^{-1/p}\right\|_{\mathscr{A},2^{j+1}Q}. \end{aligned} \tag{134}$$

Moreover, in view of (111) and (112), we can deduce that

$$\begin{aligned} K_5' &\le C \|b\|_* \|f\|_{\mathscr{M}^{p,\psi}(v,u)} \sum_{j=1}^{\infty} \frac{u\left(2^{j+1}Q\right)^{\kappa/p}}{u(Q)^{\kappa/p}} \\ &\cdot \frac{u(Q)^{1/p}}{|2^{j+1}Q|^{1/p}} \cdot \left\|v^{-1/p}\right\|_{\mathscr{A},2^{j+1}Q} \le C \|b\|_* \|f\|_{\mathscr{M}^{p,\psi}(v,u)} \\ &\cdot \sum_{j=1}^{\infty} \frac{u(Q)^{(1-\kappa)/p}}{u(2^{j+1}Q)^{(1-\kappa)/p}} \\ &\cdot \left(\frac{1}{|2^{j+1}Q|} \int_{2^{j+1}Q} u(x)^r\, dx \right)^{1/(rp)} \cdot \left\|v^{-1/p}\right\|_{\mathscr{A},2^{j+1}Q} \\ &\le C \|f\|_{\mathscr{M}^{p,\psi}(v,u)} \sum_{j=1}^{\infty} \frac{u(Q)^{(1-\kappa)/p}}{u(2^{j+1}Q)^{(1-\kappa)/p}} \\ &\le C \|f\|_{\mathscr{M}^{p,\psi}(v,u)}. \end{aligned} \tag{135}$$

The last inequality is obtained by condition (122) on (u, v) and estimate (114). It remains to estimate the last term K_6'. Applying Lemma 30(i) (use Q instead of B) and Hölder's inequality, we get

$$\begin{aligned} K_6' &\le C \cdot \frac{u(Q)^{1/p}}{\psi(u(Q))^{1/p}} \sum_{j=1}^{\infty} \frac{(j+1)\|b\|_*}{|2^{j+1}Q|} \\ &\cdot \int_{2^{j+1}Q} |f(y)|\, dy \le C \cdot \frac{u(Q)^{1/p}}{\psi(u(Q))^{1/p}} \\ &\cdot \sum_{j=1}^{\infty} \frac{(j+1)\|b\|_*}{|2^{j+1}Q|} \left(\int_{2^{j+1}Q} |f(y)|^p v(y)\, dy \right)^{1/p} \\ &\cdot \left(\int_{2^{j+1}Q} v(y)^{-p'/p}\, dy \right)^{1/p'} \le C \|f\|_{\mathscr{M}^{p,\psi}(v,u)} \\ &\cdot \frac{u(Q)^{1/p}}{\psi(u(Q))^{1/p}} \sum_{j=1}^{\infty} (j+1) \\ &\cdot \frac{\psi\left(u\left(2^{j+1}Q\right)\right)^{1/p}}{|2^{j+1}Q|} \left(\int_{2^{j+1}Q} v(y)^{-p'/p}\, dy \right)^{1/p'}. \end{aligned} \tag{136}$$

Let $\mathscr{C}(t)$ and $\mathscr{A}(t)$ be the same as before. Obviously, $\mathscr{C}(t) \le \mathscr{A}(t)$ for all $t > 0$; then, for any cube $Q \subset \mathbb{R}^n$, we have $\|f\|_{\mathscr{C},Q} \le \|f\|_{\mathscr{A},Q}$ by definition, which implies that condition (122) is stronger than condition (96). This fact together with (111) and (112) yields

$$\begin{aligned} K_6' &\le C \|f\|_{\mathscr{M}^{p,\psi}(v,u)} \sum_{j=1}^{\infty} (j+1) \cdot \frac{u(Q)^{(1-\kappa)/p}}{u(2^{j+1}Q)^{(1-\kappa)/p}} \\ &\cdot \frac{u\left(2^{j+1}Q\right)^{1/p}}{|2^{j+1}Q|} \left(\int_{2^{j+1}Q} v(y)^{-p'/p}\, dy \right)^{1/p'} \\ &\le C \|f\|_{\mathscr{M}^{p,\psi}(v,u)} \sum_{j=1}^{\infty} (j+1) \cdot \frac{u(Q)^{(1-\kappa)/p}}{u(2^{j+1}Q)^{(1-\kappa)/p}} \\ &\cdot \frac{\left|2^{j+1}Q\right|^{1/(r'p)}}{|2^{j+1}Q|} \left(\int_{2^{j+1}Q} u(y)^r\, dy \right)^{1/(rp)} \\ &\cdot \left(\int_{2^{j+1}Q} v(y)^{-p'/p}\, dy \right)^{1/p'} \le C \|f\|_{\mathscr{M}^{p,\psi}(v,u)} \\ &\cdot \sum_{j=1}^{\infty} (j+1) \cdot \frac{u(Q)^{(1-\kappa)/p}}{u(2^{j+1}Q)^{(1-\kappa)/p}}. \end{aligned} \tag{137}$$

Moreover, by our additional hypothesis on $u : u \in A_\infty$ and inequality (13) (use Q instead of B), we finally obtain

$$\sum_{j=1}^{\infty}(j+1)\cdot\frac{u(Q)^{(1-\kappa)/p}}{u(2^{j+1}Q)^{(1-\kappa)/p}} \leq C\sum_{j=1}^{\infty}(j+1)\cdot\left(\frac{|Q|}{|2^{j+1}Q|}\right)^{\delta(1-\kappa)/p} \leq C\sum_{j=1}^{\infty}(j+1)\cdot\left(\frac{1}{2^{(j+1)n}}\right)^{\delta(1-\kappa)/p} \leq C, \tag{138}$$

which in turn gives that $K_6' \leq C\|f\|_{\mathscr{M}^{p,\psi}(v,u)}$. Summing up all the estimates above and then taking the supremum over all cubes $Q \subset \mathbb{R}^n$ and all $\sigma > 0$, we therefore conclude the proof of Theorem 36. □

In particular, if we take $\psi(x) = x^\kappa$ with $0 < \kappa < 1$, then we immediately get the following two-weight, weak-type (p, p) inequalities for T_θ and $[b, T_\theta]$ in the weighted Morrey spaces.

Corollary 37. *Let $1 < p < \infty$, $0 < \kappa < 1$, and $u \in A_\infty$. Given a pair of weights (u, v), suppose that, for some $r > 1$ and for all cubes Q,*

$$\left(\frac{1}{|Q|}\int_Q u(x)^r\,dx\right)^{1/(rp)}\left(\frac{1}{|Q|}\int_Q v(x)^{-p'/p}\,dx\right)^{1/p'} \leq C < \infty. \tag{139}$$

Then, the θ-type Calderón–Zygmund operator T_θ is bounded from $\mathscr{L}^{p,\kappa}(v,u)$ into $W\mathscr{L}^{p,\kappa}(u)$.

Corollary 38. *Let $1 < p < \infty$, $0 < \kappa < 1$, $u \in A_\infty$, and $b \in BMO(\mathbb{R}^n)$. Given a pair of weights (u, v), suppose that, for some $r > 1$ and for all cubes Q,*

$$\left(\frac{1}{|Q|}\int_Q u(x)^r\,dx\right)^{1/(rp)}\left\|v^{-1/p}\right\|_{\mathscr{A},Q} \leq C < \infty, \tag{140}$$

where $\mathscr{A}(t) = t^{p'}(1+\log^+ t)^{p'}$. If θ satisfies (8), then the commutator operator $[b, T_\theta]$ is bounded from $\mathscr{L}^{p,\kappa}(v,u)$ into $W\mathscr{L}^{p,\kappa}(u)$.

Competing Interests

The author declares that there is no conflict of interests regarding the publication of this article.

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