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Research Article

Singularities for One-Parameter Null Hypersurfaces of Anti-de Sitter Spacelike Curves in Semi-Euclidean Space

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We consider one-parameter null hypersurfaces associated with spacelike curves. The spacelike curves are in anti-de Sitter 3-space while one-parameter null hypersurfaces lie in 4-dimensional semi-Euclidean space with index 2. We classify the generic singularities of the hypersurfaces, which are cuspidal edges and swallowtails. And we reveal the geometric meanings of the singularities of such hypersurfaces by the singularity theory.

1. Introduction

Semi-Euclidean space is a vector space with pseudoscalar product which is different from Euclidean space. The study of semi-Euclidean space has produced fruitful results; please see [1–5]. It is well known that there exist spacelike submanifolds, timelike submanifolds, and null submanifolds in semi-Euclidean space. Null submanifolds appear in many physics papers. For example, the null submanifolds are of interest because they provide models of different horizon types such as event horizons of Kerr black holes, Cauchy horizons, isolated horizons, Kruskal horizons, and Killing horizons [6–12]. Null submanifolds are also studied in the theory of electromagnetism.

Anti-de Sitter space is a maximally symmetric semi-Riemannian manifold with constant negative scalar curvature. This space is a very important subject in physics; it is also one of the vacuum solutions of Einstein's field equation in the theory of relativity. There is a conjecture in physics that the classical gravitation theory on anti-de Sitter space is equivalent to the conformal field theory on the ideal boundary of anti-de Sitter space. It is called the AdS/CFT correspondence or the holographic principle by E. Witten. In mathematics this conjecture is that the extrinsic geometric properties on submanifolds in anti-de Sitter space have corresponding Gauge theoretic geometric properties in its ideal boundary. Therefore, it is necessary to investigate the submanifolds in anti-de Sitter space. During the last four decades, singularity theory has enjoyed rapid development. The French mathematician R. Thom (Fields medallist) first put forward the philosophical idea of applying singularity theory to the study of differential geometry. Porteous applied the thoughts of Thom to the study of Euclidean geometry [13]. The first attempts to apply the singularity theory to non-Euclidean geometry were undertaken by S. Izumiya, the second author, and T. Sano et al.

Recently there appear several results on submanifolds in anti-de Sitter space from the viewpoint of singularity theory. The timelike hypersurfaces are studied in the anti-de Sitter space from the viewpoint of Lagrangian singularity theory [14]. In the study of submanifolds, the null submanifolds happen to be the most interesting subjects, both from the viewpoint of singularity theory and the theory of relativity [15, 16]. Fusho and Izumiya have discussed the spacelike curves in de Sitter 3-space [17]; they define the null surfaces of spacelike curves. The spacelike curves have degenerate contact with null cones at the singularities of the null surfaces. In [18], L. Chen, Q. Han, the second author, and W. Sun consider null ruled surfaces along spacelike curves in anti-de Sitter 3-space. They give the classifications of singularities of the ruled surfaces which are the codimensional two submanifolds in semi-Euclidean space with index 2. Null surfaces have been

studied in preceding literature. As we all know, the horizon of the black hole is a null hypersurface or a part. However, to the best of the authors' knowledge, no literature exists regarding the singularities of one-parameter null hypersurfaces as they relate to spacelike curves in anti-de Sitter 3-space. Thus, the current study hopes to serve such a need. Therefore, in this paper, we stick to the one-parameter null hypersurfaces, which are generated by spacelike curves in anti-de Sitter 3-space. When the parameter is fixed, the sections of one-parameter null hypersurfaces are null surfaces. Moreover, the null ruled surfaces in [18] are the sections of one-parameter null hypersurfaces. And the one-parameter null hypersurfaces can be taken as the most elementary case for the study of the lowest codimensional submanifolds in semi-Euclidean space with index 2.

A singularity is a point at which a function blows up. It is a point at which a function is at a maximum/minimum or a submanifold is no longer smooth and regular. In [19], we have discussed the singularities of normal hypersurface associated with a timelike curve. In this paper we first consider spacelike curves in anti-de Sitter 3-space and then define the one-parameter null hypersurfaces which are bundles along spacelike curves whose fibres are null lines or timelike curves. We also define the one-parameter height functions on spacelike curves and apply the versal unfolding theory of functions to discuss them; the functions can be used to investigate the geometric properties of one-parameter null hypersurfaces. In fact, one-parameter null hypersurfaces are the discriminant sets of these functions (the discriminant sets of one-parameter height functions are precisely the wavefronts of spacelike curves); the singularities of null hypersurfaces are A_k-singularities ($k \geq 2$) of these functions. The main result in this paper is Theorem 5. This theorem characterizes the contact of spacelike curves with null cones in semi-Euclidean space with index 2.

A brief description of the organization of this paper is as follows. In Section 2, we review the concepts of submanifolds in semi-Euclidean space with index 2. In Section 3, we give one-parameter height functions of a spacelike curve, by which we can obtain several geometric invariants of the spacelike curve. We also get the singularities of one-parameter null hypersurfaces, and the geometric meaning of Theorem 5 is described in this section. The preparations for the proof of Theorem 5 are in Section 4. We give the proof of Theorem 5 in Section 5. In Section 6, we give an example to illustrate the results of Theorem 5.

We will assume throughout the whole paper that all manifolds and maps are C^∞ unless the contrary is stated.

2. Preliminaries

Let $\mathbb{R}^4$ be a 4-dimensional vector space. For any two vectors $\mathbf{x} = (x_1, x_2, x_3, x_4)$ and $\mathbf{y} = (y_1, y_2, y_3, y_4)$ in $\mathbb{R}^4$, their pseudoscalar product is defined by

$$\langle \mathbf{x}, \mathbf{y} \rangle = -x_1 y_1 - x_2 y_2 + x_3 y_3 + x_4 y_4. \tag{1}$$

The space $(\mathbb{R}^4, \langle , \rangle)$ is called *semi-Euclidean* 4*-space with index* 2 and denoted by $\mathbb{R}^4_2$.

For three vectors $\mathbf{x} = (x_1, x_2, x_3, x_4)$, $\mathbf{y} = (y_1, y_2, y_3, y_4)$, and $\mathbf{z} = (z_1, z_2, z_3, z_4) \in \mathbb{R}^4_2$, we define a vector $\mathbf{x} \wedge \mathbf{y} \wedge \mathbf{z}$ by

$$\mathbf{x} \wedge \mathbf{y} \wedge \mathbf{z} = \begin{vmatrix} -\mathbf{e_1} & -\mathbf{e_2} & \mathbf{e_3} & \mathbf{e_4} \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{vmatrix}, \tag{2}$$

where $\{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}, \mathbf{e_4}\}$ is the canonical basis of $\mathbb{R}^4_2$. We have $\langle \mathbf{x}_0, \mathbf{x} \wedge \mathbf{y} \wedge \mathbf{z} \rangle = \det(\mathbf{x}_0, \mathbf{x}, \mathbf{y}, \mathbf{z})$, so $\mathbf{x} \wedge \mathbf{y} \wedge \mathbf{z}$ is pseudoorthogonal to $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$. A nonzero vector $\mathbf{x} \in \mathbb{R}^4_2$ is called *spacelike*, *null,* or *timelike* if $\langle \mathbf{x}, \mathbf{x} \rangle > 0$, $\langle \mathbf{x}, \mathbf{x} \rangle = 0$, or $\langle \mathbf{x}, \mathbf{x} \rangle < 0$, respectively. The norm of $\mathbf{x} \in \mathbb{R}^4_2$ is defined by $\|\mathbf{x}\| = (\operatorname{sign}(\mathbf{x}) \langle \mathbf{x}, \mathbf{x} \rangle)^{1/2}$, where $\operatorname{sign}(\mathbf{x})$ denotes the signature of $\mathbf{x}$ which is given by $\operatorname{sign}(\mathbf{x}) = 1, 0$ or -1 if $\mathbf{x}$ is a spacelike, null, or timelike vector, respectively.

Let $\boldsymbol{\gamma} : I \to \mathbb{R}^4_2$ be a regular curve in $\mathbb{R}^4_2$ (i.e., $\dot{\boldsymbol{\gamma}}(t) = d\boldsymbol{\gamma}/dt \neq 0$), where I is an open interval. For any $t \in I$, the curve $\boldsymbol{\gamma}$ is called *spacelike*, *null,* or *timelike* if $\langle \dot{\boldsymbol{\gamma}}(t), \dot{\boldsymbol{\gamma}}(t) \rangle > 0$, $\langle \dot{\boldsymbol{\gamma}}(t), \dot{\boldsymbol{\gamma}}(t) \rangle = 0$, or $\langle \dot{\boldsymbol{\gamma}}(t), \dot{\boldsymbol{\gamma}}(t) \rangle < 0$, respectively. We call $\boldsymbol{\gamma}$ a *nonnull curve* if $\boldsymbol{\gamma}$ is a spacelike or timelike curve. The arc-length of a nonnull curve $\boldsymbol{\gamma}$ measured from $\boldsymbol{\gamma}(t_0)(t_0 \in I)$ is $s(t) = \int_{t_0}^{t} \|\dot{\boldsymbol{\gamma}}(t)\| dt$.

The parameter s is determined such that $\|\boldsymbol{\gamma}'(s)\| = 1$ for the nonnull curve, where $\boldsymbol{\gamma}'(s) = d\boldsymbol{\gamma}/ds$ is the unit tangent vector of $\boldsymbol{\gamma}$. Some submanifolds in $\mathbb{R}^4_2$ are as follows.

The *anti-de Sitter space* is defined by

$$H^3_1 = \left\{ \mathbf{x} \in \mathbb{R}^4_2 \mid \langle \mathbf{x}, \mathbf{x} \rangle = -1 \right\} \tag{3}$$

and the *lightlike cone* by

$$\mathrm{LC}_p = \left\{ \mathbf{x} \in \mathbb{R}^4_2 \mid \langle \mathbf{x} - \mathbf{p}, \mathbf{x} - \mathbf{p} \rangle = 0 \right\}. \tag{4}$$

We also define the *one-parameter anti-de Sitter space* by

$$H^3_1\left(-\sinh^2\theta\right) = \left\{ \mathbf{x} \in \mathbb{R}^4_2 \mid \langle \mathbf{x}, \mathbf{x} \rangle = -\sinh^2\theta \right\}. \tag{5}$$

Let $\boldsymbol{\gamma} : I \to H^3_1$ be a spacelike regular curve; that is, $\boldsymbol{\gamma}$ satisfies $\langle \dot{\boldsymbol{\gamma}}(t), \dot{\boldsymbol{\gamma}}(t) \rangle > 0$, $t \in I$. Since the curve $\boldsymbol{\gamma}$ is spacelike, we can reparametrize it by the arc-length s. Then we have the tangent vector $\mathbf{t}(s) = \boldsymbol{\gamma}'(s)$; obviously $\|\mathbf{t}(s)\| = 1$. When $\langle \mathbf{t}'(s), \mathbf{t}'(s) \rangle \neq -1$, we define a unit vector:

$$\mathbf{n}(s) = \frac{\mathbf{t}'(s) - \boldsymbol{\gamma}(s)}{\|\mathbf{t}'(s) - \boldsymbol{\gamma}(s)\|}. \tag{6}$$

Let $\mathbf{e}(s) = \boldsymbol{\gamma}(s) \wedge \mathbf{t}(s) \wedge \mathbf{n}(s)$; then we have a pseudoorthonormal frame $\{\boldsymbol{\gamma}(s), \mathbf{t}(s), \mathbf{n}(s), \mathbf{e}(s)\}$ of $\mathbb{R}^4_2$ along $\boldsymbol{\gamma}$. By direct calculating, the following Frenet-Serret type is displayed, under the assumption that $\langle \mathbf{t}'(s), \mathbf{t}'(s) \rangle \neq -1$:

$$\begin{aligned} \boldsymbol{\gamma}'(s) &= \mathbf{t}(s) \\ \mathbf{t}'(s) &= \boldsymbol{\gamma}(s) + \kappa_g(s)\, \mathbf{n}(s) \\ \mathbf{n}'(s) &= -\delta\kappa_g(s)\, \mathbf{t}(s) + \delta\tau_g(s)\, \mathbf{e}(s) \\ \mathbf{e}'(s) &= \delta\tau_g(s)\, \mathbf{n}(s). \end{aligned} \tag{7}$$

Here, $\kappa_g(s) = \|\mathbf{t}'(s) - \boldsymbol{\gamma}(s)\|$ is the *geodesic curvature*,

$$\tau_g(s) = -\kappa_g^{-2}(s)\det\left(\boldsymbol{\gamma}(s), \boldsymbol{\gamma}'(s), \boldsymbol{\gamma}''(s), \boldsymbol{\gamma}'''(s)\right) \quad (8)$$

is the *geodesic torsion*, and $\delta = \operatorname{sign}(n(s))$. If $\langle \mathbf{t}'(s), \mathbf{t}'(s)\rangle = -1$, we can obtain $\kappa_g(s) = 0$; it means that $\boldsymbol{\gamma}(s)$ is a geodesic curve in H_1^3. We consider $\langle \mathbf{t}'(s), \mathbf{t}'(s)\rangle \neq -1$ (i.e., $\kappa_g(s) \neq 0$) in the following sections.

Let $\boldsymbol{\gamma} : I \to H_1^3$ be a unit speed spacelike curve; we write $\boldsymbol{\gamma}_\theta = \sinh\theta\boldsymbol{\gamma}$ and define $L_\theta^\pm : I \times \mathbb{R} \to H_1^3(-\sinh^2\theta)$ by

$$\begin{aligned} L_\theta^\pm(s,\mu) &= \sinh\theta\boldsymbol{\gamma}(s) + \mu(\mathbf{n}(s) \pm \mathbf{e}(s)) \\ &= \boldsymbol{\gamma}_\theta + \mu(\mathbf{n}(s) \pm \mathbf{e}(s)), \quad \theta \in (0, +\infty). \end{aligned} \quad (9)$$

We call $L_\theta^\pm(s,\mu)$ the *one-parameter null hypersurfaces* associated with $\boldsymbol{\gamma}(s)$. We also define the following model surface. For any $\mathbf{v}_0 \in H_1^3(-\sinh^2\theta)$,

$$\mathrm{LC}_{v_0}(\theta) = \left\{\mathbf{u} \in H_1^3\left(-\sinh^2\theta\right) \mid \langle \mathbf{u}, \mathbf{v}_0\rangle = -\sinh^2\theta\right\}. \quad (10)$$

On the other hand, let $F : H_1^3 \to \mathbb{R}$ be a submersion and let $\boldsymbol{\gamma} : I \to H_1^3$ be a spacelike curve. We say that $\boldsymbol{\gamma}$ and $F^{-1}(0)$ have k-point contact at $t = t_0$ if the function $g(t) = F \circ \boldsymbol{\gamma}(t)$ satisfies $g(t_0) = g'(t_0) = \cdots = g^{(k-1)}(t_0) = 0$, $g^{(k)}(t_0) \neq 0$. We also have that $\boldsymbol{\gamma}$ and $F^{-1}(0)$ have at least k-point contact at $t = t_0$ if the function $g(t) = F\circ\boldsymbol{\gamma}(t)$ satisfies $g(t_0) = g'(t_0) = \cdots = g^{(k-1)}(t_0) = 0$.

3. One-Parameter Height Functions and the Singularities of One-Parameter Null Surfaces

In this section we discuss a kind of Lorentzian invariant functions on a spacelike curve in H_1^3. It is useful to study the null hypersurfaces of the spacelike curve. Let $\boldsymbol{\gamma} : I \to H_1^3$ be a unit spacelike curve. We now define a function

$$H : I \times H_1^3\left(-\sinh^2\theta\right) \times (0, +\infty) \longrightarrow \mathbb{R} \quad (11)$$

by $H(s, \mathbf{v}, \theta) = \langle \boldsymbol{\gamma}_\theta(s), \mathbf{v}\rangle + \sinh^2\theta$; we call H *one-parameter height function* on the spacelike curve $\boldsymbol{\gamma}$. We denote that $h_{v,\theta}(s) = H(s, \mathbf{v}, \theta)$, $(\mathbf{v}, \theta) \in H_1^3(-\sinh^2\theta) \times (0, +\infty)$. Then, we have the following proposition.

Proposition 1. *Let $\boldsymbol{\gamma} : I \to H_1^3$ be a unit spacelike curve and $(\mathbf{v}, \theta) \in H_1^3(-\sinh^2\theta) \times (0, +\infty)$. Then one has the following:*

(1) *$h_{v,\theta}(s) = h'_{v,\theta}(s) = 0$ if and only if there exists $\mu \in \mathbb{R}$ such that $\mathbf{v} = \sinh\theta\boldsymbol{\gamma}(s) + \mu(\mathbf{n}(s) \pm \mathbf{e}(s))$.*

(2) *$h_{v,\theta}(s) = h'_{v,\theta}(s) = h''_{v,\theta}(s) = 0$ if and only if*

$$\mathbf{v} = \sinh\theta\boldsymbol{\gamma}(s) + \frac{\sinh\theta}{\delta\kappa_g(s)}(\mathbf{n}(s) \pm \mathbf{e}(s)). \quad (12)$$

(3) *$h_{v,\theta}(s) = h'_{v,\theta}(s) = h''_{v,\theta}(s) = h'''_{v,\theta}(s) = 0$ if and only if*

$$\mathbf{v} = \sinh\theta\boldsymbol{\gamma}(s) + \frac{\sinh\theta}{\delta\kappa_g(s)}(\mathbf{n}(s) \pm \mathbf{e}(s)) \quad (13)$$

and $\sigma = 0$.

(4) *$h_{v,\theta}(s) = h'_{v,\theta}(s) = h''_{v,\theta}(s) = h'''_{v,\theta}(s) = h^{(4)}_{v,\theta}(s) = 0$ if and only if*

$$\mathbf{v} = \sinh\theta\boldsymbol{\gamma}(s) + \frac{\sinh\theta}{\delta\kappa_g(s)}(\mathbf{n}(s) \pm \mathbf{e}(s)) \quad (14)$$

and $\sigma = \sigma' = 0$.

Proof. (1) Since $\mathbf{v} \in H_1^3(-\sinh^2\theta)$, we can find that $\eta, \alpha, \mu, \beta \in \mathbb{R}$ with $-\eta^2 + \alpha^2 + \delta\mu^2 - \delta\beta^2 = -\sinh^2\theta$ such that $\mathbf{v} = \eta\boldsymbol{\gamma}(s) + \alpha\mathbf{t}(s) + \mu\mathbf{n}(s) + \beta\mathbf{e}(s)$. Because

$$h_{v,\theta}(s) = \langle \sinh\theta\boldsymbol{\gamma}(s), \mathbf{v}\rangle + \sinh^2\theta = 0, \quad (15)$$

we can get $\eta = \sinh\theta$; when $h'_{v,\theta}(s) = 0$, it means that $\alpha = 0$ and $\beta = \pm\mu$. Therefore $\mathbf{v} = \sinh\theta\boldsymbol{\gamma}(s) + \mu(\mathbf{n}(s) \pm \mathbf{e}(s))$; the converse direction also holds.

(2) By (1), an easy computation shows that

$$\begin{aligned} \sinh\theta\left\langle \mathbf{t}'(s), \mathbf{v}\right\rangle &= \sinh\theta\left\langle \boldsymbol{\gamma}(s) + \kappa_g(s)\mathbf{n}(s), \mathbf{v}\right\rangle \\ &= \sinh\theta\left\langle \boldsymbol{\gamma}(s) + \kappa_g(s)\mathbf{n}(s), \sinh\theta\boldsymbol{\gamma}(s) \right. \\ &\quad \left. + \mu(\mathbf{n}(s) \pm \mathbf{e}(s))\right\rangle = 0; \end{aligned} \quad (16)$$

we get $\mu = \sinh\theta/\delta\kappa_g(s)$; therefore

$$\mathbf{v} = \sinh\theta\boldsymbol{\gamma}(s) + \left(\frac{\sinh\theta}{\left(\delta\kappa_g(s)\right)}\right)(\mathbf{n}(s) \pm \mathbf{e}(s)). \quad (17)$$

(3) Under the assumption that

$$\begin{aligned} &h_{v,\theta}(s) = h'_{v,\theta}(s) = h''_{v,\theta}(s) = 0, \\ &\frac{h'''_{v,\theta}(s)}{\sinh\theta} = \left\langle\left(1 - \delta\kappa_g^2(s)\right)\mathbf{t}(s) + \kappa'_g(s)\mathbf{n}(s) \right. \\ &\qquad \left. + \delta\kappa_g(s)\tau_g(s), \mathbf{v}\right\rangle, \end{aligned} \quad (18)$$

we can get $\sigma(s) = \kappa'_g(s) \mp \kappa_g(s)\tau_g(s) = 0$; assertion (3) follows.

(4) Based on the assumption that

$$h_{v,\theta}(s) = h'_{v,\theta}(s) = h''_{v,\theta}(s) = h'''_{v,\theta}(s) = 0, \quad (19)$$

the relation

$$\begin{aligned} \frac{h^{(4)}_{v,\theta}(s)}{\sinh\theta} &= \left\langle\left(1 - \delta\kappa_g^2(s)\right)\mathbf{t}(s) + \kappa'_g(s)\mathbf{n}(s) \right. \\ &\quad \left. - \delta\kappa_g(s)\tau_g(s)\mathbf{e}(s), \mathbf{v}\right\rangle' \end{aligned} \quad (20)$$

follows the fact that $h^{(4)}_{v,\theta}(s) = 0$ is equivalent to

$$\kappa''_g(s) \mp \kappa'_g(s)\tau_g(s) \mp \kappa_g(s)\tau'_g(s) = 0, \quad (21)$$

so $\sigma' = 0$. This proves assertion (4). □

Now, we do research on some properties of one-parameter null hypersurfaces of the spacelike curve in H_1^3. As we can know the functions $\kappa_g(s)$, $\tau_g(s)$, and $\sigma(s)$ have particular meanings. Here, we consider the case when the one-parameter null hypersurfaces have the most degenerate singularities. We have the following proposition.

Proposition 2. *Let $\boldsymbol{\gamma} : I \to H_1^3$ be a unit spacelike curve. Then one has the following:*

(1) *The set $\{L_\theta^\pm(s,\mu) \mid \mu = \sinh\theta/\delta\kappa_g(s)\}$ is the singularities of one-parameter null hypersurfaces $L_\theta^\pm(s,\mu)$.*

(2) *If $\mathbf{v}_0 = L_\theta^\pm(s, \sinh\theta/\delta\kappa_g(s))$ is a constant vector, one has $\boldsymbol{\gamma}_\theta(s) \in LC_{v_0}(\theta)$ for any $s \in I$; at the same time $\sigma(s) = 0$.*

Proof. By calculations we have

$$\frac{\partial L_\theta^\pm(s,\mu)}{\partial\theta} = -\cosh\theta\boldsymbol{\gamma}(s), \tag{22}$$

$$\frac{\partial L_\theta^\pm(s,\mu)}{\partial\mu} = \mathbf{e}(s) \pm \mathbf{n}(s), \tag{23}$$

$$\begin{aligned}\frac{\partial L_\theta^\pm(s,\mu)}{\partial s} &= \sinh\theta\mathbf{t}(s) \\ &\quad + \mu\left(-\delta\kappa_g(s)\mathbf{t}(s) + \delta\tau_g(s)\mathbf{e}(s) \pm \delta\tau_g(s)\mathbf{n}(s)\right) \\ &= \left(\sinh\theta - \delta\mu\kappa_g(s)\right)\mathbf{t}(s) \\ &\quad + \delta\mu\tau_g(s)(\mathbf{e}(s) \pm \mathbf{n}(s)).\end{aligned} \tag{24}$$

(1) If the above three vectors are linearly dependent, we can get the singularities of $L_\theta^\pm(s,\mu)$ if and only if $\sinh\theta - \delta\mu\kappa_g(s) = 0$. Therefore, assertion (1) holds.

(2) For any fixed $\theta \in (0,+\infty)$, if

$$f(s) = \sinh\theta\boldsymbol{\gamma}(s) + \mu(s)(\mathbf{n}(s) \pm \mathbf{e}(s)) \tag{25}$$

is a constant, then

$$\begin{aligned}\frac{df}{ds} &= \left(\sinh\theta - \delta\mu(s)\kappa_g(s)\right)\mathbf{t}(s) \\ &\quad + \left(\mu'(s) - \delta\mu(s)\tau_g(s)\right)(\mathbf{n}(s) \pm \mathbf{e}(s)) = 0.\end{aligned} \tag{26}$$

Since

$$\mu(s) = \frac{\sinh\theta}{\delta\kappa_g(s)}, \quad \mu'(s) - \delta\mu(s)\tau_g(s) = 0, \tag{27}$$

then

$$\sigma = \kappa_g'(s) \mp \kappa_g(s)\tau_g(s) = 0. \tag{28}$$

We have

$$\begin{aligned}&\langle\boldsymbol{\gamma}_\theta(s), \mathbf{v}_0\rangle \\ &\quad = \left\langle \boldsymbol{\gamma}_\theta(s), \sinh\theta\boldsymbol{\gamma}(s) + \frac{\sinh\theta}{\delta\kappa_g(s)}(\mathbf{n}(s) \pm \mathbf{e}(s))\right\rangle \\ &\quad = -\sinh^2\theta.\end{aligned} \tag{29}$$

This completes the proof. □

4. Unfoldings of One-Parameter Height Functions

In this section we classify singularities of the one-parameter null hypersurfaces along $\boldsymbol{\gamma}$ as an application of the unfolding theory of functions.

Let $F : (\mathbb{R} \times \mathbb{R}^r, (s_0, \mathbf{x}_0)) \to \mathbb{R}$ be a function germ; $f(s) = F_{x_0}(s, \mathbf{x}_0)$. We call F an *r-parameter unfolding* of f. If $f^{(p)}(s_0) = 0$ for all $1 \le p \le k$ and $f^{(k+1)}(s_0) \ne 0$, we say that f has A_k-singularity at s_0. We also say that f has $A_{\ge k}$-singularity at s_0 if $f^{(p)}(s_0) = 0$ for all $1 \le p \le k$. Let F be an r-parameter unfolding of f and f has A_k-singularity ($k \ge 1$) at s_0; we define the $(k-1)$-jet of the partial derivative $\partial F/\partial x_i$ at s_0 as

$$j^{(k-1)}\frac{\partial F}{\partial x_i}(s, \mathbf{x}_0)(s_0) = \sum_{j=1}^{k+1}\alpha_{ji}(s - s_0)^j, \quad (i = 1, \ldots, r). \tag{30}$$

If the rank of $k \times r$ matrix $(\alpha_{0i}, \alpha_{ji})$ is k ($k \le r$), then F is called a *versal unfolding* of f, where $\alpha_{0i} = (\partial F/\partial x_i)(s_0, \mathbf{x}_0)$. The *discriminant set* of F is defined by

$$D_F = \left\{\mathbf{x} \in \mathbb{R}^r \mid \exists s \in \mathbb{R},\ F(s,\mathbf{x}) = \frac{\partial F}{\partial s}(s,\mathbf{x}) = 0\right\}. \tag{31}$$

There has been the following famous result [20].

Theorem 3. *Let $F : (\mathbb{R} \times \mathbb{R}^r, (s_0, \mathbf{x}_0)) \to \mathbb{R}$ be an r-parameter unfolding of $f(s)$ which has A_k-singularity at s_0; suppose that F is a versal unfolding of f. Then one has the following:*

(a) *If $k = 1$, then D_F is locally diffeomorphic to $\{0\} \times \mathbb{R}^{r-1}$.*

(b) *If $k = 2$, then D_F is locally diffeomorphic to $C \times \mathbb{R}^{r-2}$.*

(c) *If $k = 3$, then D_F is locally diffeomorphic to $SW \times \mathbb{R}^{r-3}$.*

By Proposition 1, the discriminant set of the timelike height function $H(s, \mathbf{v}, \theta)$ is given by

$$D_H = \{\sinh\theta\boldsymbol{\gamma}(s) + \mu(\mathbf{n}(s) \pm \mathbf{e}(s)) \mid s, \mu \in I,\ \theta \in (0,+\infty)\}. \tag{32}$$

Proposition 4. *Let $H(s, \mathbf{v}, \theta)$ be a one-parameter height function on the spacelike curve $\boldsymbol{\gamma}, \mathbf{v} \in D_H$. If h_v has A_k-singularity at s ($k = 1, 2, 3, 4$), then H is a versal unfolding of h_v.*

Proof. Let $\boldsymbol{\gamma}(s) = (x_1(s), x_2(s), x_3(s), x_4(s)) \in H_1^3$ and $\mathbf{v} = (v_1, v_2, v_3, v_4) \in H_1^3(-\sinh^2\theta)$.

Then

$$H(s, \mathbf{v}, \theta) = \sinh\theta\left(-x_1v_1 - x_2v_2 + x_3v_3 + x_4v_4 + \sinh\theta\right), \quad \theta \in (0, +\infty). \tag{33}$$

Let $v_1 = \pm\sqrt{-v_2^2 + v_3^2 + v_4^2 + \sinh^2\theta}$, so

$$\begin{aligned}
\frac{\partial H}{\partial v_2}(s, \mathbf{v}) &= \sinh\theta\left(-x_2 - \frac{v_2}{v_1}x_1\right),\\
\frac{\partial^2 H}{\partial s\partial v_2}(s, \mathbf{v}) &= \sinh\theta\left(-x_2' - \frac{v_2}{v_1}x_1'\right),\\
\frac{\partial^3 H}{\partial^2 s\partial v_2}(s, \mathbf{v}) &= \sinh\theta\left(-x_2'' - \frac{v_2}{v_1}x_1''\right),\\
\frac{\partial H}{\partial v_i}(s, \mathbf{v}) &= \sinh\theta\left(x_i + \frac{v_i}{v_1}x_1\right),\\
\frac{\partial^2 H}{\partial s\partial v_i}(s, \mathbf{v}) &= \sinh\theta\left(x_i' + \frac{v_i}{v_1}x_1'\right),\\
\frac{\partial^3 H}{\partial^2 s\partial v_i}(s, \mathbf{v}) &= \sinh\theta\left(x_i'' + \frac{v_i}{v_1}x_1''\right) \quad (i = 3, 4),\\
\frac{\partial H}{\partial \theta}(s, \mathbf{v}) &= \varphi_\theta(s),\\
\frac{\partial^2 H}{\partial s\partial \theta}(s, \mathbf{v}) &= \varphi_\theta'(s),\\
\frac{\partial^3 H}{\partial^2 s\partial \theta}(s, \mathbf{v}) &= \varphi_\theta''(s).
\end{aligned} \tag{34}$$

The 2-*jet* of $(\partial H/\partial v_2)(s, \mathbf{v})$ at s_0 is given by

$$\sinh\theta\left(-x_2's - \frac{1}{2}x_2''s^2\right) + \frac{v_2}{v_1}\left(-x_1's - \frac{1}{2}x_1''s^2\right); \tag{35}$$

the 2-*jet* of $(\partial H/\partial v_i)(s, \mathbf{v})$ at s_0 is given by

$$\sinh\theta\left(x_i's + \frac{1}{2}x_i''s^2\right) + \frac{v_i}{v_1}\left(x_1's + \frac{1}{2}x_1''s^2\right) \quad (i = 3, 4). \tag{36}$$

(1) By Proposition 1, h_v has the A_1-singularity at s_0 if and only if $\mathbf{v} = \sinh\theta\boldsymbol{\gamma}(s) + \mu(\mathbf{n}(s) \pm \mathbf{e}(s))$. Since the curve $\boldsymbol{\gamma}(s)$ is regular, the rank of

$$\left(-x_2 - \frac{v_2}{v_1}x_1, x_3 + \frac{v_3}{v_1}x_1, x_4 + \frac{v_4}{v_1}x_1, \frac{\varphi_\theta}{\sinh\theta}\right) \tag{37}$$

is 1.

(2) We can get that h has the A_2-singularity at s_0 if and only if

$$\mathbf{v} = \sinh\theta\boldsymbol{\gamma}(s) + \frac{\sinh\theta}{\delta\kappa_g(s)}(\mathbf{n}(s) \pm \mathbf{e}(s)) \tag{38}$$

and $\sigma \neq 0$. When h has the $A_{\geq 2}$-singularity at s_0, we require the 2×4 matrix

$$\begin{pmatrix} -x_2 - \frac{v_2}{v_1}x_1 & x_3 + \frac{v_3}{v_1}x_1 & x_4 + \frac{v_4}{v_1}x_1 & \frac{\varphi_\theta}{\sinh\theta}\\ -x_2' - \frac{v_2}{v_1}x_1 & x_3' + \frac{v_3}{v_1}x_1' & x_4' + \frac{v_4}{v_1}x_1' & \frac{\varphi_\theta'}{\sinh\theta}\end{pmatrix} \tag{39}$$

to have rank 2, which follows from the proof of the next case.

(3) It also follows from Proposition 1 that h has the $A_{\geq 3}$-singularity at s_0 if and only if

$$\mathbf{v} = \sinh\theta\boldsymbol{\gamma}(s) + \frac{\sinh\theta}{\delta\kappa_g(s)}(\mathbf{n}(s) \pm \mathbf{e}(s)) \tag{40}$$

and $\sigma = 0$, $\sigma' \neq 0$.

We require the 3×4 matrix

$$\begin{pmatrix} -x_2 - \frac{v_2}{v_1}x_1 & x_3 + \frac{v_3}{v_1}x_1 & x_4 + \frac{v_4}{v_1}x_1 & \frac{\varphi_\theta}{\sinh\theta}\\ -x_2' - \frac{v_2}{v_1}x_1' & x_3' + \frac{v_3}{v_1}x_1' & x_4' + \frac{v_4}{v_1}x_1' & \frac{\varphi_\theta'}{\sinh\theta}\\ -x_2'' - \frac{v_2}{v_1}x_1'' & x_3'' + \frac{v_3}{v_1}x_1'' & x_4'' + \frac{v_4}{v_1}x_1'' & \frac{\varphi_\theta''}{\sinh\theta}\end{pmatrix} \tag{41}$$

to have rank 3.

Let 3×3 matrix

$$A = \begin{pmatrix} -x_2 - \frac{v_2}{v_1}x_1 & x_3 + \frac{v_3}{v_1}x_1 & x_4 + \frac{v_4}{v_1}x_1\\ -x_2' - \frac{v_2}{v_1}x_1' & x_3' + \frac{v_3}{v_1}x_1' & x_4' + \frac{v_4}{v_1}x_1'\\ -x_2'' - \frac{v_2}{v_1}x_1'' & x_3'' + \frac{v_3}{v_1}x_1'' & x_4'' + \frac{v_4}{v_1}x_1''\end{pmatrix}. \tag{42}$$

We denote

$$A(i, j, k) = \det\begin{pmatrix} x_i & x_j & x_k\\ x_i' & x_j' & x_k'\\ x_i'' & x_j'' & x_k''\end{pmatrix}; \tag{43}$$

then

$$\begin{aligned}
\det A &= -A(2, 3, 4) - \frac{v_2}{v_1}A(1, 3, 4) - \frac{v_3}{v_1}A(2, 1, 4) - \frac{v_4}{v_1}A(2, 3, 1)\\
&= \pm\frac{1}{v_1}\left\langle (v_1, v_2, v_3, v_4), \begin{vmatrix} -\mathbf{e}_1 & -\mathbf{e}_2 & \mathbf{e}_3 & \mathbf{e}_4\\ x_1 & x_2 & x_3 & x_4\\ x_1' & x_2' & x_3' & x_4'\\ x_1'' & x_2'' & x_3'' & x_4''\end{vmatrix}\right\rangle\\
&= \pm\frac{1}{v_1}\left\langle \mathbf{v}, \boldsymbol{\gamma}(s) \wedge \boldsymbol{\gamma}'(s) \wedge \boldsymbol{\gamma}''(s)\right\rangle.
\end{aligned} \tag{44}$$

Since $\mathbf{v} \in D_H$ is a singular point, then

$$\mathbf{v} = \sinh\theta\boldsymbol{\gamma}(s) + \frac{\sinh\theta}{\delta\kappa_g(s)}(\mathbf{n}(s) \pm \mathbf{e}(s)), \tag{45}$$

and we have

$$\begin{aligned}\boldsymbol{\gamma}(s)\wedge\boldsymbol{\gamma}'(s)\wedge\boldsymbol{\gamma}''(s) &= \boldsymbol{\gamma}(s)\wedge\boldsymbol{\gamma}'(s)\\ &\quad\wedge\left(\boldsymbol{\gamma}(s)+\kappa_g(s)\,\mathbf{n}(s)\right)\\ &= \kappa_g(s)\left(\boldsymbol{\gamma}(s)\wedge\mathbf{t}(s)\wedge\mathbf{n}(s)\right)\\ &= \kappa_g(s)\,\mathbf{e}(s).\end{aligned} \tag{46}$$

Therefore

$$\begin{aligned}\det A &= \pm\frac{1}{\nu_1}\left\langle \sinh\theta\boldsymbol{\gamma}(s)\right.\\ &\quad\left.+\frac{\sinh\theta}{\delta\kappa_g(s)}\left(\mathbf{n}(s)\pm\mathbf{e}(s)\right),\kappa_g(s)\,\mathbf{e}(s)\right\rangle = \pm\frac{\sinh\theta}{\nu_1}\\ &\neq 0.\end{aligned} \tag{47}$$

In summary, H is a versal unfolding of h_ν; this completes the proof. □

5. Main Result

The main result in this paper is in this section. We now consider the following conditions:

(A1) The number of points p of $\boldsymbol{\gamma}_\theta(s)$, where $\mathrm{LC}_{\nu_0}(\theta)$ at p have four-point contact with the curve $\boldsymbol{\gamma}_\theta$, is finite.

(A2) There is no point p of $\boldsymbol{\gamma}_\theta(s)$, where $\mathrm{LC}_{\nu_0}(\theta)$ at p have five-point contact or greater with the curve $\boldsymbol{\gamma}_\theta$.

Our main result is as follows.

Theorem 5. *Let $\boldsymbol{\gamma}: I\to H_1^3$ be a unit regular spacelike curve, $\mathbf{v}_0 = L_\theta^\pm(s_0,\mu_0)$, and*

$$LC_{\nu_0}(\theta) = \left\{\mathbf{u}\in H_1^3\left(-\sinh^2\theta\right) \mid \langle\mathbf{u},\mathbf{v}_0\rangle = -\sinh^2\theta\right\}. \tag{48}$$

Then one has the following.

(1) $LC_{\nu_0}(\theta)$ and $\boldsymbol{\gamma}_\theta$ have at least 2-point contact at s_0.

(2) $LC_{\nu_0}(\theta)$ and $\boldsymbol{\gamma}_\theta$ have 3-point contact at s_0 if and only if

$$\mathbf{v}_0 = \sinh\theta\boldsymbol{\gamma}(s_0) + \frac{\sinh\theta}{\delta\kappa_g(s_0)}\left(\mathbf{n}(s_0)\pm\mathbf{e}(s_0)\right) \tag{49}$$

and $\sigma(s_0) = \kappa_g'(s_0)\mp\kappa_g(s_0)\tau_g(s_0)\neq 0$. Under this condition the germ of image $L_\theta^\pm$ at (s_0,μ_0) is diffeomorphic to the cuspidal edge $C\times\mathbb{R}$ (Figure 1).

(3) $LC_{\nu_0}(\theta)$ and $\boldsymbol{\gamma}_\theta$ have 4-point contact at s_0 if and only if

$$\mathbf{v}_0 = \sinh\theta\boldsymbol{\gamma}(s_0) + \frac{\sinh\theta}{\delta\kappa_g(s_0)}\left(\mathbf{n}(s_0)\pm\mathbf{e}(s_0)\right), \tag{50}$$

$\sigma(s_0) = 0$ and $\sigma'(s_0)\neq 0$. Under this condition the germ of image $L_\theta^\pm$ at (s_0,μ_0) is diffeomorphic to the swallowtail SW (Figure 2).

Here $C = \{(x_1,x_2) \mid x_1^2 = x_2^3\}$ and $SW = \{(x_1,x_2,x_3) \mid x_1 = 3u^4+u^2v, x_2 = 4u^3+2uv, x_3 = v\}$.

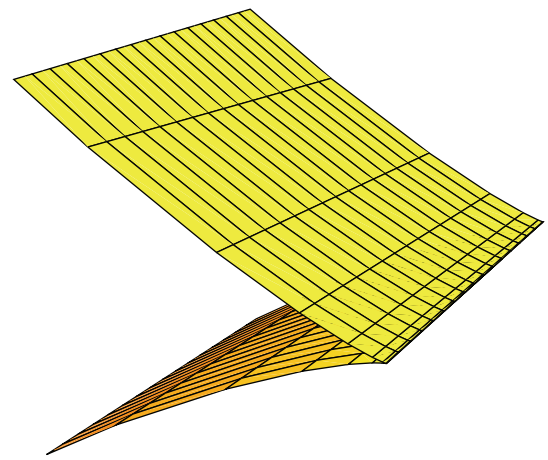

Figure 1: Cuspidal edge.

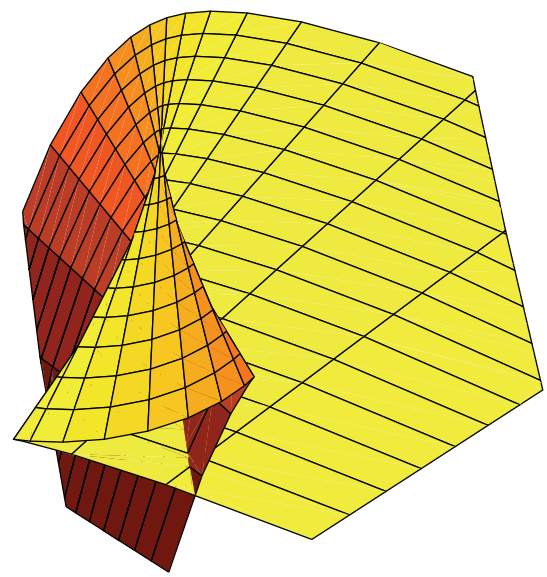

Figure 2: Swallowtail.

Proof. Let $\boldsymbol{\gamma}: I\to H_1^3$ be a spacelike regular curve and $\langle\mathbf{t}'(s),\mathbf{t}'(s)\rangle\neq -1$. As $\mathbf{v}_0 = L_{\theta_0}^\pm(s_0,\mu_0)$, we give a function

$$H: H_1^3\left(-\sinh^2\theta_0\right)\longrightarrow\mathbb{R} \tag{51}$$

by $H(u) = \langle u,v_0\rangle + \sinh^2\theta_0$; then we assume that

$$h_{\nu_0,\theta_0}(s) = H\left(\boldsymbol{\gamma}_{\theta_0}(s)\right). \tag{52}$$

Because $H^{-1}(0) = \mathrm{LC}_{\nu_0}(\theta_0)$ and 0 is a regular value of H, $\boldsymbol{\gamma}_{\theta_0}$ and $\mathrm{LC}_{\nu_0}(\theta_0)$ have $(k+1)$-point contact at s_0 if and only if $h_{\nu_0,\theta_0}(s)$ has the A_k-singularity at s_0. By Proposition 1, Theorem 3, and Proposition 4, we get the results. □

6. Example

In this section, we construct the one-parameter null hypersurfaces associated with a spacelike curve and two sections of

the one-parameter null hypersurfaces. The two sections are null surfaces and they are also the wavefronts of spacelike curves. By calculating, we get the singularities of null surfaces. It is useful to understand the one-parameter null hypersurfaces.

Let $\boldsymbol{\gamma}(s) = (\sqrt{2}\cosh(2s), \sqrt{3}\cosh(\sqrt{7}s) + 2\sinh(\sqrt{7}s), \sqrt{2}\sinh(2s), \sqrt{3}\sinh(\sqrt{7}s)+2\cosh(\sqrt{7}s))$ be a spacelike curve in H_1^3, where s is the arc-length parameter. Then

$$\begin{aligned}\mathbf{t}(s) &= \left(2\sqrt{2}\sinh(2s), \sqrt{21}\sinh\left(\sqrt{7}s\right) + 2\sqrt{7}\right.\\ &\quad\left.\cdot\cosh\left(\sqrt{7}s\right), 2\sqrt{2}\cosh(2s), \sqrt{21}\cosh\left(\sqrt{7}s\right) + 2\sqrt{7}\right.\\ &\quad\left.\cdot\sinh\left(\sqrt{7}s\right)\right),\\ \mathbf{n}(s) &= \left(\cosh(2s), \sqrt{6}\cosh\left(\sqrt{7}s\right) + 2\sqrt{2}\right.\\ &\quad\left.\cdot\sinh\left(\sqrt{7}s\right), \sinh(2s), \sqrt{6}\sinh\left(\sqrt{7}s\right) + 2\sqrt{2}\right.\\ &\quad\left.\cdot\cosh\left(\sqrt{7}s\right)\right).\end{aligned} \tag{53}$$

We have

$$\begin{aligned}\mathbf{e}(s) &= \boldsymbol{\gamma}(s) \wedge \mathbf{t}(s) \wedge \mathbf{n}(s)\\ &= \left(\sqrt{7}\sinh(2s), 2\sqrt{6}\sinh\left(\sqrt{7}s\right)\right.\\ &\quad + 4\sqrt{2}\cosh\left(\sqrt{7}s\right), \sqrt{7}\cosh(2s), 2\sqrt{6}\cosh\left(\sqrt{7}s\right)\\ &\quad\left. + 4\sqrt{2}\sinh\left(\sqrt{7}s\right)\right).\end{aligned} \tag{54}$$

$L_\theta^{\pm}(s,\mu) = \sinh\theta\boldsymbol{\gamma}(s)+\mu(\mathbf{n}(s)\pm\mathbf{e}(s))$ be the one-parameter null hypersurfaces of $\boldsymbol{\gamma}(s)$. At the moment, we can calculate that $\kappa_g(s) = 3\sqrt{2}$ and $\tau_g(s) = 2\sqrt{7}$; two sections of $L_\theta^{\pm}(s,\mu)$ with $\theta_1 = \operatorname{arcsinh}(1/2)$, $\theta_2 = \operatorname{arcsinh}(\sqrt{2}/2)$ are as follows:

$$\begin{aligned}L_{\theta_1}^{+}(s,\mu) &= \frac{1}{2}\boldsymbol{\gamma}(s) + \mu(\mathbf{n}(s) + \mathbf{e}(s)) = \left(\left(\frac{\sqrt{2}}{2} + \mu\right)\right.\\ &\quad\cdot\cosh(2s) + \sqrt{7}\mu\sinh(2s), \left(\frac{\sqrt{3}}{2} + \sqrt{6}\mu + 4\sqrt{2}\mu\right)\\ &\quad\cdot\cosh\left(\sqrt{7}s\right) + \left(1 + 2\sqrt{2}\mu + 2\sqrt{6}\mu\right)\\ &\quad\cdot\sinh\left(\sqrt{7}s\right), \sqrt{7}\mu\cosh(2s) + \left(\frac{\sqrt{2}}{2} + \mu\right)\\ &\quad\cdot\sinh(2s), \left(1 + 2\sqrt{2}\mu + 2\sqrt{6}\mu\right)\\ &\quad\cdot\cosh\left(\sqrt{7}s\right) + \left(\frac{\sqrt{3}}{2} + \sqrt{6}\mu + 4\sqrt{2}\mu\right)\\ &\quad\left.\cdot\sinh\left(\sqrt{7}s\right)\right),\end{aligned}$$

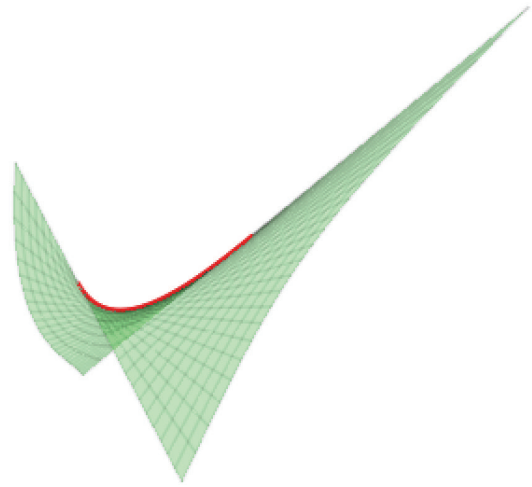

Figure 3: Projection of $L_{\theta_1}^{+}(s,\mu)$ on $x_2x_3x_4$-space.

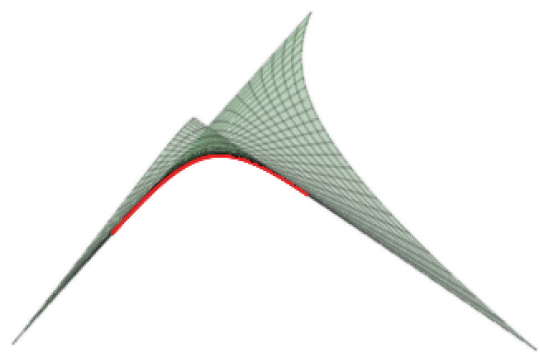

Figure 4: Projection of $L_{\theta_2}^{-}(s,\mu)$ on $x_1x_2x_3$-space.

$$\begin{aligned}L_{\theta_2}^{-}(s,\mu) &= \frac{\sqrt{2}}{2}\boldsymbol{\gamma}(s) + \mu(\mathbf{n}(s) - \mathbf{e}(s)) = \left((1+\mu)\right.\\ &\quad\cdot\cosh(2s) - \sqrt{7}\mu\sinh(2s), \left(\frac{\sqrt{6}}{2} + \sqrt{6}\mu - 4\sqrt{2}\mu\right)\\ &\quad\cdot\cosh\left(\sqrt{7}s\right) + \left(\sqrt{2} + 2\sqrt{2}\mu - 2\sqrt{6}\mu\right)\sinh\left(\sqrt{7}s\right),\\ &\quad - \sqrt{7}\mu\cosh(2s) + (1+\mu)\\ &\quad\cdot\sinh(2s), \left(\sqrt{2} + 2\sqrt{2}\mu - 2\sqrt{6}\mu\right)\cosh\left(\sqrt{7}s\right)\\ &\quad\left. + \left(\frac{\sqrt{6}}{2} + \sqrt{6}\mu - 4\sqrt{2}\mu\right)\sinh\left(\sqrt{7}s\right)\right).\end{aligned} \tag{55}$$

The pictures of θ_1-null hypersurface $L_{\theta_1}^{+}(s,\mu)$ and its singularities $L_{\theta_1}^{+}(s,\sqrt{2}/12)$ can be seen in Figure 3. And the pictures of θ_2-null hypersurface $L_{\theta_2}^{-}(s,\mu)$ and its singularities $L_{\theta_2}^{-}(s,1/6)$ can be seen in Figure 4.

Competing Interests

The authors declare that they have no competing interests.

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