Hindawi Publishing Corporation
Journal of Function Spaces
Volume 2016, Article ID 1710260, 6 pages
http://dx.doi.org/10.1155/2016/1710260

Research Article

Trace Operators on Wiener Amalgam Spaces

Jayson Cunanan and Yohei Tsutsui

Department of Mathematical Sciences, Faculty of Science, Shinshu University, Asahi 3-1-1, Matsumoto, Nagano 390-8621, Japan

Correspondence should be addressed to Jayson Cunanan; jcunanan@shinshu-u.ac.jp

Received 24 February 2016; Accepted 27 March 2016

Academic Editor: HuoXiong Wu

The paper deals with trace operators of Wiener amalgam spaces using frequency uniform decomposition operators and maximal inequalities, obtaining sharp results. Additionally, we provide the embedding between standard and anisotropic Wiener amalgam spaces.

1. Introduction

The aim of this paper is to study the trace problem: what can be said about the trace operator $\mathbb{T}$,

$$\mathbb{T} : f(x) \longrightarrow f(\overline{x}, 0), \quad \overline{x} = (x_1, x_2, \ldots, x_{n-1}), \tag{1}$$

as a mapping from $W_s^{p,q}(\mathbb{R}^n)$ to $W_s^{p,q}(\mathbb{R}^{n-1})$. We note that, for a tempered distribution f defined on $\mathbb{R}^n$, $f(x,0)$ has no straightforward meaning and the question is how to define the trace for a class of tempered distributions. One can resort to the Schwartz function ϕ, which has a pointwise trace $\phi(\overline{x}, 0)$. It can be extended to (quasi-)Banach function spaces which contain the Schwartz space $\mathcal{S}$ as a dense subspace.

Our setting is on Wiener amalgam spaces. These spaces, together with modulation spaces, were introduced by Feichtinger [1–3] in the 80s and are now widely used function spaces for various problems in PDE and harmonic analysis [4–10]. They resemble Triebel-Lizorkin spaces in the sense that we are taking $L^p(\ell^q)$ norms but differ with the decomposition operator being used. Instead of the dyadic decomposition operators $\Delta_k \sim \mathcal{F}^{-1}\chi_{\{\xi:|\xi|\sim 2^k\}}\mathcal{F}$ used for Triebel-Lizorkin spaces, Wiener amalgam spaces use frequency uniform decomposition operators $\square_k \sim \mathcal{F}^{-1}\chi_{Q_k}\mathcal{F}$, where Q_k denotes a unit cube with center k and $\cup_{k\in\mathbb{Z}^n} Q_k = \mathbb{R}^n$.

The concept of trace operator plays an important role in studying the existence and uniqueness of solutions to boundary value problems, that is, to partial differential equations with prescribed boundary conditions [11, 12]. The trace operator makes it possible to extend the notion of restriction of a function to the boundary of its domain to "generalized" functions in various function spaces with regularity. Now, we give a formal definition for the trace operators.

Definition 1. Let X and Y be quasi-Banach function spaces defined on $\mathbb{R}^n$ and $\mathbb{R}^{n-1}$, respectively. Assume that the Schwartz class $\mathcal{S}$ is dense in X. Denote

$$\mathbb{T} : f(x) \longrightarrow f(\overline{x}, 0), \quad f \in \mathcal{S}. \tag{2}$$

Assuming that there exists a constant $C > 0$ such that

$$\|\mathbb{T}f\|_Y \le C\|f\|_X, \quad \forall f \in \mathcal{S}, \tag{3}$$

one can extend $\mathbb{T} : X \to Y$ by the density of $\mathcal{S}$ in X and we write $f(\overline{x}, 0) = \mathbb{T}f$, which is said to be the trace of $f \in X$. Moreover, if there exists a continuous linear operator $\mathbb{T}^{-1} : Y \to X$ such that $\mathbb{T}\mathbb{T}^{-1}$ is the identity operator on Y, then $\mathbb{T}$ is said to be a trace-retraction from X onto Y.

For (α-) modulation spaces, Besov spaces, and Triebel-Lizorkin spaces, trace theorems have been extensively studied [12–14]. Feichtinger et al. [13] considered the trace theorems on anisotropic modulation spaces $M_s^{p,q,r}$ with $0 < p, q, r < \infty$, $s \in \mathbb{R}$ and they obtained $\mathbb{T}M_s^{p,q,p\wedge q\wedge 1}(\mathbb{R}^n) = M_s^{p,q}(\mathbb{R}^{n-1})$. In [15, 16], we find that, for $0 < p, q \le \infty$, and $s - 1/p > (n-1)(1/p - 1)$, we have $\mathbb{T}B_s^{p,q}(\mathbb{R}^n) = B_{s-1/p}^{p,q}(\mathbb{R}^{n-1})$ and $\mathbb{T}F_s^{p,q}(\mathbb{R}^n) = F_{s-1/p}^{p,p}(\mathbb{R}^{n-1})$ (the case $F^{\infty,q}$ is omitted). The use

of atoms as a framework in studying trace problems can be found in [16] and the references within.

Our main results are the following.

Theorem 2. *Let $n \geq 2$, $0 < p, q < \infty$, $s \in \mathbb{R}$. Then*

$$\mathbb{T} : f(x) \longrightarrow f(\overline{x}, 0), \quad \overline{x} = (x_1, x_2, \ldots, x_{n-1}) \tag{4}$$

is a trace-retraction from $W_s^{p,q,1\wedge q}(\mathbb{R}^n)$ to $W_s^{p,q}(\mathbb{R}^{n-1})$.

In view of the embedding in Theorem 6(II-ii), we immediately have the following corollary.

Corollary 3. *Let $n \geq 2$, $0 < p, q < \infty$, $s \geq 0$. Then for any $\epsilon > 0$*

$$\mathbb{T} : W_{s+1/(1\wedge q)-1/q+\epsilon}^{p,q}(\mathbb{R}^n) \longrightarrow W_s^{p,q}(\mathbb{R}^{n-1}). \tag{5}$$

We remark that Corollary 3 is an improvement of an older trace theorem found in [14] and that our result is sharp at least for $1 < p, q < \infty$. Moreover, our result shows independence of p. This is due to the pointwise estimates we were able to prove in Section 3. An interesting observation is that the trace theorem of Triebel-Lizorkin spaces stated above shows independence in q. This difference might be due to the decomposition operators used in the norm of each of the function spaces.

The paper is organised as follows. In Section 2, the embedding between standard and anisotropic Wiener amalgam spaces is given. We also define notations, function spaces, and some lemmas to be used throughout this paper. In Section 3, we prove our main result, Theorem 2, and the sharpness of Corollary 3.

2. Preliminaries

Notations. The Schwartz class of test functions on $\mathbb{R}^n$ will be denoted by $\mathcal{S} := \mathcal{S}(\mathbb{R}^n)$ and its dual and the space of tempered distributions will be denoted by $\mathcal{S}' := \mathcal{S}'(\mathbb{R}^n)$. $L^p(\mathbb{R}^n)$ norm is given by $\|f\|_{L^p} = (\int_{\mathbb{R}^n} |f(x)|^p dx)^{1/p}$ whenever $1 \leq p < \infty$ and $\|f\|_{L^\infty} = \operatorname{ess\,sup}_{x\in\mathbb{R}^n} |f(x)|$. The Fourier transform of a function $f \in \mathcal{S}(\mathbb{R}^n)$ is given by

$$\mathcal{F} f(\xi) = \widehat{f}(\xi) = \int_{\mathbb{R}^n} e^{-i2\pi x\cdot\xi} f(x)\, dx \tag{6}$$

which is an isomorphism of the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ onto itself that extends to the tempered distributions $\mathcal{S}'(\mathbb{R}^n)$ by duality. The inverse Fourier transform is given by $\mathcal{F}^{-1} f(x) = \check{f}(x) = \int_{\mathbb{R}^n} e^{i2\pi\xi\cdot x} f(\xi) d\xi$. Given $1 \leq p \leq \infty$, we denote by p' the conjugate exponent of p (i.e., $1/p + 1/p' = 1$). We use the notation $u \lesssim v$ to denote $u \leq cv$ for a positive constant c independent of u and v. We write $a \wedge b := \min(a, b)$ and $a \vee b := \max(a, b)$. We now define the function spaces in this paper.

Let $\eta : \mathbb{R} \to [0, 1]$ be a smooth bump function satisfying

$$\eta(\xi) := \begin{cases} 1, & |\xi| \leq 1 \\ \text{smooth}, & 1 < |\xi| \leq 2 \\ 0, & |\xi| \geq 2. \end{cases} \tag{7}$$

We write, for $k = (k_1, \ldots, k_n)$ and $\xi = (\xi_1, \ldots, \xi_n)$,

$$\phi_{k_i} = \eta(2(\xi_i - k_i)). \tag{8}$$

Put

$$\varphi_k(\xi) = \frac{\phi_{k_1}(\xi_1) \cdots \phi_{k_n}(\xi_n)}{\sum_{k\in\mathbb{Z}^n} \phi_{k_1}(\xi_1) \cdots \phi_{k_n}(\xi_n)}, \quad k \in \mathbb{Z}^n. \tag{9}$$

Definition 4 (Wiener amalgam spaces). For $0 < p, q \leq \infty$, and $s \in \mathbb{R}$, the Wiener amalgam space $W_s^{p,q}$ consists of all tempered distributions $f \in \mathcal{S}'$ for which the following is finite:

$$\|f\|_{W_s^{p,q}} = \left\| \left\| \{\langle k\rangle^s \square_k f\} \right\|_{\ell^q} \right\|_{L^p}, \tag{10}$$

with $\square_k f = \mathcal{F}^{-1}(\varphi_k \widehat{f})$.

We note that (10) is a quasi-norm if $0 < p, q \leq \infty$ and norm if $1 \leq p, q \leq \infty$. Moreover, (10) is independent of the choice of $\varphi = \{\varphi_k\}_{k\in\mathbb{Z}^n}$. We refer the reader to [1, 2, 17] for equivalent definitions (continuous versions).

We write $\overline{x} = (x_1, x_2, \ldots, x_{n-1})$ and define the anisotropic Wiener amalgam spaces $W^{p,q,r}$ by the following norm:

$$\|f\|_{W_s^{p,q,r}(\mathbb{R}^n)} = \left\| \left(\sum_{k_n\in\mathbb{Z}} \left(\sum_{\overline{k}\in\mathbb{Z}^{n-1}} \langle \overline{k}\rangle^{sq} |\square_k f|^q \right)^{r/q} \right)^{1/r} \right\|_{L^p(\mathbb{R}^n)}. \tag{11}$$

Similarly, for $\overline{\overline{x}} = (x_1, x_2, \ldots, x_{n-2})$, we define

$$\|f\|_{W_s^{p,q,r,r}(\mathbb{R}^n)} = \left\| \left(\sum_{(k_{n-1}k_n)\in\mathbb{Z}^2} \left(\sum_{\overline{\overline{k}}\in\mathbb{Z}^{n-2}} \langle \overline{\overline{k}}\rangle^{sq} |\square_k f|^q \right)^{r/q} \right)^{1/r} \right\|_{L^p(\mathbb{R}^n)}. \tag{12}$$

Comparing amalgam spaces $W_s^{p,q}$ with anisotropic amalgam spaces $W_s^{p,q,r}$ we see that $W_s^{p,q}$ is rotational invariant but $W_s^{p,q,r}$ is not. Using the almost orthogonality of φ we see that $W_s^{p,q,r}$ is independent of φ. Moreover, recalling that $\|f\|_{W_s^{p,q,r}}$ is the function sequence $\{\square_k f\}_{k\in\mathbb{Z}^n}$ equipped with the $L^p \ell_{k_n}^r \ell_{\overline{k}}^q$ norm, it is easy to see that $W_s^{p,q,r}$ is a quasi-Banach space for any $s \in \mathbb{R}$, $p, q, r \in (0, \infty]$ and a Banach space for any $s \in \mathbb{R}$, $1 \leq p, q, r \leq \infty$. Moreover, the Schwartz space is dense in $W_s^{p,q,r}$ if $p, q, r < \infty$. The proofs are similar to those of amalgam spaces in [1, 2, 17].

We collect properties of Wiener amalgam spaces in the following lemma.

Lemma 5. *Let $p, q, p_i, q_i \in [1, \infty]$ for $i = 1, 2$ and $s_j \in \mathbb{R}$ for $j = 1, 2$. Then one has the following:*

(1) $\mathcal{S}(\mathbb{R}^n) \hookrightarrow W^{p,q}(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$;

(2) *$\mathcal{S}$ is dense in $W^{p,q}$ if p and $q < \infty$;*

(3) *if $q_1 \leq q_2$ and $p_1 \leq p_2$, then $W^{p_1,q_1} \hookrightarrow W^{p_2,q_2}$;*

(4) *if* $s_1 \geq s_2$, *then* $W^{p,q}_{s_1} \hookrightarrow W^{p,q}_{s_2}$;

(5) *(complex interpolation) let* $0 < \theta < 1$, $1/p = \theta/p_1 + (1-\theta)/p_2$, $1/q = \theta/q_1 + (1-\theta)/q_2$, *and* $s = \theta s_1 + (1-\theta)s_2$. *Then*

$$\left[W^{p_1,q_1}_{s_1}, W^{p_2,q_2}_{s_2}\right]_{[\theta]} = W^{p,q}_s. \tag{13}$$

The proofs of these statements can be found in [1, 3, 17, 18].

Theorem 6 (embedding: $W^{p,q}_s \hookrightarrow W^{p,q,r}_{s'}$). *Let* $p, q, r \in (0, \infty]$ *and* $s \geq 0$.

(I) *The case* $r = q$.

(I-i) *The case* $r = q = \infty$:

$$W^{p,\infty}_s \hookrightarrow W^{p,\infty,\infty}_s. \tag{14}$$

(I-ii) *The case* $r = q < \infty$:

$$W^{p,q}_s \hookrightarrow W^{p,q,q}_s. \tag{15}$$

(II) *The case* $r < q$.

(II-i) *The case* $q = \infty$. *If* $s > 1/r$, *then*

$$W^{p,\infty}_s \hookrightarrow W^{p,\infty,r}_{s'}, \tag{16}$$

for any $s' \in (-\infty, s - 1/r)$.

(II-ii) *The case* $q < \infty$. *If* $s > (1/r - 1/q)$, *then*

$$W^{p,q}_s \hookrightarrow W^{p,q,r}_{s'}, \tag{17}$$

for any $s' \in (-\infty, s - (1/r - 1/q))$.

(III) *The case* $q < r$.

(III-i) *The case* $r = \infty$:

$$W^{p,q}_s \hookrightarrow W^{p,q,\infty}_s. \tag{18}$$

(III-ii) *The case* $r < \infty$:

$$W^{p,q}_s \hookrightarrow W^{p,q,r}_s. \tag{19}$$

Proof. For part (I), it suffices to show the following estimates.

(I-i) Consider

$$\sup_{k_n} \sup_{\bar{k}} \langle \bar{k} \rangle^s |\square_k f| \leq \sup_k \langle k \rangle^s |\square_k f|. \tag{20}$$

(I-ii) Consider

$$\left(\sum_{k_n} \sum_{\bar{k}} \langle \bar{k} \rangle^{sq} |\square_k f|^q\right)^{1/q} \leq \left(\sum_k \langle k \rangle^{sq} |\square_k f|^q\right)^{1/q}. \tag{21}$$

(II-i) Let $s' := s - 1/r - \varepsilon$, $(\varepsilon > 0)$. We may assume that $s' \geq 0$:

$$\begin{aligned} &\left(\sum_{k_n} \sup_{\bar{k}} \langle \bar{k} \rangle^{s'r} |\square_k f|^r\right)^{1/r} \\ &\quad \leq \left(\sup_k \langle k \rangle^s |\square_k f|\right) \left(\sum_{k_n} \sup_{\bar{k}} \langle k \rangle^{-sr} \langle \bar{k} \rangle^{s'r}\right)^{1/r}. \end{aligned} \tag{22}$$

The last term is equivalent to

$$\begin{aligned} &\left(\sum_{m \in \mathbb{Z}} \left(\sup_{t \geq 1} \left(\frac{1}{t + |m|}\right)^s t^{s'}\right)^r\right)^{1/r} \\ &\quad \leq \left(\sum_{m \in \mathbb{Z}} \left(\frac{1}{1 + |m|}\right)^{(s-s')r} \sup_{t \geq 1} t^{(s'-s')r}\right)^{1/r} < \infty, \end{aligned} \tag{23}$$

where $(s - s')r = 1 + \varepsilon r > 1$ and $s' \geq 0$ have been used.

(II-ii) Let $s' := s - (1/r - 1/q) - \varepsilon$, $(\varepsilon > 0)$. It suffices to show the embedding in the case $s' \geq 0$. Remark that $q/r \in (1, \infty)$ and $(q/r)' = 1/(r(1/r - 1/q))$. Let $\alpha := 1 - r/q + \varepsilon r$:

$$\begin{aligned} &\left[\sum_{k_n} \left\{\sum_{\bar{k}} \langle \bar{k} \rangle^{s'q} |\square_k f|^q\right\}^{r/q}\right]^{1/r} \\ &\quad = \left[\sum_{k_n} \left\{\sum_{\bar{k}} \langle \bar{k} \rangle^{s'q} \langle k_n \rangle^{\alpha q/r} |\square_k f|^q\right\}^{r/q} \langle k_n \rangle^{-\alpha}\right]^{1/r} \\ &\quad \leq \left[\left\{\sum_{k_n} \sum_{\bar{k}} \langle \bar{k} \rangle^{s'q} \langle k_n \rangle^{\alpha q/r} |\square_k f|^q\right\}^{r/q} \right. \\ &\qquad \left. \cdot \left(\sum_{k_n} \langle k_n \rangle^{-\alpha(q/r)'}\right)^{1/(q/r)'}\right]^{1/r} \\ &\quad \lesssim \left(\sum_k \langle \bar{k} \rangle^{s'q} \langle k_n \rangle^{\alpha q/r} |\square_k f|^q\right)^{1/q} \\ &\quad = \left\{\sum_k \langle k \rangle^{sq} |\square_k f|^q \left(\langle \bar{k} \rangle^{s'} \langle k_n \rangle^{\alpha/r} \langle k \rangle^{-s}\right)^q\right\}^{1/q} \\ &\quad \leq \left[\sup_k \langle \bar{k} \rangle^{s'} \langle k_n \rangle^{\alpha/r} \langle k \rangle^{-s}\right] \left\{\sum_k \langle k \rangle^{sq} |\square_k f|^q\right\}^{1/q}. \end{aligned} \tag{24}$$

Here, we have used $\alpha(q/r)' = 1 + \varepsilon/(1/r - 1/q) > 1$. Because $\alpha/r = 1/r - 1/q + \varepsilon = s - s'$, $s - s' \geq 0$, and $s' \geq 0$,

$$\langle \bar{k} \rangle^{s'} \langle k_n \rangle^{\alpha/r} \langle k \rangle^{-s} = \left(\frac{\langle k_n \rangle}{\langle k \rangle}\right)^{s-s'} \left(\frac{\langle \bar{k} \rangle}{\langle k \rangle}\right)^{s'} \lesssim 1. \tag{25}$$

(III-i) Consider

$$\sup_{k_n}\left(\sum_{\bar{k}}\langle\bar{k}\rangle^{sq}\left|\square_k f\right|^q\right)^{1/q}\le\left(\sum_{k}\langle k\rangle^{sq}\left|\square_k f\right|^q\right)^{1/q}. \quad (26)$$

Here, we have used $s\ge 0$.

(III-ii) Using the embedding $\ell^q\hookrightarrow\ell^r$,

$$\begin{aligned}&\left\{\sum_{k_n}\left(\sum_{\bar{k}}\langle\bar{k}\rangle^{sq}\left|\square_k f\right|^q\right)^{r/q}\right\}^{1/r}\\&\quad\le\left(\sum_{k}\langle\bar{k}\rangle^{sq}\left|\square_k f\right|^q\right)^{1/q}\le\left(\sum_{k}\langle k\rangle^{sq}\left|\square_k f\right|^q\right)^{1/q}.\end{aligned} \quad (27)$$

In the last inequality, we need $s\ge 0$. □

Lemma 7 (Triebel [12]). *Let $0<p<\infty$ and $0<q\le\infty$. Let $\Omega=\{\Omega_k\}_{k\in\mathbb{Z}^n}$ be a sequence of compact subsets of $\mathbb{R}^n$. Let d_k be the diameter of Ω_k. If $0<r<\min(p,q)$, then there exists a constant c such that*

$$\left\|\sup_{z\in\mathbb{R}^n}\frac{\left|f_k(\cdot-z)\right|}{1+\left|d_k z\right|^{n/r}}\right\|_{L^p(\ell^q)}\le c\left\|f_k\right\|_{L^p(\ell^q)} \quad (28)$$

holds for all $f\in L^p_\Omega(\ell^q)$, where $f=\{f_k\}$, $\|f_k\|_{L^p(\ell^q)}=\|\|f_k(\cdot)\|_{\ell^q}\|_{L^p}$, and

$$\begin{aligned}L^p_\Omega(\ell^q)=\Big\{&f\mid f=\{f_k\}_{k\in\mathbb{Z}^n}\subset\mathcal{S}',\ \operatorname{supp}\mathcal{F}f_k\\&\subset\Omega_k,\ \left\|f_k\right\|_{L^p(\ell^q)}<\infty\Big\}.\end{aligned} \quad (29)$$

Definition 8 (maximal functions). Let $b>0$ and $f\in\mathcal{S}$. Then

$$\square_k^* f(x):=\sup_{y\in\mathbb{Z}^n}\frac{\left|\square_k f(x-y)\right|}{1+\left|y\right|^b}\quad x\in\mathbb{R}^n,\ k\in\mathbb{Z}^n. \quad (30)$$

Proposition 9. *Let $0<p<\infty$ and $0<q\le\infty$, $b>n/\min(p,q)$. Then*

$$\left\|\left(\sum_{k\in\mathbb{Z}^n}\langle k\rangle^{sq}\left|\square_k^* f\right|^q\right)^{1/q}\right\|_{L^p(\mathbb{R}^n)}, \quad (31)$$

$$\left\|\left(\sum_{k_n\in\mathbb{Z}}\left(\sum_{\bar{k}\in\mathbb{Z}^{n-1}}\langle\bar{k}\rangle^{sq}\left|\square_k^* f\right|^q\right)^{r/q}\right)^{1/r}\right\|_{L^p(\mathbb{R}^n)} \quad (32)$$

are equivalent norms in $W_s^{p,q}(\mathbb{R}^n)$ and $W_s^{p,q,r}(\mathbb{R}^n)$, respectively.

The proof is a direct consequence of Lemma 7, taking $f_k=\square_k f$. See also [14, Proposition].

3. Proof of the Main Results

First, we narrate the idea of the proof. We give an equivalent formulation for $\square_{\bar{k}}(\mathbb{T}f)(\bar{x})$, a function in $\mathbb{R}^{n-1}$, via some $\square_{\bar{k},l}f(\bar{x},0)$, a function in $\mathbb{R}^n$. Then we compute for pointwise estimates between the corresponding ℓ^q norms and $\ell^r_{k_n}\ell^q_{\bar{k}}$ norms for cases $0<q<1$ and $1\le q<\infty$, separately. Finally, taking $L^p(\mathbb{R}^{n-1})$ norms and using our equivalent norms in Proposition 9, we arrive to our conclusion.

We denote by $\mathcal{F}_{\bar{x}}(\mathcal{F}^{-1}_{\bar{\xi}})$ the partial (inverse) Fourier transform on $\bar{x}(\bar{\xi})\in\mathbb{R}^{n-1}$. Write $\{\varphi_{\bar{k}}\}_{\bar{k}\in\mathbb{Z}^{n-1}}$ as versions of (9) in $\mathbb{R}^{n-1}$. By the support property of $\varphi_{\bar{k}}$, we observe

$$\begin{aligned}\square_{\bar{k}}(\mathbb{T}f)(\bar{x})&=\left(\mathcal{F}^{-1}_{\bar{\xi}}\varphi_{\bar{k}}\mathcal{F}_{\bar{x}}\right)(\mathbb{T}f)(\bar{x})\\&=\sum_{l\in\mathbb{Z}^n}\left\{\mathcal{F}^{-1}_{\bar{\xi}}\varphi_{\bar{k}}\mathcal{F}_{\bar{x}}\left[\left(\mathcal{F}^{-1}\varphi_l\mathcal{F}f\right)(\bar{y},0)\right]\right\}(\bar{x})\\&=\sum_{l\in\mathbb{Z}^n}\chi_{(|\bar{k}-\bar{l}|\le 1)}\left(\mathcal{F}^{-1}\psi_{\bar{k},l}\mathcal{F}f\right)(\bar{x},0)\\&=\sum_{l\in\mathbb{Z}^n}\chi_{(|\bar{k}-\bar{l}|\le 1)}\square_{\bar{k},l}f(\bar{x},0),\end{aligned} \quad (33)$$

where $\psi_{\bar{k},l}(\xi)=\varphi_{\bar{k}}(\bar{\xi})\varphi_l(\xi)$, $l=(\bar{l},l_n)$, and $\square_{\bar{k},l}f:=\mathcal{F}^{-1}\psi_{\bar{k},l}\mathcal{F}f$. Note that the left-hand side is a function in $\mathbb{R}^{n-1}$ while the right-hand side is a function in $\mathbb{R}^n$.

Recall our maximal function (30) and take $y_1=y_2=\cdots=y_{n-1}=0$, $y_n=x_n$; we have, for $|x_n|\le 1$,

$$\left|\square_k f(\bar{x},0)\right|\lesssim\square_k^* f(x). \quad (34)$$

Proof of Theorem 2. We start by taking the ℓ^q-norm of (33). We write

$$\begin{aligned}&\left(\sum_{\bar{k}\in\mathbb{Z}^{n-1}}\langle\bar{k}\rangle^{sq}\left|\square_{\bar{k}}(\mathbb{T}f)(\bar{x})\right|^q\right)^{1/q}\\&\quad=\left(\sum_{\bar{k}\in\mathbb{Z}^{n-1}}\langle\bar{k}\rangle^{sq}\left(\sum_{l\in\mathbb{Z}^n}\chi_{(|\bar{k}-\bar{l}|\le 1)}\square_{\bar{k},l}f(\bar{x},0)\right)^q\right)^{1/q}.\end{aligned} \quad (35)$$

For $0<q<1$, we estimate (35) by

$$\begin{aligned}&\left(\sum_{\bar{k}\in\mathbb{Z}^{n-1}}\langle\bar{k}\rangle^{sq}\left|\square_{\bar{k}}(\mathbb{T}f)(\bar{x})\right|^q\right)^{1/q}\\&\lesssim\left(\sum_{l\in\mathbb{Z}^n}\sum_{\bar{k}\in\mathbb{Z}^{n-1}}\langle\bar{l}\rangle^{sq}\chi_{(|\bar{k}-\bar{l}|\le 1)}\left|\square_{\bar{k},l}f(\bar{x},0)\right|^q\right)^{1/q}\\&=\left(\sum_{l_n\in\mathbb{Z}}\sum_{\bar{l}\in\mathbb{Z}^{n-1}}\langle\bar{l}\rangle^{sq}\sum_{\bar{k}\in\mathbb{Z}^{n-1}}\chi_{(|\bar{k}-\bar{l}|\le 1)}\left|\square_{\bar{k},l}f(\bar{x},0)\right|^q\right)^{1/q}.\end{aligned} \quad (36)$$

Note that $\sum_{\bar{k}\in\mathbb{Z}^{n-1}}\chi_{(|\bar{k}-\bar{l}|\le 1)}|\square_{\bar{k},l}f(\bar{x},0)|^q=\sum_{j=1}^{n-1}|\square_{\overline{l\pm e_j},l}f(\bar{x},0)|^q$, where e_j is the jth column of the identity matrix. In the

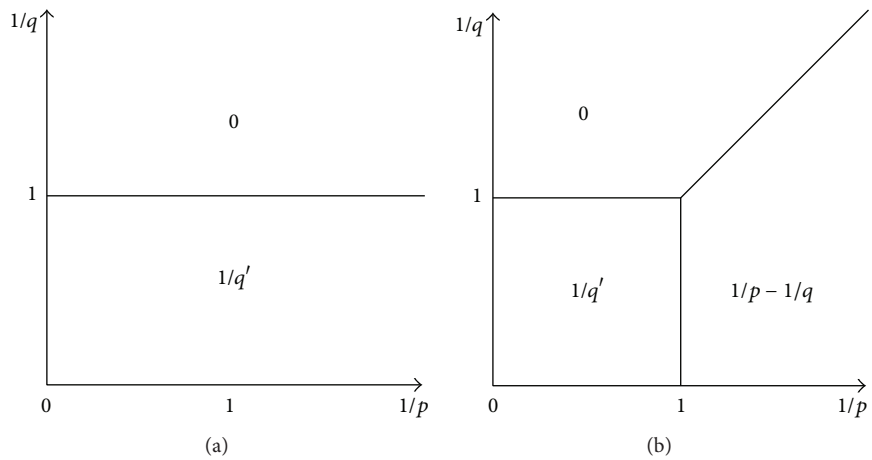

Figure 1: Comparison between the critical regularity index s for $\mathbb{T}W^{p,q}_{s+\epsilon}(\mathbb{R}^n) = W^{p,q}(\mathbb{R}^{n-1})$ (a) and $\mathbb{T}M^{p,q}_{s+\epsilon}(\mathbb{R}^n) = M^{p,q}(\mathbb{R}^{n-1})$ (b).

sequel, it suffices to consider only the case $j = 1$. Moreover, we write $\tilde{\square}_l f := \square_{\overline{l\pm e_1},l} f$ for some ψ_l satisfying (9). Using (34) we have

$$\left(\sum_{l_n\in\mathbb{Z}}\sum_{\bar{l}\in\mathbb{Z}^{n-1}}\langle\bar{l}\rangle^{sq}\left|\square_{\overline{l\pm e_1},l} f(\bar{x},0)\right|^q\right)^{1/q} \lesssim \left(\sum_{l_n\in\mathbb{Z}}\sum_{\bar{l}\in\mathbb{Z}^{n-1}}\langle\bar{l}\rangle^{sq}\left|\tilde{\square}^*_l f(\bar{x},x_n)\right|^q\right)^{1/q}. \tag{37}$$

Combining (36) and (37), then taking the $L^p(\mathbb{R}^{n-1})$-norm, and raising to pth power give

$$\|f(\bar{x},0)\|^p_{W^{p,q}_s(\mathbb{R}^{n-1})} \lesssim \left\|\left(\sum_{l\in\mathbb{Z}^n}\langle\bar{l}\rangle^{sq}\tilde{\square}^{*q}_l f(\bar{x},x_n)\right)^{1/q}\right\|^p_{L^p(\mathbb{R}^{n-1})}. \tag{38}$$

Integrating over $x_n \in [0,1]$,

$$\begin{aligned}\|f(\bar{x},0)\|_{W^{p,q}_s(\mathbb{R}^{n-1})} &\lesssim \left\|\left(\sum_{l\in\mathbb{Z}^n}\langle\bar{l}\rangle^{sq}\tilde{\square}^{*q}_l f(\bar{x},x_n)\right)^{1/q}\right\|_{L^p(\mathbb{R}^n)} \\ &\lesssim \|f\|_{W^{p,q,q}_s(\mathbb{R}^n)}.\end{aligned} \tag{39}$$

Note that the last inequality follows from Proposition 9.

For $1 \le q \le \infty$, we use Minkowski's inequality to give an upper bound of (35) as follows:

$$\begin{aligned}&\left(\sum_{\bar{k}\in\mathbb{Z}^{n-1}}\langle\bar{k}\rangle^{sq}\left|\square_{\bar{k}}(\mathbb{T}f)(\bar{x})\right|^q\right)^{1/q} \\ &\lesssim \left(\sum_{\bar{k},\bar{l}\in\mathbb{Z}^{n-1}}\langle\bar{k}\rangle^{sq}\left(\sum_{l_n\in\mathbb{Z}}\chi_{(|\bar{k}-\bar{l}|\le 1)}\square_{\bar{k},l} f(\bar{x},0)\right)^q\right)^{1/q} \\ &\lesssim \sum_{l_n\in\mathbb{Z}}\left(\sum_{\bar{k},\bar{l}\in\mathbb{Z}^{n-1}}\langle\bar{k}\rangle^{sq}\chi_{(|\bar{k}-\bar{l}|\le 1)}\left|\square_{\bar{k},l} f(\bar{x},0)\right|^q\right)^{1/q} \\ &\lesssim \sum_{l_n\in\mathbb{Z}}\left(\sum_{\bar{l}\in\mathbb{Z}^{n-1}}\langle\bar{l}\rangle^{sq}\left|\tilde{\square}^*_l f(\bar{x},x_n)\right|^q\right)^{1/q}.\end{aligned} \tag{40}$$

Repeating the arguments above on (40) gives us the estimate

$$\|f(\bar{x},0)\|_{W^{p,q}_s(\mathbb{R}^{n-1})} \lesssim \|f\|_{W^{p,q,1}_s(\mathbb{R}^n)}. \tag{41}$$

Hence, we arrive to our desired estimates.

Let $\eta' \in \mathcal{S}(\mathbb{R})$ be a function with $\operatorname{supp}\eta' \subset (-1/4, 1/4)$ and $(\mathcal{F}^{-1}_{\xi_n})\eta'(0) = 1$. For any $f \in W^{p,q}_s(\mathbb{R}^{n-1})$, we define $g(x) = (\mathbb{T}^{-1}f)(x) := [(\mathcal{F}^{-1}_{\xi_n})\eta'(x_n)]f(\bar{x})$. We easily see that $g(\bar{x},0) = f(\bar{x})$ and $\square_k g = 0$ when $|k_n| \ge 3$. Moreover, we can decompose $\square_k g = \square_{\bar{k}} f \cdot \square_{k_n}(\mathcal{F}^{-1}_{\xi_n}\eta')$ due to the way φ_k is defined in (9). Now we do an estimate

$$\begin{aligned}\|g\|_{W^{p,q,q\wedge 1}_s(\mathbb{R}^n)} &= \left\|\left(\sum_{k_n\in\mathbb{Z}}\left(\sum_{\bar{k}\in\mathbb{Z}^{n-1}}\langle\bar{k}\rangle^{sq}|\square_k g|^q\right)^{q\wedge 1/q}\right)^{1/q\wedge 1}\right\|_{L^p(\mathbb{R}^n)} \\ &= \left\|\left(\sum_{\bar{k}\in\mathbb{Z}^{n-1}}\langle\bar{k}\rangle^{sq}|\square_{\bar{k}} f|^q\right)^{1/q}\right. \\ &\quad\left.\cdot\left(\sum_{|k_n|\le 2}\left|\square_{k_n}(\mathcal{F}^{-1}_{\xi_n}\eta')\right|^{1\wedge q}\right)^{1/q\wedge 1}\right\|_{L^p(\mathbb{R}^n)} \\ &\lesssim \|f\|_{W^{p,q}_s(\mathbb{R}^{n-1})}.\end{aligned} \tag{42}$$

Thus, $\mathbb{T}^{-1} : W^{p,q}_s(\mathbb{R}^{n-1}) \to W^{p,q,q\wedge 1}_s(\mathbb{R}^n)$. □

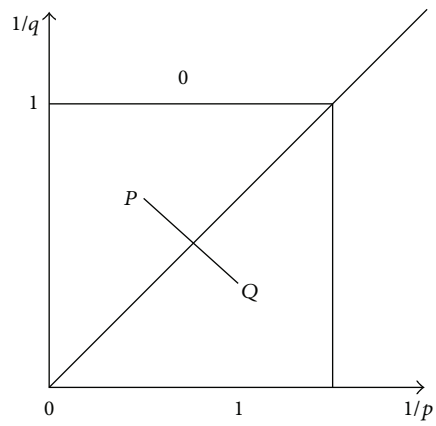

Figure 2: Contradiction argument using interpolation.

As the end of this paper, we discuss the optimality of Corollary 3. We recall the counterexample given in [13]. For $1 < p, q < \infty$, there exists a function which shows

$$\mathbb{T} : M^{p,q}_{1/q'}\left(\mathbb{R}^n\right) \nrightarrow M^{p,q}_0\left(\mathbb{R}^{n-1}\right). \tag{43}$$

Since $M^{q,q} = W^{q,q}$, we also have $\mathbb{T} : W^{q,q}_{1/q'}(\mathbb{R}^n) \nrightarrow W^{q,q}_0(\mathbb{R}^{n-1})$. Hence, Corollary 3 is sharp for $p = q$, $1 < p, q < \infty$ (refer to Figure 1). We now claim that it is also sharp for all $1 < p, q < \infty$. Contrary to our claim, suppose $s = 1/q'$ implies $\mathbb{T}W^{p,q}_s(\mathbb{R}^n) = W^{p,q}(\mathbb{R}^{n-1})$. Then, by interpolation with the estimate for a point $Q(p_1, q_1)$ with $s = 1/q_1'$, one would obtain an improvement for the segment connecting $P(p, q)$ and $Q(p_1, q_1)$ (refer to Figure 2), which is not possible.

Competing Interests

The authors declare that they have no competing interests.

Acknowledgments

This work was supported by JSPS, through "Program to Disseminate Tenure Tracking System." Yohei Tsutsui was also partially supported by JSPS, through Grant-in-Aid for Young Scientists (B) (no. 15K20919).

References

[1] H. G. Feichtinger, "Banach convolution algebras of wiener's type," in *Proceedings of the Conference Function, Series, Operators*, Colloquia Mathematica Societatis János Bolyai, pp. 509–524, North-Holland, 1980.

[2] H. G. Feichtinger, "Banach spaces of distributions of Wiener's type and interpolation," in *Functional Analysis and Approximation*, vol. 60, pp. 153–165, Birkhäuser, Basel, Switzerland, 1981.

[3] H. G. Feichtinger, "Modulation spaces on locally compact abelian groups," Tech. Rep., University of Vienna, Vienna, Austria, 1983.

[4] R. Bělohlávek, *Fuzzy Relational Systems: Foundations and Principles*, Kluwer Academic/Plenum Press, New York, NY, USA, 2002.

[5] A. Benyi and T. Oh, "Modulation spaces, Wiener amalgam spaces, and Brownian motions," *Advances in Mathematics*, vol. 228, no. 5, pp. 2943–2981, 2011.

[6] E. Cordero and F. Nicola, "Strichartz estimates in Wiener amalgam spaces for the Schrödinger equation," *Mathematische Nachrichten*, vol. 281, no. 1, pp. 25–41, 2008.

[7] E. Cordero and F. Nicola, "Remarks on Fourier multipliers and applications to the wave equation," *Journal of Mathematical Analysis and Applications*, vol. 353, no. 2, pp. 583–591, 2009.

[8] J. Cunanan, M. Sugimoto, and M. Kobayashi, "Inclusion relations between L^p-Sobolev and Wiener amalgam spaces," *Journal of Functional Analysis*, vol. 419, no. 2, pp. 738–747, 2014.

[9] K. Grochenig and C. Heil, "Modulation spaces and pseudodifferential operators," *Integral Equations and Operator Theory*, vol. 34, no. 4, pp. 439–457, 1999.

[10] B. Wang and C. Huang, "Frequency-uniform decomposition method for the generalized BO, KdV and NLS equations," *Journal of Differential Equations*, vol. 239, no. 1, pp. 213–250, 2007.

[11] L. C. Evans, *Partial Differential Equations*, American Mathematical Society, Providence, RI, USA, 1998.

[12] H. Triebel, *Theory of Function Spaces*, Birkhäauser, Boston, Mass, USA, 1983.

[13] H. G. Feichtinger, C. Huang, and B. Wang, "Trace operators for modulation, α-modulation and Besov spaces," *Applied and Computational Harmonic Analysis*, vol. 30, no. 1, pp. 110–127, 2011.

[14] H. Triebel, "Modulation spaces on the Euclidean n-space," *Zeitschrift für Analysis und ihre Anwendungen*, vol. 2, no. 5, pp. 443–457, 1983.

[15] M. Frazier and B. Jawerth, "Decomposition of Besov spaces," *Indiana University Mathematics Journal*, vol. 34, no. 4, pp. 777–799, 1985.

[16] H. Triebel, *Theory of Function Spaces II*, Birkhäuser, Basel, Switzerland, 1992.

[17] H. G. Feichtinger, "Generalized amalgams, with applications to Fourier transform," *Canadian Journal of Mathematics*, vol. 42, no. 3, pp. 395–409, 1990.

[18] J. Toft, "Continuity properties for modulation spaces, with applications to pseudo-differential calculus-II," *Annals of Global Analysis and Geometry*, vol. 26, no. 1, pp. 73–106, 2004.